Analytical solution for coupled non-Fickian diffusion-thermoelasticity and thermoelastic wave propagation analysis

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Abstract. The time history analysis and propagation of molar concentration, temperature, and displacement waves are studied in details using an analytical method. The method is applied to coupled non-Fickian diffusion-thermoelasticity analysis of a strip. The governing equations are derived using non-Fickian theory of diffusion and classic theories for coupled thermoelasticity. Molar concentration and thermoelastic wave propagations are considered to be of finite speed. The governing equations are first transferred to the frequency domain using Laplace transform technique. The unknown parameters are then obtained in analytical forms proposed by the presented method. By employing the Talbot technique, the unknown parameters are eventually determined in time domain. It can be concluded that the presented analytical method has a high capability for dynamic and transient analysis of coupled diffusion-thermoelasticity problems. The wave fronts in displacement, temperature, and molar concentration fields can be tracked at various time instants employing the presented analytical method.

1. Introduction

Diffusion can be defined as the random walk of an ensemble of particles, from regions of high concentration to regions of lower concentration. The coupling between the thermoelastic fields and concentration of diffusive gas takes place when a solid body is immersed in a gas. Thermo-diffusion in an elastic solid is due

to the field of temperature, mass diffusion, and that of strain [1].

At present, much attention is paid to the governing equations and boundary conditions by using the chemical potential definition and the first and the second laws of thermodynamics. Recently, Kumar et al. [2] presented analytical solutions to the time fractional diffusion equation using Homotopy Perturbation Transform Method (HPTM). This method is a combined form of the Laplace transform and homotopy perturbation methods. The variational principle is usually used to derive a fully coupled multifield formulation [3-6]. The variational method has been regarded as the base of analysis and computations in the coupled fields. In non-Fickian theory of diffusion, it

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is assumed that the molar concentration is propagated with finite speed. There are some published works in which the non-Fickian diffusion problems have been studied using numerical methods [7-9]. Kuang [3,10] proposed an inertial entropy concept and an inertial concentration or chemical potential concept. Also, Kuang [4] considered that variation of temperature should be supported by the extra heat and proposed an inertial entropy theory from which a temperature wave equation with a finite propagation velocity was obtained. So many generalized thermoelastic theories have been proposed to allow a finite velocity for the propagation of a thermal wave. There are mainly 4 generalized theories: Lord-Shulman (LS) [11], Green-Lindsay (GL) [12], Green-Naghdi (GN) [13], and the inertial entropy [4].

In the recent years, some research works have been carried out for coupled thermoelasticity analysis based on the non-classical theories in coupled thermoelasticity using analytical [14,15] as well as numerical [16,17] methods.

Lee [18] presented the thermoelasticity problem of a multilayered hollow cylinder whose boundaries were subjected to time-dependent temperatures, adiabatic and clamped. In the case of an infinitely long cylinder, numerical results of multilayered hollow cylinder were calculated with finite difference and Laplace transform methods. In another research, Xia et al. [19] studied an isotropic semi-infinite plate subjected to a moving heat source by employing the finite element method directly in the time domain.

Sharyiat [20] carried out nonlinear generalized (with second sound effect) and classical thermoelasticity analyses for functionally graded thick cylinders subjected to various thermomechanical shocks at their inner and outer surfaces employing Hermitian elements. Recently, Sharyiat et al. [21,22] investigated some aspects of the vibration and wave propagation phenomena in thick-hollow functionally graded cylinders under thermomechanical loads using a numerical method. Bagri and Eslami [23,24] presented a unified formulation in which a few theories of coupled thermoelasticity were considered. They employed their unified formulation for dynamic and transient analysis of coupled thermoelasticity in cylinders. Sherief and Abd El-Latief [25] studied the fractional order theory of thermoelasticity for a 2D problem for a half-space and solved it with Laplace and exponential Fourier transforms techniques. Hosseini et al. [26] presented an analytical method for one-dimensional analysis of coupled non-Fickian diffusion and mechanics. Kumar and Chawla [27] studied the fundamental solution to two-dimensional problem in orthotropic thermodiffusive elastic medium. Some various problems for the theory of generalized thermoelastic diffusion have also been studied during the last few years [28-30]. The

effects of rotation and diffusion on the plane-wave propagation were presented by Singh et al. [31] in which a thermoelastic half-space without energy dissipation problem was solved using GN theory (Type II). Allam et al. [32] studied a thermoelastic diffusion interaction in an infinitely long annular cylinder using the theory of thermoelastic diffusion with one relaxation time.

