ON NEW FAMILY OF KIES BURR III DISTRIBUTION:

Development, Properties, Characterizations and Applications

Fiaz Ahmad Bhatti and Munir Ahmad
National College of Business Administration and Economics, Lahore Pakistan.
Email: fiazahmad72@gmail.com, munirahmaddr@yahoo.co.uk

ABSTRACT

In this paper, a new lifetime family of distribution called new family of Kies Burr III (NFKBIII) distribution is developed from T-X family technique. The NFKBIII distribution is very flexible and its hazard rate function accommodates various shapes such as increasing, decreasing, increasing-decreasing-increasing and bathtub. The density function of the NFKBIII is arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical shaped. Some structural and mathematical properties including quantiles, sub-models, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, residual life functions and reliability measures are derived. Two characterizations for the NFKBIII distribution are studied. The maximum likelihood estimates (MLE) for unknown parameters of NFKBIII distribution are obtained. A simulation study is performed to evaluate the behavior of the maximum likelihood estimators. The NFKBIII distribution is applied to two real data sets to illustrate its potentiality and utility. The adequacy of the NFKBIII distribution is tested via different goodness of fit statistics.

Key Words: Moments; Reliability; Characterizations; Maximum Likelihood Estimation

1. INTRODUCTION

Many univariate continuous distributions have been established in recent decades but many data sets from reliability, life testing, risk analysis, finance, ecology, climatology, geology, hydrology and other fields do not fit to these distributions. Therefore, applications of the modified distributions to problems in these fields are a vibrant necessity of day.

The modified, generalized and extended distributions are attained by adding one or more parameters or the introduction of some transformation to the parent distribution. Therefore, the new proposed distributions provide best fit than the sub and competing models.

Burr [1] proposed a family of 12 distributions by fitting cumulative frequency functions to frequency data called Burr family. Burr distributions III, VI, X and XII have wide applications. Burr-III (BIII) distribution is commonly applied to model risk data in business and finance, crop rice in market, failure time data in life testing and reliability and ozone data in environmental sciences.

Many modified, generalized and extended types of BIII distribution are presented in statistical literature such as two parameter family of distributions (Mielke; [2]), inverse Burr (Kleiber and Kotz; [3]), BIII type (Gove et al.;[4]), extended Burr III (Shao et al.; [5]), Dagum (Benjamin et al.; [6]) modified BIII( Ali et al.;[7]), McDonald BIII (Gomes et al.; [8]), interpolating family (Sinner et al.;[9]), mixture of two BIII (Moisheer; [10]), generalized gamma BIII (Olobatuyi et al.; [11]), four parameter gamma BIII (Cordeiro et al.; [12]), odd BIII family (Jamil et al.[13]), Kumaraswamy odd Burr G family (Nasir et al. [14]) and generalized BIII (Kehinde et al.; [15]).

The main motivation of this article is to develop and study a flexible lifetime family of BIII type distribution with two extra shape parameters and two location parameters called the NFKBIII distribution. The shapes of NFKBIII density are arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical shapes. The hazard rate function for the NFKBIII distribution has various shapes such as increasing, decreasing, increasing-decreasing-increasing and bathtub. The NFKBIII distribution is the best model for modeling data such as times to failures of items in life testing, maximum annual flood discharges in hydrology and other various fields. The NFKBIII distribution offers better fits than sub and competing models.

The article is organized follows. In Section 2, the NFKBIII distribution is derived from T-X family technique, transformation and compounding mixture of distributions. Structural properties, quantile function, sub-models and various plots for density and hazard rate functions are discussed. In Section 3, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, residual life functions and reliability measures are derived. The characterizations for the NFKBIII distribution are studied in Section 4. In Section 5, the maximum likelihood estimates (MLEs) for unknown parameters of the NFKBIII distribution are obtained. In Section 6, a simulation study is performed to assess the behavior of the maximum likelihood estimators. In Section 7, the potentiality and utility of the NFKBIII distribution is illustrated via its applications to two real data sets: times to failures of devices and maximum annual flood discharges. The adequacy of the NFKBIII distribution is tested via different goodness of fit statistics. The ultimate comments are given in Section 8.

### 2. DEVELOPMENT OF NFKBIII DISTRIBUTION

The cumulative distribution function (cdf) of the generalized uniform distribution is given by

\[
G(x; a, b, \kappa) = \frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - a^{\kappa}}, \quad x \in [a, b], a > 0, b > 0, \kappa > 0. \tag{1}
\]

The odds ratio for the generalized uniform random variable X is

\[
W(G(x)) = \frac{G(x; a, b, \kappa)}{G(x; a, b, \kappa)} = \frac{x^{\kappa} - a^{\kappa}}{b^{\kappa} - x^{\kappa}}. \tag{2}
\]

Gurvich et al. [27] replaced ‘x’ with odds ratio in the Weibull distribution for the development of a class of extended Weibull distributions. Alzaatreh et al. [28] developed the cdf of the T-X family of distributions as

\[
F(x) = \int_a^{W(G(x))} r(t) dt, \tag{3}
\]

where \(W(G(x))\) is a function of \(G(x)\) and \(r(t)\) is the pdf of a non-negative random variable.

Bourguignon et al. [29] inserted the odds ratio of a baseline distribution in place of ‘x’ in the cdf of the Weibull distribution for the development of a new family of distributions.

The NFKBIII is developed by inserting the odds ratio for the generalized uniform in place of ‘x’ in the cdf of MBIII distribution. The cdf for the NFKBIII distribution is obtained as

\[
F(x) = \int_0^{W(G(x))} \alpha \beta r^{-\beta - 1}(1 + \gamma t^{-\beta})^{\frac{\alpha}{\beta} - 1} dt, \tag{4}
\]

or
\begin{align*}
F(x; \alpha, \beta, \gamma, \lambda) &= \int_0^{x^e-a^e} \frac{a \beta t^{-\beta-1} (1+\gamma t^{-\beta})^{-\alpha-1}}{b^e-x^e} dt, \\
or \quad F(x) &= \left[ 1 + \gamma \left( \frac{b^e-x^e}{x^e-a^e} \right)^\beta \right]^{\frac{\alpha}{\gamma}}, a \leq x < b,
\end{align*}

where \( a, b, \alpha, \beta, \kappa \) and \( \gamma \) are positive parameters of which \( a, b \) are location parameters and \( \alpha, \beta, \kappa \) are shape parameters. Clearly, \( F(x) \) is a strictly increasing and differential cdf on \((a, b)\).

The pdf of the NFKBIII distribution is
\begin{align*}
f(x) &= \alpha \beta \left( b^e-x^e \right) \kappa x^{e-1} \left[ \frac{b^e-x^e}{x^e-a^e} \right]^{\beta-1} \left[ 1 + \gamma \left( \frac{b^e-x^e}{x^e-a^e} \right)^\beta \right]^{\frac{\alpha-1}{\gamma}}, x > a.
\end{align*}

\textbf{2.1 Transformation and Compounding}

The NFKBIII model is also developed via (i) transformation between the ratio of exponential and gamma random variables and (ii) compounding generalized inverse Kies (GNIK) and gamma distributions.

\textbf{2.1 (i)} Let \( Z_1 \) be a random variable having exponential distribution with parameter value 1 and \( Z_2 \) be a random variable with gamma i.e., \( Z_2 \sim \text{gamma} \left( \frac{\alpha}{\gamma}, 1 \right) \), then using relationship \( Z_1 = \gamma \left( \frac{b^e-X^e}{X^e-a^e} \right)^\beta Z_2, \) we have

\( X = \left[ \left( a^e + b^e \frac{Z_1}{\gamma Z_2} \right)^{-\frac{1}{\beta}} \right]^{\frac{1}{\kappa}} \left[ 1 + \left( \frac{Z_1}{\gamma Z_2} \right)^{-\frac{1}{\beta}} \right]^{-\frac{1}{\gamma}} \sim \text{NFKBIII} \left( a, b, \alpha, \beta, \gamma, \kappa \right). \)

\textbf{(ii)} Let \( X \) be a random variable with GNIK distribution i.e. \( X \sim \text{GNIK} \left( x; a, b, \beta, \kappa, \gamma, \theta \right) \) and \( \theta \) be a random variable with gamma distribution, i.e. \( \theta \sim \text{gamma} \left( \theta; \alpha, \gamma \right) \), then after simplifying the integral,

\( f(x; a, b, \alpha, \beta, \gamma, \kappa) = \int_0^\infty \text{GNIK} \left( x / a, b, \beta, \kappa, \gamma, \theta \right) g \left( \theta / \alpha, \gamma \right) d\theta. \)

we have \( X \sim \text{NFKBIII} \left( a, b, \alpha, \beta, \gamma, \kappa \right). \)

\textbf{2.2 Structural Properties}

The survival, hazard, cumulative hazard, reverse hazard functions and the Mills ratio of a random variable \( X \) with the NFKBIII distribution are given, respectively, by

\( S(x) = 1 - \left[ 1 + \gamma \left( \frac{b^e-x^e}{x^e-a^e} \right)^\beta \right]^{\frac{\alpha}{\gamma}}, x \geq a, \)

\textbf{3}

\[ h(x) = \alpha \beta \kappa \left( b^x - a^x \right) x^{x-1} \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta + 1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^\alpha \theta \right]^{-1}, x > a, \quad (7) \]

\[ H(x) = -\ln \left( 1 - \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^\alpha \theta \right]^{-1} \right), x \geq a, \quad (8) \]

\[ \tau(x) = \frac{f(x)}{F(x)} = \alpha \beta \left( b^x - a^x \right) \kappa x^{x-1} \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta + 1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^\alpha \theta \right]^{-1} x > a, \quad (9) \]

and

\[ m(x) = \frac{1 - F(x)}{f(x)} = \frac{1 - \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^\alpha \theta \right]^{-1}}{\alpha \beta \left( b^x - a^x \right) \kappa x^{x-1} \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta + 1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^\alpha \theta \right]^{-1}. \quad (10) \]

The elasticity \( e(x) = x \tau(x) = \frac{d \ln F(x)}{d \ln x} \), for the NFKBIII distribution is

\[ e(x) = \alpha \beta \left( b^x - a^x \right) \kappa x^{x-1} \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta + 1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^\alpha \theta \right]^{-1}. \quad (11) \]

The quantile function of NFKBIII distribution is \( x_q = \frac{\alpha^x \gamma^{-\frac{1}{\beta}} \left( q^{-\frac{\gamma}{\alpha}} - 1 \right)^{\frac{1}{\beta}} + b^x}{\gamma^{-\frac{1}{\beta}} \left( q^{-\frac{\gamma}{\alpha}} - 1 \right)^{\frac{1}{\beta}} - 1} \) and its random number generator is \( X = \frac{\alpha^x \gamma^{-\frac{1}{\beta}} \left( Z^{-\frac{\gamma}{\alpha}} - 1 \right)^{\frac{1}{\beta}} + b^x}{\gamma^{-\frac{1}{\beta}} \left( Z^{-\frac{\gamma}{\alpha}} - 1 \right)^{\frac{1}{\beta}} - 1} \), where the random variable \( Z \) has the uniform distribution on \((0, 1)\).
2.3 Sub-Models

The NFKBIII distribution has applications in life testing, reliability concept, survival analysis and hydrology. The NFKBIII distribution has the subsequent nested models (Table 1).

2.4 Plots for the NFKBIII Density and Hazard Rate Functions

Fig. 1 shows that the shapes of the NFKBIII density are arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical (Fig. 1). The shapes of failure rate function for the NFKBIII distribution are increasing, decreasing, increasing-decreasing-increasing and bathtub (Fig. 2).

3. MATHEMATICAL PROPERTIES

Some descriptive measures for the NFKBIII distribution such as ordinary and incomplete moments, inequality curves, means deviations, residual life functions and reliability measures are established in this section.

3.1 Moments of the NFKBIII Distribution

The $r^{th}$ moment about origin of X with the NFKBIII distribution is

$$\mu'_r = E(X^r) = \int^b_a x^r f(x)dx,$$

$$E(X^r) = \alpha \beta \left(b^k - a^k\right) \kappa \int^b_a x^r x^{k-1} \left(b^k - x^k\right)^{\beta-1} \left[1 + \gamma \left(b^k - x^k\right)^{\beta} \right] \frac{\alpha - 1}{\gamma} dx,$$

Letting $\gamma \left(b^k - x^k\right)^{\beta} = w$ and $x = \left[\left(b^k - x^k\right) \frac{\alpha - 1}{1 + \gamma w^{1/\beta}} + b^k\right]^{1/\kappa}$, we arrive at

$$E(X^r) = \alpha \left(\frac{\alpha - 1}{1 + \gamma w^{1/\beta}} + b^k\right)^{r/\kappa} \left[1 + w\right]^{\alpha - 1} dw,$$

$$E(X^r) = \alpha \frac{\gamma}{\kappa} \sum_{\ell=0}^r \left(\frac{\kappa}{\ell}\right) \left(a^k - b^k\right)^{\ell} b^{(r-\ell)k} \sum_{v=0}^{\alpha} \left(-1\right)^v \frac{\ell^v}{v!} \gamma^\rho \left[w^\rho \left[1 + w\right]^{\alpha - 1} dw\right].$$

Observe that

$$\mu'_r = \alpha \frac{\gamma}{\kappa} \sum_{\ell=0}^r \left(\frac{\kappa}{\ell}\right) \left(a^k - b^k\right)^{\ell} b^{(r-\ell)k} \sum_{v=0}^{\alpha} \left(-1\right)^v \frac{\ell^v}{v!} \gamma^\rho \left[1 - \frac{\alpha}{\beta} + \frac{\nu}{\beta}\right] B\left(1 - \frac{\nu}{\beta}, \frac{\alpha}{\beta} + \frac{\nu}{\beta}\right), \quad r = 1, 2, 3, \ldots, (12)$$

where $\left(\ell\right)_k = \frac{\Gamma\left(\ell + k\right)}{\Gamma\left(\ell\right)}$ is Pochhammer symbol.
The factorial moments \( E[X]_n = \sum_{r=1}^{n} \Theta_r E(X^r) \) for the NFKBIII distribution are

\[
E[X]_n = \sum_{r=1}^{n} \Theta_r \frac{\alpha \sum_{\ell=0}^{r} \binom{r}{\ell} (a - b)^\ell b^{r-\ell}}{\gamma} \sum_{k=0}^{\infty} \frac{(-1)^k \binom{r}{k} \gamma^k \beta}{k!} B\left(1 - \frac{k}{\beta}, \frac{\alpha + k}{\beta}\right),
\]

where \( [Z]_q = Z(Z+1)(Z+2)....(Z+i-1) \) and \( \Theta_r \) is Stirling number of the first kind.

The Mellin transform is used to obtain moments of a probability distribution. By definition, the Mellin transform is

\[
M\{f(x); s\} = f^s(s) = \int_0^\infty f(x) x^{s-1} dx.
\]

The Mellin transform of X with the NFKBIII distribution is

\[
M\{f(x); s\} = \alpha \sum_{\ell=0}^{s-1} \sum_{k=0}^{\infty} \binom{s-1}{\ell} (-1)^k \binom{r}{k} \gamma^k \beta^{-1} B\left(1 - \frac{k}{\beta}, \frac{\alpha + k}{\beta}\right),
\]

The \( r^{th} \) moment about means, Pearson’s measures for skewness and kurtosis, moment generating function and cumulants of X for the NFKBIII distribution are attained from the relations

\[
\mu_r = \sum_{i=1}^{r} \binom{r}{i} \mu_1^i \mu_{r-i}, \quad \gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}}, \quad \beta_2 = \frac{\mu_4}{(\mu_2)^2} \cdot M_X(t) = E[e^{itX}] = \sum_{r=1}^{\infty} \frac{r^r}{r!} E(X)^r,
\]

and \( k_r = \mu'_r - \sum_{c=1}^{r-1} \binom{r-1}{c-1} k_c \mu'_{r-c} \).

Table 2 displays the numerical descriptive measures such as median, mean, standard deviation, skewness and kurtosis of the NFKBIII distribution for carefully chosen parameter values to describe their effect on these descriptive measures.

### 3.2 Moments of Order Statistics

Moments of order statistics have uses in life testing and reliability. Moments of order statistics are also aimed to anticipate the failure of future items obtained after few initial failures.