Recently, several variational principles for coupled temperature-diffusion-mechanics were presented by Sue and Shen [33]. In another work [34], they presented an approximate analytical method for one-dimensional analysis of coupled non-Fickian diffusion and mechanics without considering temperature effects. Hosseini et al. [35] presented a 2D dynamic analysis of coupled non-Fickian diffusion-elasticity for a 2D domain in functionally graded materials using Meshless Local Petrov-Galerkin (MLPG) method. The propagation of wave fronts in both molar concentration and displacement fields were obtained at various times. In another work, Hosseini et al. [36] employed the MLPG method for 2D coupled non-Fickian diffusion-thermoelasticity based on GN theory of coupled thermoelasticity. They studied the dynamic behaviors of molar concentrations, temperature, and displacement fields.

The main purpose of this work is to present an effective analytical method to study the coupled non-Fickian diffusion-thermoelasticity based on the inertial entropy theory [4] in a strip subjected to shock loading. The molar concentration, temperature, and displacements are obtained in analytical forms using the presented analytical method. In this way, the wave fronts of molar concentration, displacement, and temperature can be tracked at various time instants. It is assumed that the temperature, diffusion, and elastic waves are propagated at finite speeds.

2. Governing equations

The constitutive equations for the coupled non-Fickian diffusion-thermoelasticity can be written based on the classical theory of coupled thermoelasticity [33] in the following forms:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \alpha_{ij}c - \beta_{1ij}(T - T_0), \qquad (1a)$$

$$\tilde{\mu} = \alpha_{ij}\varepsilon_{ij} + \beta c + \tau_1 T, \qquad (1b)$$

$$\tilde{s} = \beta_{1ij}\varepsilon_{ij} + \tau_1 c + \frac{A}{T_0}T, \qquad (1c)$$

where σ_{ij}, ε_{ij}, c, $\tilde{s}$, A, T, and $\tilde{\mu}$ are the stress, strain, molar concentration, entropy, temperature constant, temperature, and chemical potential (J/mol), respectively. The terms C_{ijkl}, α_{ij}, β_{1ij}, τ_1, and β stand for the elastic coefficients, mechanical diffusion coefficient,

mechanical-thermo coefficient, thermo-chemical coefficient, and the chemical potential constant, respectively.

The equation of motion in elasticity, the non-Fickian general diffusion equations, and entropy balance equation can be taken into account as follows [33]:

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i, \tag{2a}$$

$$\dot{\tilde{\mu}} + \dot{\tilde{\mu}}^{(a)} = \frac{I}{c_0} - \frac{\beta' J_{i,i}}{c_0}, \tag{2b}$$

$$\dot{\tilde{s}} + \dot{\tilde{s}}^{(a)} = \frac{\dot{r}}{T} - \frac{q_{i,i}}{T}, \tag{2c}$$

where f_i, ρ, u_i, $\tilde{\mu}^{(a)}$, $\tilde{s}^{(a)}$, I, r, q_i, and c_0 are the body force, density, displacement, inertial chemical potential, inertial entropy, diffusion source term, body heat source strength, the heat flow, and reference concentration, respectively. The term J_i is the diffusion flux (mol/m³s), which satisfies $\mathbf{J} = -D_0 \nabla \mathbf{c}$, and D_0 is the diffusion coefficient. $\tilde{\mu}^{(a)}$ and $\tilde{s}^{(a)}$ are assumed to be proportional to the rates of concentration and temperature, respectively. $\tilde{\mu}^{(a)} = \gamma \dot{c}$, and γ is a coefficient. β' is equal to RT, where R is the universal gas constant and T is the absolute temperature. β' and β satisfy $\beta = \beta'/c_0$. $\tilde{s}^{(a)} = \frac{\hat{c}\hat{\omega}}{T_0}\dot{T}$, $\hat{c}$ is the specific heat and $\hat{\omega}$ is a constant coefficient with dimension of time. Eqs. (2) can be rewritten in the following forms by neglecting the body force, diffusion source term, and body heat source strength:

$$\sigma_{ij,j} = \rho \ddot{u}_i, \tag{3a}$$

$$\dot{\tilde{\mu}} + \dot{\tilde{\mu}}^{(a)} = -\frac{\beta' J_{i,i}}{c_0} = -\beta J_{i,i}, \tag{3b}$$

$$\dot{\tilde{s}} + \dot{\tilde{s}}^{(a)} = -\frac{q_{i,i}}{T_0}. \tag{3c}$$

For a strip made of homogenous isotropic materials, the governing equations can be developed as follows:

$$(2\mu + \lambda)\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial c}{\partial x} - \beta_1\frac{\partial \theta}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{4a}$$

$$\sigma = (2\mu + \lambda)\frac{\partial u}{\partial x} - \alpha c - \beta_1 \theta, \tag{4b}$$

$$D_0\frac{\partial^2 c}{\partial x^2} = \frac{\alpha}{\beta}\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial c}{\partial t} + \tau_0\frac{\partial^2 c}{\partial t^2} + \frac{\tau_1}{\beta}\frac{\partial \theta}{\partial t}, \tag{4c}$$

$$\frac{k}{T_0}\frac{\partial^2 \theta}{\partial x^2} = \beta_1\frac{\partial^2 u}{\partial x \partial t} + \tau_1\frac{\partial c}{\partial t} + \frac{\hat{c}\hat{\omega}}{T_0}\frac{\partial^2 \theta}{\partial t^2} + \frac{A}{T_0}\frac{\partial \theta}{\partial t}, \tag{4d}$$

where $\theta = T - T_0$, $\tau_0 = \gamma/\beta$ is the relaxation time, and λ and μ are Lame's constants. Eq. (4c) for molar concentration balance is concluded by combining Eq. (1) and Eq. (3) with the expressions of J_i and $\tilde{\mu}^{(a)}$.

Obviously, there exist the coupled terms between the non-Fickian diffusion and thermoelasticity in Eqs. (4). It is noted that if the coupled term in Eq. (4c) is not considered, i.e., $\alpha = 0$ and $\tau_1 = 0$, Eq. (4c) will become the pure non-Fickian diffusion.

The following homogenous initial conditions are assumed for the problem:

$$u(x,t) = \frac{\partial u(x,t)}{\partial t} = 0, \qquad \text{at} \quad t = 0, \tag{5a}$$

$$c(x,t) = \frac{\partial c(x,t)}{\partial t} = 0, \qquad \text{at} \quad t = 0, \tag{5b}$$

$$\theta(x,t) = \frac{\partial \theta(x,t)}{\partial t} = 0, \qquad \text{at} \quad t = 0. \tag{5c}$$

The governing equations are solved for a strip of which the boundary is excited by a shock loading for molar concentration and temperature in the form of Heaviside unit step function of time:

$$\sigma(x,t) = 0, \qquad \text{at} \quad x = 0, \tag{6a}$$

$$c(x,t) = c_1 H(t), \qquad \text{at} \quad x = 0, \tag{6b}$$

$$\theta(x,t) = \theta_1 H(t), \qquad \text{at} \quad x = 0, \tag{6c}$$

$$u(x,t) = 0, \qquad \text{when} \quad x = 1.5 \text{ m}, \tag{6d}$$

$$c(x,t) = 0, \qquad \text{when} \quad x = 1.5 \text{ m}, \tag{6e}$$

$$\theta(x,t) = 0, \qquad \text{when} \quad x = 1.5 \text{ m}, \tag{6f}$$

where $H(t)$ is the Heaviside function.