The pdf for \( m^{th} \) order statistic \( X_{m:n} \) is

\[
f(x_{m:n}) = \frac{1}{B(m, n-m+1)} \left[ F(x) \right]^{m-1} \left[ 1 - F(x) \right]^{n-m} f(x).
\]

The pdf of \( X_{m:n} \) for the NFKBIII distribution is
\[
\begin{align*}
\hat{f_{_{m,n}}}(x) &= \left\{ \begin{array}{ll}
1 & B(m,n-m+1) \\
\frac{1}{B(m,n-m+1) \sum_{i=0}^{m-n} (-1)^i \left( \begin{array}{c}
2m-n \\\n2i-n
\end{array} \right)} \\
\alpha \beta \left( \frac{b^x - a^x}{x^x - a^x} \right)^{\beta-1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta+1} \right]^{\frac{1}{\gamma}} & \frac{a}{r(m+1)}
\end{array} \right. 
\end{align*}
\]

Moments about the origin of \( X_{m,n} \) for the NFKBII distribution are

\[
E \left( X^r_{m,n} \right) = \int_a^b x^r f \left( x_{m,n} \right) dx,
\]

\[
E \left( X^r_{m,n} \right) = \frac{\alpha}{\gamma} \sum_{i=0}^{m-n} \sum_{v=0}^{r} \left( \begin{array}{c}
r \\
v
\end{array} \right) \left( a^x - b^x \right)^v b^{(r-v)\left( n-m \right)} \left( n-m \right)^{(r-v)} \left( \frac{n-m}{v!} \right) B \left( \frac{1 - \frac{v}{\beta}, \frac{\alpha}{(m+i) + \frac{v}{\beta}} \right)
\]

\[
E \left( X^r_{m,n} \right) = \frac{\alpha}{\gamma} B \left( m, n-m+1 \right) \sum_{i=0}^{m-n} \sum_{v=0}^{r} \left( \begin{array}{c}
r \\
v
\end{array} \right) \left( a^x - b^x \right)^v b^{(r-v)\left( n-m \right)} \left( n-m \right)^{(r-v)} \left( \frac{n-m}{v!} \right) B \left( \frac{1 - \frac{v}{\beta}, \frac{\alpha}{(m+i) + \frac{v}{\beta}} \right), r = 1, 2, 3, ...
\]

3.3 Incomplete Moments

Bonferroni and Lorenz curves can be easily computed using first incomplete moment. The life testing features such as residual life and mean inactivity life functions can be obtained from incomplete moments. The lower incomplete moments for the random variable \( X \) with the NFKBII distribution are

\[
M_r \left( z \right) = E \left( X^{(r)} \right) = \alpha \beta \left( \frac{b^x - a^x}{x^x - a^x} \right)^{\beta-1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta+1} \right]^{\frac{1}{\gamma}} dx.
\]

Letting \( \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta} = w \) and \( x = \left\{ \left( a^x - b^x \right)^{1\gamma-1} + b^x \right\} \),

we arrive at

\[
E(X^r) = \frac{\alpha}{\gamma} \int_0^\infty \left\{ \left( a^x - b^x \right)^{1\gamma-1} + b^x \right\} \left\{ 1 + \left( \frac{1}{\gamma-1} w \right) \right\}^{-1} dw,
\]

\[
E \left( X^{(r)} \right) = \frac{\alpha}{\gamma} \sum_{v=0}^{r} \left( \begin{array}{c}
v \\
0
\end{array} \right) \left( a^x - b^x \right)^v b^{(r-v)\left( n-m \right)} \left( n-m \right)^{(r-v)} \left( \frac{n-m}{v!} \right) B \left( \frac{1 - \frac{v}{\beta}, \frac{\alpha}{(m+i) + \frac{v}{\beta}} \right), r = 1, 2, 3, ...
\]

Observe that
\[
M_r'(z) = \left\{ \frac{\alpha}{\gamma} \sum_{v=0}^{\infty} \binom{\frac{v}{\gamma}}{\frac{r}{\gamma}} (a^v - b^v)^v \beta^{v(1-r)} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \gamma^{\beta-1} \right\} \left[ \beta \left( 1 - \frac{a^v - z^v}{a^v - a^v} \right)^\beta - 1 \right], \quad r = 1, 2, 3, \ldots, \tag{19}
\]

where \( B(\cdot; \cdot) \) is incomplete beta function.

The upper incomplete moments for the random variable \( X \) with the NFKBIII distribution are

\[
E_{X>z}(X^r) = \int_z^\infty x^r \alpha \beta (b^v - a^v)^v \frac{(b^v - x^v)^{\beta-1}}{(x^v - a^v)^{\beta+1}} \left[ 1 + \gamma \left( \frac{b^v - x^v}{x^v - a^v} \right)^\gamma \right] dx,
\]

\[
E_{X>z}(X^r) = \frac{\alpha}{\gamma} \sum_{v=0}^{\infty} \binom{\frac{v}{\gamma}}{\frac{r}{\gamma}} (a^v - b^v)^v \beta^{v(1-r)} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \gamma^{\beta-1} \left[ \beta \left( 1 - \frac{a^v - z^v}{a^v - a^v} \right)^\beta - 1 \right]. \tag{20}
\]

The mean deviation about the mean is \( MD_x = E|X - \mu| = 2 \mu F(\mu) - 2 \mu M'(\mu) \) and mean deviation about the median is \( MD_m = E|X - M| = 2MF(M) - 2MM'(M) \), where \( \mu = E(X) \) and \( M = Q(0.5) \). Bonferroni and Lorenz curves for a specified probability \( p \) are computed by \( B(p) = \frac{M'(q)}{p \mu} \) and \( L(p) = \frac{M'(q)}{\mu} \), where \( q = Q(p) \).

### 3.4 Residual Life functions

The \( n \)-th moment \( m_n(z) \) of residual life for \( X \) with the NFKBIII distribution is

\[
m_n(z) = E[(X - z)^n | X > z] = \frac{1}{S(z)} \int_z^\infty (x - z)^n f(x) dx,
\]

\[
m_n(z) = \frac{1}{S(z)} \sum_{v=0}^{\infty} \binom{n}{x} (-z)^{x-v} E_{X>z}(X^x),
\]

\[
m_n(z) = \frac{1}{S(z)} \sum_{v=0}^{\infty} \binom{n}{x} (-z)^{x-v} \frac{\alpha}{\gamma} \sum_{v=0}^{\infty} \binom{\frac{v}{\gamma}}{\frac{x}{\gamma}} (a^v - b^v)^v \beta^{v(1-r)} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \gamma^{\beta-1} \left[ \beta \left( 1 - \frac{a^v - z^v}{a^v - a^v} \right)^\beta - 1 \right]. \tag{21}
\]

The residual life (MRL) function \( m_1(z) \) of a component at time \( z \), or average remaining lifetime is also called the life expectancy given by

\[
m_1(z) = \frac{1}{S(z)} \sum_{v=0}^{\infty} \binom{v}{x} (-z)^{x-v} \frac{\alpha}{\gamma} \sum_{v=0}^{\infty} \binom{\frac{v}{\gamma}}{\frac{x}{\gamma}} (a^v - b^v)^v \beta^{v(1-r)} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \gamma^{\beta-1} \left[ \beta \left( 1 - \frac{a^v - z^v}{a^v - a^v} \right)^\beta - 1 \right]. \tag{22}
\]

The \( n \)-th moment of reverse residual life \( M_n(z) \) for \( X \) with the NFKBIII distribution is

\[
M_n(z) = E[(z - X)^n | X > z] = \frac{1}{F(z)} \int_x^\infty (z - x)^n \ f(x) dx
\]
\[ M_n(z) = \frac{1}{F(z)} \sum_{x=0}^{n} (-1)^x \binom{n}{x} z^{-x} E_{x \leq z}(X^x), \]

\[ M_s(z) = \frac{1}{F(z)} \sum_{x=0}^{s} (-1)^x \binom{s}{x} \left[ \sum_{t=0}^{x} (-1)^t \binom{x}{t} z^t b^{(x-t)} \right] b^{(s-x)} \left[ \frac{z^\gamma}{\gamma} \right]^x \left[ \frac{z^\beta}{\beta} \right]^x \left[ \frac{z^\alpha}{\alpha} \right]^x \left[ 1 - \frac{z}{z-a} \right] \right] \]  

\[ M_1(z) = \frac{1}{F(z)} \sum_{x=0}^{1} (-1)^x \binom{1}{x} \left[ \sum_{t=0}^{x} (-1)^t \binom{x}{t} z^t b^{(x-t)} \right] b^{(1-x)} \left[ \frac{z^\gamma}{\gamma} \right]^x \left[ \frac{z^\beta}{\beta} \right]^x \left[ \frac{z^\alpha}{\alpha} \right]^x \left[ 1 - \frac{z}{z-a} \right] \right] \]  

The waiting time \( z \) for the failure of a component has passed with condition that this failure had happened in the interval \([0, z]\) is called mean waiting time (MWT) or mean inactivity time. The waiting time \( z \) for the failure of a component of \( X \) with the NFKBIII distribution is defined by

\[ M_1(z) = \frac{1}{F(z)} \sum_{x=0}^{1} (-1)^x \binom{1}{x} \left[ \sum_{t=0}^{x} (-1)^t \binom{x}{t} z^t b^{(x-t)} \right] b^{(1-x)} \left[ \frac{z^\gamma}{\gamma} \right]^x \left[ \frac{z^\beta}{\beta} \right]^x \left[ \frac{z^\alpha}{\alpha} \right]^x \left[ 1 - \frac{z}{z-a} \right] \right] \]  

### 3.5 Stress-strength Reliability for the NFKBIII Distribution

Let \( X_1 \) be stress and \( X_2 \) be stress and \( X_1 \) follows NFKBIII distribution \((\alpha_1, \beta, \gamma, \kappa, a, b)\) and \( X_2 \) follows NFKBIII distribution \((\alpha_2, \beta, \gamma, \kappa, a, b)\), then \( R = \Pr(X_2 < X_1) = \int_a^b f_2(x)F_1(x) \, dx \) is reliability parameter (Kotz et al.; [32]). The reliability of the component is computed as

\[ R = \int_a^b \alpha_1 \beta \left( b^x - a^x \right) \kappa x^x \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta+1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^\beta \right]^{\alpha_1 - 1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^\beta \right]^{\alpha_2 / \gamma} dx, \]

\[ R = \frac{\alpha_1}{\alpha_1 + \alpha_2} \]  

Therefore \( R \) is independent of \( a, b, \beta, \kappa \) and \( \gamma \).

### 3.6 Estimation of Multicomponent Stress-Strength System Reliability with NFKBIII Distribution

Consider a system that has \( m \) identical components out of which \( s \) components are functioning. The strengths of \( m \) components are \( X_i, i = 1, 2, \ldots, m \) with common cdf \( F \) while, the stress \( Y \) imposed on the components has cdf \( G \). The strengths \( X_i, i = 1, 2, \ldots, m \) and stress \( Y \) are i.i.d. distributed. The probability that system operates properly is reliability of the system i.e.

\[ R_{s,m} = P\{\text{strengths} (X_i; i = 1, 2, \ldots, m) > \text{stress} (Y)\}, \]

\[ R_{s,m} = P\{\text{at the minimum "s" of} (X_i; i = 1, 2, \ldots, m) \text{exceed} Y\}. \]  

\[ R_{s,m} = \sum_{l=s}^{m} \binom{m}{l} \int_{-\infty}^{\infty} \left[ 1 - F(y) \right]^l [F(y)]^{m-l} \, dG(y) \quad (\text{Bhattacharyya and Johnson; [33])} \]
Let $X \sim \text{NFKBIII}(\alpha_1, \beta, \gamma, \kappa, a, b)$ and $Y \sim \text{NFKBIII}(\alpha_2, \beta, \gamma, \kappa, a, b)$ such that $\alpha_1, \alpha_2$ are unknown shape parameters and $a, b$ are common location parameters. $X$ and $Y$ are independently distributed. The reliability that system operates properly in multicomponent stress-strength for the NFKBIII distribution is

$$R_{s,m} = \sum_{\ell=s}^{m} \left( \int_{0}^{1} \left( 1 - u^\ell \right) u^{Y(m-\ell)} du, \text{ where } Y = \frac{\alpha_2}{\alpha_1}. \right.$$  

Again letting $u^Y = w,$

$$R_{s,m} = \sum_{\ell=s}^{m} \left( \int_{0}^{1} \left( 1 - u^\ell \right) w^{(m-\ell)} \frac{1}{Y} w^{-1} dw, \right.$$  

is reliability in multicomponent stress-strength model (Bhattacharyya and Johnson; [33]).

### 4. CHARACTERIZATIONS

In this section, two essential characterizations for the NFKBIII distribution are planned via: (i) conditional expectation and (ii) ratio of truncated moments.

#### 4.1 Characterization Based on Conditional Expectation

Here the NFKBIII distribution is characterized via conditional expectation.

**Proposition 4.1.1:** Let $X : \Omega \rightarrow (a, b)$ be a continuous random variable with cdf $F(x), (0 < F(x) < 1$ for $x \geq a$), then for $\alpha > \gamma$, $X$ has cdf (4) if and only if

$$E \left[ \left( \frac{X^\kappa - a^\kappa}{b^\kappa - X^\kappa} \right)^\beta \mid X < z \right] = \frac{1}{(\alpha - \gamma)} \left[ 1 + \alpha \left( \frac{z^\kappa - a^\kappa}{b^\kappa - z^\kappa} \right)^{-\beta} \right], \text{ for } z > a. \tag{29}$$  

**Proof** If (5) is pdf of $X$, then

$$E \left[ \left( \frac{b^\kappa - X^\kappa}{X^\kappa - a^\kappa} \right)^\beta \mid X < z \right] = (F(z))^{-1} \int_{a}^{z} \left( \frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa} \right)^\beta f(x) dx$$
\[ (F(z))^{-1} \int_a^z \left( \frac{b^x - x^x}{x^x - a^x} \right) \beta \, dx \times \frac{\alpha}{\gamma + \beta} \gamma (b^x - a^x) \kappa x^{x-1} \left( \frac{b^x - x^x}{x^x - a^x} \right) \beta - 1 \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right) \right] \frac{\alpha}{\gamma} \beta - 1 \, dx \]

After integrating and simplifying, we arrive at

\[ E \left[ \left( \frac{b^x - X^x}{X^x - a^x} \right)^\beta \right| X < z = \frac{1}{(\alpha - \gamma)} \left[ 1 + \alpha \left( \frac{b^x - z^x}{z^x - a^x} \right) \right], \text{ for } z > a. \]

Conversely if proposition 4.1.1 holds then

\[ \int_a^z \left( \frac{b^x - x^x}{x^x - a^x} \right) \beta f(x) \, dx = F(z) \frac{\alpha}{\gamma} \beta - 1 \left[ 1 + \alpha \left( \frac{b^x - z^x}{z^x - a^x} \right) \right]. \tag{30} \]

Differentiate (30) with respect to \( t \), we achieve

\[ \left( \frac{b^x - z^x}{z^x - a^x} \right) \beta f(z) = \frac{f(z)}{(\alpha - \gamma)} \left[ 1 + \alpha \left( \frac{b^x - z^x}{z^x - a^x} \right) \right] - F(z) \frac{\alpha}{\gamma} \beta - 1 \left[ \alpha \beta (b^x - a^x) \kappa x^{x-1} \left( \frac{b^x - z^x}{z^x - a^x} \right)^{\beta - 1} \right] \]

After simplification and integration, we work out \( F(z) = \left[ 1 + \gamma \left( \frac{b^x - z^x}{z^x - a^x} \right) \right] \), \( z \geq a \).

4.2 Characterization of the NFKBIII Distribution through Ratio of Truncated Moments

The NFKBIII distribution is characterized using Theorem G (Glänzel; [34]) from a simple relationship between two truncated moments of functions of \( X \).

**Proposition 4.2.1:** Let \( X : \Omega \rightarrow (a, b) \) be a continuous random variable. Let \( h(x) = \frac{1}{\alpha} \left[ 1 + \gamma \left( \frac{b-x}{x^x - a^x} \right)^{\beta - 1} \right], \, x > a \)

and \( h(x) = 2\alpha^{-1} \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta - 1} \left[ 1 + \gamma \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta - 1} \right], \, x > a \). According to theorem G, the random variable \( X \) has pdf (5) if and only if that the function \( p(x) \) has the form \( p(x) = \left( \frac{x^x - a^x}{b^x - x^x} \right)^{\beta}, \, x > a \).

**Proof.** For random variable \( X \) with pdf (5), then

\[ (1 - F(x)) E \left[ h(x) \right| X \geq x = \left( \frac{b^x - x^x}{x^x - a^x} \right)^{\beta}, \, x > a, \]
\[
(1 - F(x))E(h_2(x) | X \geq x) = \left(\frac{b^x - x^x}{x^x - a^x}\right)^{2\beta}, \ x > a,
\]
\[
E[h_1(x) | X \geq x] = p(x) = \left(\frac{x^x - a^x}{b^x - x^x}\right)^\beta, \ x > a
\]
and
\[
p'(x) = \beta \kappa (b^x - a^x)x^{x-1}\left(x^x - a^x\right)^{\beta-1}(b^x - x^x)^{-\beta-1}, \ x > a.
\]
The differential equation
\[
s'(x) = \frac{p'(x)h_2(x)}{p(x)h_2(x) - h_1(x)} = 2\beta (b^x - a^x) \frac{\kappa x^{x-1}}{(b^x - x^x)^2} \left[\frac{x^x - a^x}{b^x - x^x}\right]^\gamma
\]
has solution
\[
s(x) = \ln \left(\frac{x^x - a^x}{b^x - x^x}\right)^{2\beta}, \ x > a. \text{ So, in the light of theorem G, } X \text{ has pdf (5)}
\]

**Corollary 4.2.1:** Let \( X : \Omega \rightarrow (a, b) \) be a continuous random variable and let

\[
h_2(x) = 2\alpha^{-1} \left(\frac{b^x - x^x}{x^x - a^x}\right)^\beta \left[1 + \gamma \left(\frac{b^x - x^x}{x^x - a^x}\right)^\beta\right]^{\gamma + 1}, \ x > a. \text{ The pdf of } X \text{ is (5) if and only if there exists functions}
\]

\( p(x) \) and \( h_1(x) \) (defined in Theorem G), satisfying the differential equation

\[
\frac{p'(x)}{p(x)h_2(x) - h_1(x)} = \alpha \beta (b^x - a^x) \frac{\kappa x^{x-1}}{(b^x - x^x)^2} \left[\frac{x^x - a^x}{b^x - x^x}\right]^{\beta-1} \left[1 + \gamma \left(\frac{b^x - x^x}{x^x - a^x}\right)^\beta\right]^{\gamma + 1}. \ (31)
\]

**Remarks 4.2.1:** The solution of (31) is

\[
p(x) = \left(\frac{x^x - a^x}{b^x - x^x}\right)^{2\beta} \int \left[-\alpha \beta (b^x - a^x) \frac{\kappa x^{x-1}}{(b^x - x^x)^2} \left[1 + \gamma \left(\frac{b^x - x^x}{x^x - a^x}\right)^\beta\right]^{\gamma - 1} h_1(x) \right] dx + D,
\]
where \( D \) is constant.

**5. Maximum Likelihood Estimation**

In this section, parameters estimates are derived using maximum likelihood method. The log likelihood function for the NFKBIII distribution with the vector of parameters \( \Phi = (a, b, \alpha, \beta, \gamma, \kappa) \) is
\[ \ln L(x_i, \Phi) = n \ln \alpha + n \ln \beta + n \ln (b^x - a^x) - \left( \beta + 1 \right) \sum_{i=1}^{n} \ln \left( x_i^x - a^x \right) \]
\[ + \left( \beta - 1 \right) \sum_{i=1}^{n} \ln \left( b^x - x_i^x \right) - \left( \alpha \gamma + 1 \right) \sum_{i=1}^{n} \ln \left[ 1 + \gamma \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right)^\beta \right] \]
\[ = 0, \quad (32) \]

where \( a \) and \( b \) are assumed to be known, since its minimum and maximum likelihood are equal to minimum and maximum order statistics. The MLEs of the parameters for the NFKBIII distribution can be computed from the simultaneously solution of the following nonlinear equations:

\[ \frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \frac{1}{\gamma} \sum_{i=1}^{n} \ln \left[ 1 + \gamma \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right)^\beta \right] = 0, \quad (33) \]

\[ \frac{\partial \ln L}{\partial \beta} = \frac{\sum_{i=1}^{n} \ln \left( x_i^x - a^x \right) + \sum_{i=1}^{n} \ln \left( b^x - x_i^x \right) + (\alpha + \gamma) \left[ 1 + \gamma \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right)^\beta \right]^{-1} \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right) \ln \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right) = 0, \quad (34) \]

\[ \frac{\partial \ln L}{\partial \gamma} = \alpha \gamma \sum_{i=1}^{n} \ln \left[ 1 + \gamma \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right)^\beta \right] - \left( \alpha \gamma + 1 \right) \sum_{i=1}^{n} \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right)^\beta \left[ 1 + \gamma \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right)^\beta \right]^{-1} = 0, \quad (35) \]

\[ \frac{\partial \ln L}{\partial \kappa} = n \left[ \frac{b^x \ln b - a^x \ln a}{b^x - a^x} \right] - (\beta + 1) \sum_{i=1}^{n} \left[ \frac{x_i^x \ln x_i - a^x \ln a}{x_i^x - a^x} \right] + (\beta - 1) \sum_{i=1}^{n} \left[ \frac{b^x \ln b - x_i^x \ln x_i}{b^x - x_i^x} \right]
\]
\[ - (\alpha + \beta) \sum_{i=1}^{n} \left( a^x \ln a + b^x \ln b - 2x_i^x \ln x_i \right) \left( \frac{b^x - x_i^x}{x_i^x - a^x} \right)^{-\beta} + \gamma \right]^{-1}. \quad (36) \]

The above equations 33-36 can be solved either directly or using the R (optim and maxLik functions), SAS (PROC NLMIXED) and Ox program (sub-routine Max BFGS) or using non-linear optimization approaches such as the quasi-Newton procedure.

### 6. Simulation Study

In this section, the behavior of the MLEs of the NFKBIII parameters is assessed with respect to sample size \( n \). The steps for simulation to assess the behavior are as follows. Generate 10000 samples of sizes \( n \) from the NFKBIII distribution using the inverse cdf method. Calculate the MLEs for 10000 samples, say \((\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\kappa})\) for \( i=1, 2, \ldots, 10000 \) from non-linear optimization technique with constraint matching to range of parameters. \((0.10, 4, 1.2, 0.4, 1.1, 1.2), (0.5, 5, 1.5, 0.5, 1.3, 1.5)\) and \((1, 6, 2, 0.8, 1.5, 1.75)\) are taken as the true parameter values \((a, b, \alpha, \beta, \gamma, \kappa)\). Calculate the means, biases and mean squared errors (MSE) of MLEs.

For this purpose, we have chosen various arbitrarily parameters and \( n=50,100,150 \) sample sizes. All codes are written in R and the results are summarized in Table 3. The results clearly show that when the sample size \( n \) increases, the estimated MSE decrease and estimated biases drop to zero. MSE of estimated parameters increases, as shape parameter rises. This reveals that MLEs for NFKBIII distribution are reliable.

### 7. Applications

The potentiality and utility of use of NFKBIII distribution is established by applying it to two data sets: failure times of devices (Aarset, [35]) data and maximum annual flood discharges. The NFKBIII distribution is compared with KMBIII, NKBIII, KBIII, NIKL, KIL, modified Burr XII (MBXII), Burr XII (BXII), modified Burr III (MBIII), Burr III (BIII), Weibull and inverse Weibull distributions. R-package is applied to compute goodness of fit criteria such as “Cramer-von Mises (W*) , Anderson Darling (A*), Kolmogorov-Smirnov statistics with p-values [KS(p-values),
Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and estimate of likelihood ratio statistics (−ℓ) values for times to failures of 50 components and maximum annual flood discharges. Chen and Balakrishnan [36] described in detail about the statistics W* and A*.

A model is best model for which the values for goodness of fit criteria are smaller. The MLEs for unknown parameters and goodness of fit criteria values for the NFKBIII, KMBIII, NKBIII, KBIII, NIKL, KIL, modified Burr XII (MBXII), Burr XII (BXII), modified Burr III (MBIII), Burr III (BIII), Weibull and inverse Weibull models are computed.

7.1 Times to Failure: The times to failures of 50 components (Aarset,[35]) are: 0.10, 0.20, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 86, 86. The Aarset data set is recognized as bathtub shaped.

The MLEs along with standard errors (in parentheses) and goodness of fit criteria such as W*, A*, KS (p-values) are summarized in Table 4. The values of goodness-of-fit criteria such as AIC, CAIC, BIC, HQIC and −ℓ are written in Table 5.

The NFKBIII distribution is best fitted than KMBIII, NKBIII, KBIII, NIKL, KIL, MBXII, BXII, MBIII, BIII, Weibull and inverse Weibull distributions because the values of all criteria are smaller for the NFKBIII distribution. We can identify that the NFKBIII distribution is closer fit to empirical data (Fig. 3).

7.2 Maximum Annual Flood Discharges: The data for 47 years of the North Saskachevan River (Edmonton) about maximum annual flood discharges (1000 ft³/sec) are: 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.

The MLEs along with standard errors (in parentheses) and goodness of fit criteria such as W*, A*, KS (p-values) are summarized in Table 6. The values of goodness-of-fit criteria such as AIC, CAIC, BIC, HQIC and −ℓ are written in Table 7.

The NFKBIII distribution is best fitted than KMBIII, NKBIII, KBIII, NIKL, KIL, MBXII, BXII, MBIII, BIII, Weibull and inverse Weibull distributions as the values of all criteria are smaller for the NFKBIII distribution. We can identify that the NFKBIII distribution is closer fit to empirical data (Fig. 4).

8. CONCLUDING REMARKS

We have derived the NFKBIII distribution from the T-X family technique, transformation and compounding mixture of distributions. The NFKBIII density has arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical shapes. The hazard rate function for the NFKBIII distribution has various shapes such as increasing, decreasing, increasing-decreasing-increasing and bathtub. Different statistical properties such as quantile function, sub-models, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, moments for residual life functions and reliability measures have derived. Two characterizations of the NFKBIII distribution have studied. The maximum likelihood estimates (MLE) for unknown parameters of NFKBIII distribution have computed. A simulation study has accomplished to evaluate the behavior of the maximum likelihood estimators. The potentiality and utility of the NFKBIII distribution has demonstrated via its applications to times to failures of 50 devices and maximum annual flood discharges. The adequacy of the NFKBIII distribution is tested via different goodness of fit criteria. The goodness of fit statistics has shown that the NFKBIII distribution is best fit model. We have displayed that the NFKBIII distribution is empirically best for lifetime applications.
REFERENCES


---

**Fig.1: Plots of pdf of NFKBIII Distribution**

NFKBIII-Distribution for $a=0.1, b=5$

- $\alpha=1.10, \beta=2.50, \gamma=0.50, \kappa=0.50$
- $\alpha=1.10, \beta=1.70, \gamma=1.65, \kappa=3.00$
- $\alpha=1.10, \beta=3.50, \gamma=1.20, \kappa=1.00$
- $\alpha=0.95, \beta=0.95, \gamma=0.95, \kappa=1.00$
- $\alpha=1.60, \beta=0.95, \gamma=0.95, \kappa=1.10$
- $\alpha=1.70, \beta=2.70, \gamma=4.90, \kappa=0.65$
- $\alpha=1.60, \beta=1.50, \gamma=2.50, \kappa=0.40$
- $\alpha=0.75, \beta=4.60, \gamma=3.50, \kappa=0.40$
- $\alpha=0.80, \beta=4.10, \gamma=2.90, \kappa=0.40$
**Fig. 2: Plots of hrf of NFKBIII Distribution**

NFKBIII Distribution for $a=0.1, b=3$

**Fig. 3: Fitted pdf, cdf, survival and pp plots of the NFKBIII distribution for device failure times**

**Fig. 4: Fitted pdf, cdf, survival and pp plots of the NFKBIII distribution for maximum annual flood discharges**
Table 1: Sub-Models of the NFKBIII Distribution

<table>
<thead>
<tr>
<th>#</th>
<th>X</th>
<th>a</th>
<th>b</th>
<th>α</th>
<th>β</th>
<th>γ</th>
<th>κ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>κ</td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>1</td>
<td>κ</td>
</tr>
<tr>
<td>4</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>α</td>
<td>β</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>κ</td>
</tr>
<tr>
<td>7</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>κ</td>
</tr>
<tr>
<td>8</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>α</td>
<td>β</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>κ</td>
</tr>
<tr>
<td>12</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>κ</td>
</tr>
<tr>
<td>13</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>0</td>
</tr>
<tr>
<td>18</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>0</td>
</tr>
<tr>
<td>19</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>0</td>
</tr>
<tr>
<td>21</td>
<td>X</td>
<td>a</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>κ</td>
</tr>
<tr>
<td>22</td>
<td>X</td>
<td>0</td>
<td>b</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>κ</td>
</tr>
</tbody>
</table>

- New Family of Kies Burr III (NFKBIII)
- Kies Modified Burr III (KMBIII)
- New Kies Burr III (NKBIII)
- Kies Burr III (KBIII)
- Reduced Kies Burr III (RKBIII)
- New Kies Burr III (NKBIII)
- Reduced Kies Burr III (RKBIII)
- New Kies Modified Inverse Lomax (NKML)
- New Modified Kies Burr III (RNKBIII)
- Reduced New Kies Burr III (RNKBIII)
- New Inverse Kies (NIK)
- New Reduced Inverse Kies (NRK)
- New Modified Inverse Kies (NMIK)
- Reduced New Inverse Kies (RNK)
- Reduced New Inverse Kies (RNK)
- Reduced New Inverse Kies (RNK)
- Generalized Inverse Kies
- Kies (Kies,[30], and Kumar & Dharmaja ;[31])
- Modified Burr III Power (MBIII-Power)
Table 2: Median, mean, standard deviation, skewness and Kurtosis of the NFKBIII Distribution

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Median</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a, \beta, \gamma, \kappa, a = 0.1, b = 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5,0.5,0.5,0.5</td>
<td>0.4897</td>
<td>1.4750</td>
<td>1.7079</td>
<td>0.9765</td>
<td>2.3803</td>
</tr>
<tr>
<td>0.5,1.5,1.5,0.5</td>
<td>0.6763</td>
<td>1.1133</td>
<td>1.1156</td>
<td>1.2380</td>
<td>3.7107</td>
</tr>
<tr>
<td>1.0,5,0.5,0.5</td>
<td>2.1126</td>
<td>2.3588</td>
<td>1.8197</td>
<td>0.1678</td>
<td>1.4319</td>
</tr>
<tr>
<td>1,1,1,2</td>
<td>3.5333</td>
<td>3.3328</td>
<td>1.1766</td>
<td>-0.5611</td>
<td>2.3903</td>
</tr>
<tr>
<td>0.5,1,1,1</td>
<td>1.3210</td>
<td>1.7305</td>
<td>1.4595</td>
<td>0.6413</td>
<td>2.1471</td>
</tr>
<tr>
<td>1,1,1,1</td>
<td>2.5460</td>
<td>2.5475</td>
<td>1.4138</td>
<td>0.0019</td>
<td>1.8006</td>
</tr>
<tr>
<td>2,1,1,1</td>
<td>3.3649</td>
<td>3.3649</td>
<td>1.1546</td>
<td>-0.5638</td>
<td>2.3977</td>
</tr>
<tr>
<td>1.5,1,1,1</td>
<td>3.1834</td>
<td>3.0379</td>
<td>1.2827</td>
<td>-0.3377</td>
<td>2.0496</td>
</tr>
<tr>
<td>1,1,10,5</td>
<td>1.6268</td>
<td>1.9349</td>
<td>1.4408</td>
<td>0.4984</td>
<td>2.0057</td>
</tr>
<tr>
<td>1.5,1,5,1,5,1</td>
<td>2.8765</td>
<td>2.8088</td>
<td>1.1137</td>
<td>-0.2179</td>
<td>2.2432</td>
</tr>
<tr>
<td>1.5,1,5,1,5,1.5</td>
<td>3.4298</td>
<td>3.2977</td>
<td>0.9797</td>
<td>-0.5674</td>
<td>2.7456</td>
</tr>
<tr>
<td>1.5,1,5,1,5,2.5</td>
<td>3.9849</td>
<td>3.8405</td>
<td>0.7525</td>
<td>-0.9578</td>
<td>3.7872</td>
</tr>
<tr>
<td>1.5,1,5,1,5,3</td>
<td>4.1375</td>
<td>4.0009</td>
<td>0.6698</td>
<td>-1.0776</td>
<td>4.2189</td>
</tr>
<tr>
<td>1.5,1,5,1,5,0.5</td>
<td>1.9744</td>
<td>2.0897</td>
<td>1.1797</td>
<td>0.3375</td>
<td>2.2034</td>
</tr>
<tr>
<td>2,1,5,1,5,0.5</td>
<td>2.5717</td>
<td>2.5902</td>
<td>1.0939</td>
<td>0.0430</td>
<td>2.1806</td>
</tr>
<tr>
<td>2,2,2,2</td>
<td>3.8261</td>
<td>3.7287</td>
<td>0.6639</td>
<td>-0.8429</td>
<td>3.8245</td>
</tr>
<tr>
<td>2,5,1,5,1,2.5</td>
<td>4.2032</td>
<td>4.0413</td>
<td>0.6924</td>
<td>-1.2421</td>
<td>4.6981</td>
</tr>
<tr>
<td>2,5,1,5,2,5,5</td>
<td>4.0501</td>
<td>3.9854</td>
<td>0.4675</td>
<td>-0.9543</td>
<td>4.6041</td>
</tr>
<tr>
<td>5,2,5,2,5,2.5</td>
<td>4.2659</td>
<td>4.2409</td>
<td>0.3136</td>
<td>-0.5782</td>
<td>3.6917</td>
</tr>
<tr>
<td>3,2,5,2,4,0,65,1.5</td>
<td>3.7515</td>
<td>3.7577</td>
<td>0.4326</td>
<td>0.0016</td>
<td>2.7084</td>
</tr>
<tr>
<td>4,5,2,4,0,65,1.5</td>
<td>3.8735</td>
<td>3.8788</td>
<td>0.3956</td>
<td>0.0010</td>
<td>2.6739</td>
</tr>
<tr>
<td>5,2,5,2,5,2.5</td>
<td>4.2659</td>
<td>4.2409</td>
<td>0.3136</td>
<td>-0.5782</td>
<td>3.6917</td>
</tr>
<tr>
<td>5,2,5,2,3,0,5</td>
<td>3.2661</td>
<td>3.2192</td>
<td>0.9057</td>
<td>-0.2531</td>
<td>2.4020</td>
</tr>
<tr>
<td>6,2,1,5,0.5</td>
<td>3.0029</td>
<td>3.0224</td>
<td>0.7507</td>
<td>0.0597</td>
<td>2.5103</td>
</tr>
<tr>
<td>6,2,1,5,0.5</td>
<td>4.3075</td>
<td>4.2556</td>
<td>0.4691</td>
<td>-0.8193</td>
<td>3.6134</td>
</tr>
<tr>
<td>5,1,5,1,5,1</td>
<td>4.2268</td>
<td>4.1482</td>
<td>0.5200</td>
<td>-0.8090</td>
<td>3.5801</td>
</tr>
<tr>
<td>5,0,5,1,5,1</td>
<td>4.9222</td>
<td>4.6458</td>
<td>0.6553</td>
<td>-3.0134</td>
<td>13.2449</td>
</tr>
<tr>
<td>5,1,5,1,5,1</td>
<td>4.5453</td>
<td>4.3729</td>
<td>0.5804</td>
<td>-1.5137</td>
<td>5.6080</td>
</tr>
</tbody>
</table>
Table 3: Means, Bias and MSEs of the NFKBIII distribution (0.10, 4, 1.2, 0.4, 1.1, 1.2), (0.5, 5, 1.5, 0.5, 1.3, 1.5) and (1, 6, 2, 0.8, 1.5, 1.75)

<table>
<thead>
<tr>
<th>Sample</th>
<th>Statistics</th>
<th>$a = 0.10$</th>
<th>$b = 4$</th>
<th>$\alpha = 1.2$</th>
<th>$\beta = 0.4$</th>
<th>$\gamma = 1.1$</th>
<th>$\kappa = 1.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=50</td>
<td>Means</td>
<td>0.1002</td>
<td>3.9997</td>
<td>1.2416</td>
<td>0.4132</td>
<td>1.2034</td>
<td>1.2803</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>2e-04</td>
<td>-3e-04</td>
<td>0.0416</td>
<td>0.0132</td>
<td>0.1034</td>
<td>0.0803</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0</td>
<td>0</td>
<td>0.011</td>
<td>0.0015</td>
<td>0.027</td>
<td>0.0215</td>
</tr>
<tr>
<td>n=100</td>
<td>Means</td>
<td>0.10</td>
<td>4</td>
<td>1.2191</td>
<td>0.4099</td>
<td>1.1843</td>
<td>1.2772</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>0</td>
<td>0</td>
<td>0.0191</td>
<td>0.0099</td>
<td>0.0843</td>
<td>0.0772</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0</td>
<td>0</td>
<td>0.0046</td>
<td>7e-04</td>
<td>0.018</td>
<td>0.0193</td>
</tr>
<tr>
<td>n=150</td>
<td>Means</td>
<td>0.1</td>
<td>4</td>
<td>1.2092</td>
<td>0.4083</td>
<td>1.1722</td>
<td>1.2714</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>0</td>
<td>0</td>
<td>0.0092</td>
<td>0.0083</td>
<td>0.0722</td>
<td>0.0714</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0</td>
<td>0</td>
<td>0.0024</td>
<td>4e-04</td>
<td>0.0133</td>
<td>0.0165</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample</th>
<th>Statistics</th>
<th>$a = 0.5$</th>
<th>$b = 5$</th>
<th>$\alpha = 1.5$</th>
<th>$\beta = 0.5$</th>
<th>$\gamma = 1.3$</th>
<th>$\kappa = 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=50</td>
<td>Means</td>
<td>0.5015</td>
<td>4.9989</td>
<td>1.5386</td>
<td>0.5069</td>
<td>1.4244</td>
<td>1.6339</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>0.0015</td>
<td>-0.0011</td>
<td>0.0386</td>
<td>0.0069</td>
<td>0.1244</td>
<td>0.1339</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>3e-04</td>
<td>0</td>
<td>0.0232</td>
<td>0.001</td>
<td>0.0456</td>
<td>0.0418</td>
</tr>
<tr>
<td>n=100</td>
<td>Means</td>
<td>0.5003</td>
<td>4.9997</td>
<td>1.519</td>
<td>0.5011</td>
<td>1.4072</td>
<td>1.6386</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>3e-04</td>
<td>-3e-04</td>
<td>0.019</td>
<td>0.0011</td>
<td>0.1072</td>
<td>0.1386</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0</td>
<td>0</td>
<td>0.0095</td>
<td>2e-04</td>
<td>0.0298</td>
<td>0.0402</td>
</tr>
</tbody>
</table>
### Table 4: MLEs and their standard errors (in parentheses) for times to failure of devices

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\kappa$</th>
<th>$a$</th>
<th>$b$</th>
<th>$W^*$</th>
<th>$A^*$</th>
<th>KS (p-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NFKBII</td>
<td>1.0619</td>
<td>5.84</td>
<td>2.0538</td>
<td>0.8496</td>
<td>0.10</td>
<td>86</td>
<td>0.0454792</td>
<td>0.414671</td>
<td>0.076 (0.9445)</td>
</tr>
<tr>
<td>KMBIII</td>
<td>0.0619</td>
<td>-0.16</td>
<td>0.0538</td>
<td>0.0496</td>
<td>0.0119</td>
<td>0.0908</td>
<td>0.0365</td>
<td>0.0125</td>
<td>0.0835</td>
</tr>
<tr>
<td>NKBIII</td>
<td>1.0177</td>
<td>5.9591</td>
<td>2.0144</td>
<td>0.8165</td>
<td>0.0025</td>
<td>0.0218</td>
<td>0.0201</td>
<td>0.0041</td>
<td>0.0586</td>
</tr>
<tr>
<td>KBIII</td>
<td>0.0073</td>
<td>-0.0139</td>
<td>3e-04</td>
<td>0.01</td>
<td>0.0073</td>
<td>0.0066</td>
<td>0.0116</td>
<td>0.0018</td>
<td>0.0417</td>
</tr>
</tbody>
</table>

Sample | Statistics | $a = 1.0$ | $b = 6$ | $\alpha = 2.0$ | $\beta = 0.8$ | $\gamma = 1.5$ | $\kappa = 1.75$ | $\beta = 0.8$ | $\gamma = 1.5$ | $\kappa = 1.75$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>n=150</td>
<td>Means</td>
<td>0.5001</td>
<td>4.9999</td>
<td>1.5108</td>
<td>0.5001</td>
<td>1.3944</td>
<td>1.6333</td>
<td>1.3944</td>
<td>1.3944</td>
<td>1.6333</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>1e-04</td>
<td>-1e-04</td>
<td>0.0108</td>
<td>0.0108</td>
<td>0.0944</td>
<td>0.1333</td>
<td>0.0944</td>
<td>0.1333</td>
<td>0.1333</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0</td>
<td>0</td>
<td>0.0049</td>
<td>0.0049</td>
<td>0.0037</td>
<td>0.0357</td>
<td>0.0049</td>
<td>0.0357</td>
<td>0.0357</td>
</tr>
<tr>
<td>n=50</td>
<td>Means</td>
<td>1.0619</td>
<td>5.84</td>
<td>2.0538</td>
<td>0.8496</td>
<td>1.7006</td>
<td>1.8917</td>
<td>1.7006</td>
<td>1.8917</td>
<td>1.8917</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>0.0619</td>
<td>-0.16</td>
<td>0.0538</td>
<td>0.0496</td>
<td>0.2006</td>
<td>0.1417</td>
<td>0.2006</td>
<td>0.1417</td>
<td>0.1417</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.0119</td>
<td>0.0908</td>
<td>0.0365</td>
<td>0.0125</td>
<td>0.0835</td>
<td>0.0652</td>
<td>0.0835</td>
<td>0.0652</td>
<td>0.0652</td>
</tr>
<tr>
<td>n=100</td>
<td>Means</td>
<td>1.0177</td>
<td>5.9591</td>
<td>2.0144</td>
<td>0.8165</td>
<td>1.6366</td>
<td>1.8777</td>
<td>1.6366</td>
<td>1.8777</td>
<td>1.8777</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>0.0177</td>
<td>-0.0409</td>
<td>0.0144</td>
<td>0.0182</td>
<td>0.1366</td>
<td>0.1277</td>
<td>0.1366</td>
<td>0.1277</td>
<td>0.1277</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.0026</td>
<td>0.0218</td>
<td>0.0205</td>
<td>0.0041</td>
<td>0.0586</td>
<td>0.0542</td>
<td>0.0586</td>
<td>0.0542</td>
<td>0.0542</td>
</tr>
<tr>
<td>n=150</td>
<td>Means</td>
<td>1.0073</td>
<td>5.9861</td>
<td>2.0003</td>
<td>0.81</td>
<td>1.6132</td>
<td>1.8685</td>
<td>1.6132</td>
<td>1.8685</td>
<td>1.8685</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>0.0073</td>
<td>-0.0139</td>
<td>3e-04</td>
<td>0.01</td>
<td>0.1132</td>
<td>0.1185</td>
<td>0.1132</td>
<td>0.1185</td>
<td>0.1185</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>8e-04</td>
<td>0.0066</td>
<td>0.0116</td>
<td>0.0018</td>
<td>0.0417</td>
<td>0.0426</td>
<td>0.0417</td>
<td>0.0426</td>
<td>0.0426</td>
</tr>
</tbody>
</table>
Table 5: Goodness-of-fit statistics for times to failure of devices

<table>
<thead>
<tr>
<th>Model</th>
<th>AIC</th>
<th>CAIC</th>
<th>BIC</th>
<th>HQIC</th>
<th>$-\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NFKBIII</td>
<td>403.3062</td>
<td>404.2365</td>
<td>410.791</td>
<td>406.1348</td>
<td>197.6531</td>
</tr>
<tr>
<td>KMBIII</td>
<td>408.9108</td>
<td>409.4563</td>
<td>414.524</td>
<td>411.0322</td>
<td>201.4554</td>
</tr>
<tr>
<td>NKBIII</td>
<td>407.7712</td>
<td>408.3166</td>
<td>413.385</td>
<td>409.8926</td>
<td>200.8586</td>
</tr>
<tr>
<td>KBIII</td>
<td>407.3655</td>
<td>407.6322</td>
<td>411.079</td>
<td>408.7798</td>
<td>201.6828</td>
</tr>
<tr>
<td>NIKL</td>
<td>427.3011</td>
<td>427.5677</td>
<td>431.043</td>
<td>428.7153</td>
<td>211.6506</td>
</tr>
<tr>
<td>KIL</td>
<td>425.3018</td>
<td>425.3887</td>
<td>427.173</td>
<td>426.0089</td>
<td>211.6509</td>
</tr>
<tr>
<td>MBXII</td>
<td>577.3329</td>
<td>577.8546</td>
<td>583.069</td>
<td>579.5172</td>
<td>285.6664</td>
</tr>
<tr>
<td>BXII</td>
<td>548.6714</td>
<td>548.9267</td>
<td>552.495</td>
<td>550.1276</td>
<td>272.3357</td>
</tr>
<tr>
<td>MBBIII</td>
<td>478.7943</td>
<td>479.316</td>
<td>484.5304</td>
<td>480.9786</td>
<td>236.3972</td>
</tr>
<tr>
<td>BIII</td>
<td>525.2932</td>
<td>525.5485</td>
<td>529.1172</td>
<td>526.7494</td>
<td>260.6466</td>
</tr>
<tr>
<td>Weibull</td>
<td>485.9593</td>
<td>486.2146</td>
<td>489.7833</td>
<td>487.4155</td>
<td>240.9796</td>
</tr>
<tr>
<td>Inverse Weibull</td>
<td>533.973</td>
<td>534.2283</td>
<td>537.797</td>
<td>535.4292</td>
<td>264.9865</td>
</tr>
</tbody>
</table>

Table 6: MLEs and their standard errors (in parentheses) for maximum annual flood discharges

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\kappa$</th>
<th>$\alpha$</th>
<th>$b$</th>
<th>$W^*$</th>
<th>$A^*$</th>
<th>K-S (p-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NFKBIII</td>
<td>0.30254078</td>
<td>1.70483835</td>
<td>0.35210310</td>
<td>0.02101155 (2.8260695)</td>
<td>19.885</td>
<td>185.560</td>
<td>0.01576999</td>
<td>0.125909</td>
<td>0.0571 (0.9982)</td>
</tr>
<tr>
<td>KMBIII</td>
<td>0.005113609</td>
<td>8.88992999</td>
<td>0.133291960 (0.041725229)</td>
<td>1</td>
<td>19.885</td>
<td>185.560</td>
<td>0.611651</td>
<td>3.719855</td>
<td>0.2657 (0.003024)</td>
</tr>
<tr>
<td>NKBIII</td>
<td>0.005549908 (0.003225763)</td>
<td>67.4929706 (40.584567979)</td>
<td>1</td>
<td>1.619899847 (0.042015909)</td>
<td>19.885</td>
<td>185.560</td>
<td>0.3781306</td>
<td>2.386018</td>
<td>0.3031 (0.004277)</td>
</tr>
<tr>
<td>KBIII</td>
<td>0.1659637 (0.05820467)</td>
<td>3.0726999 (0.96621453)</td>
<td>1</td>
<td>1</td>
<td>19.885</td>
<td>185.560</td>
<td>0.1799505</td>
<td>1.224586</td>
<td>0.2379 (0.01094)</td>
</tr>
<tr>
<td>NIKL</td>
<td>0.7983516990 (0.1614468)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>19.885</td>
<td>185.560</td>
<td>0.01863201</td>
<td>0.1529237</td>
<td>0.239 (0.01046)</td>
</tr>
<tr>
<td>KIL</td>
<td>0.4679514 (0.06899535)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>19.885</td>
<td>185.560</td>
<td>0.03613052</td>
<td>0.2757191</td>
<td>0.2982 (0.0005609)</td>
</tr>
<tr>
<td>MBXII</td>
<td>0.0113894 (0.01673093)</td>
<td>124.9109988 (760.63939777)</td>
<td>5.4099658 (37.86181308)</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>0.05350551</td>
<td>0.3691381</td>
<td>0.5461 (7.386e-13)</td>
</tr>
<tr>
<td>BXII</td>
<td>0.07560631 (0.3409857)</td>
<td>3.48218254 (15.7016695)</td>
<td>1</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>0.05340923</td>
<td>0.368504</td>
<td>0.5449 (8.376e-13)</td>
</tr>
</tbody>
</table>
Table 7: Goodness-of-fit statistics for maximum annual flood discharges

<table>
<thead>
<tr>
<th>Model</th>
<th>AIC</th>
<th>CAIC</th>
<th>BIC</th>
<th>HQIC</th>
<th>$-\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NFKBIII</td>
<td>407.7306</td>
<td>408.7062</td>
<td>415.0452</td>
<td>410.4707</td>
<td>199.8653</td>
</tr>
<tr>
<td>KMBIII</td>
<td>519.4343</td>
<td>520.0057</td>
<td>524.9202</td>
<td>521.4894</td>
<td>256.7171</td>
</tr>
<tr>
<td>NKBIII</td>
<td>434.2349</td>
<td>434.8064</td>
<td>439.7209</td>
<td>436.29</td>
<td>214.1175</td>
</tr>
<tr>
<td>KBIII</td>
<td>423.3242</td>
<td>423.6033</td>
<td>426.9815</td>
<td>424.6943</td>
<td>209.6621</td>
</tr>
<tr>
<td>NKIL</td>
<td>423.6303</td>
<td>423.9093</td>
<td>427.2875</td>
<td>425.0003</td>
<td>209.8151</td>
</tr>
<tr>
<td>KIL</td>
<td>437.3848</td>
<td>437.3848</td>
<td>437.4757</td>
<td>439.2134</td>
<td>217.6924</td>
</tr>
<tr>
<td>MBXII</td>
<td>595.1127</td>
<td>595.6582</td>
<td>600.7263</td>
<td>597.2341</td>
<td>294.5564</td>
</tr>
<tr>
<td>BXII</td>
<td>592.772</td>
<td>593.0386</td>
<td>596.5144</td>
<td>594.1862</td>
<td>294.386</td>
</tr>
<tr>
<td>MBIII</td>
<td>436.2281</td>
<td>436.7736</td>
<td>441.8417</td>
<td>438.3495</td>
<td>215.1141</td>
</tr>
<tr>
<td>BIII</td>
<td>434.2277</td>
<td>434.4944</td>
<td>437.9701</td>
<td>435.642</td>
<td>215.1139</td>
</tr>
<tr>
<td>Weibull</td>
<td>458.1291</td>
<td>458.3958</td>
<td>461.8715</td>
<td>459.5434</td>
<td>227.0646</td>
</tr>
<tr>
<td>Inverse Weibull</td>
<td>434.2272</td>
<td>434.4938</td>
<td>437.9696</td>
<td>435.6414</td>
<td>215.1136</td>
</tr>
</tbody>
</table>

Biographies

**Fiaz Ahmad Bhatti** is currently working as Assistant Professor at Govt. Postgraduate College, Satellite Town, Gujranwala; Pakistan. He completed his MSc from Quad-i- Azam University, Islamabad, Pakistan. He completed his MPhil from Allama Iqbal Open University, Islamabad; Pakistan. He received the PhD degree in Statistics from National College of Business Administration and Economics, Lahore; Pakistan. He has published 12 research articles. His research interests include distribution theory, reliability analysis and Multivariate Analysis.

**Dr. Munir Ahmad** is currently working as Rector of National College of Business Administration and Economics, Lahore; Pakistan. He is chief editor of Pakistan journal of Statistics. He received the PhD degree in Statistics from Iowa State University, USA. He has published over 150 research articles. His research interests include distribution theory, reliability analysis, rank set sampling, quality control and statistical inference.

*Corresponding author, Tel.: +92 3338106672

Email address: fiazahmad72@gmail.com (Fiaz Ahmad Bhatti).