The application of Laplace transform in Eqs. (4a), (4c), and (4d) with respect to time yields:

$$(2\mu + \lambda)\frac{\partial^2 \bar{u}(x,s)}{\partial x^2} - \alpha\frac{\partial \bar{c}(x,s)}{\partial x} - \beta_1\frac{\partial \bar{\theta}(x,s)}{\partial x}$$
$$= \rho s^2 \bar{u}(x,s), \tag{7}$$

$$D_0\frac{\partial^2 \bar{c}(x,s)}{\partial x^2} = \frac{\alpha}{\beta}s\frac{\partial \bar{u}(x,s)}{\partial x} + s\bar{c}(x,s)$$
$$+ \tau_0 s^2 \bar{c}(x,s) + \frac{\tau_1}{\beta}s\bar{\theta}(x,s), \tag{8}$$

$$\frac{k}{T_0}\frac{\partial^2 \bar{\theta}(x,s)}{\partial x^2} = \beta_1 s\frac{\partial \bar{u}(x,s)}{\partial x} + \tau_1 s\bar{c}(x,s)$$
$$+ \frac{\hat{c}\hat{\omega}}{T_0}s^2 \bar{\theta}(x,s) + \frac{A}{T_0}s\bar{\theta}(x,s), \tag{9}$$

where the terms $\bar{u}(x,s)$, $\bar{c}(x,s)$, and $\bar{\theta}(x,s)$ are displacement, molar concentration, and temperature in Laplace domain.

Also, the boundary conditions may be written in Laplace domain as:

$$\bar{\sigma}(x, s) = 0, \qquad \text{at} \quad x = 0, \tag{10a}$$

$$\bar{c}(x, s) = \frac{c_1}{s}, \qquad \text{at} \quad x = 0, \tag{10b}$$

$$\bar{\theta}(x, s) = \frac{\theta_1}{s}, \qquad \text{at } x = 0, \tag{10c}$$

$$\bar{u}(x, s) = 0, \qquad \text{when } x = 1.5 \text{ m}, \tag{10d}$$

$$\bar{c}(x, s) = 0, \qquad \text{when } x = 1.5 \text{ m}, \tag{10e}$$

$$\bar{\theta}(x, s) = 0, \qquad \text{when } x = 1.5 \text{ m}. \tag{10f}$$

To find the dynamic behaviors of molar concentration, temperature, and displacement from Eqs. (7)-(9), an analytical method is employed, which is explained in the next section.

3. Analytical solution

The solution to Eqs. (7), (8), and (9) can be expressed as the following Taylor series in terms of $(x - 1)$; they are analytical at $(x - 1)$:

$$\bar{u}(x, s) = \sum_{n=0}^{\infty} A_n(s)(x - 1)^n, \tag{11}$$

$$\bar{c}(x, s) = \sum_{n=0}^{\infty} B_n(s)(x - 1)^n, \tag{12}$$

$$\bar{\theta}(x, s) = \sum_{n=0}^{\infty} D_n(s)(x - 1)^n, \tag{13}$$

where $A_n(s)$, $B_n(s)$, and $D_n(s)$ are unknown coefficients. By substituting Eqs. (11), (12), and (13) in Eqs. (7), (8), and (9), the following equations can be derived:

$$(\lambda + 2\mu) \sum_{n=2}^{\infty} A_n(s)n(n - 1)(x - 1)^{n-2}$$

$$- \alpha \sum_{n=1}^{\infty} B_n(s)n(x - 1)^{n-1}$$

$$- \beta_1 \sum_{n=1}^{\infty} D_n(s)n(x - 1)^{n-1}$$

$$- \rho s^2 \sum_{n=0}^{\infty} A_n(s)(x - 1)^n = 0, \tag{14}$$

$$\frac{\alpha}{\beta}s \sum_{n=1}^{\infty} A_n(s)n(x - 1)^{n-1} + s \sum_{n=0}^{\infty} B_n(s)(x - 1)^n$$

$$+ \tau_0 s^2 \sum_{n=0}^{\infty} B_n(s)(x - 1)^n + \frac{\tau_1}{\beta}s \sum_{n=0}^{\infty} D_n(s)(x - 1)^n$$

$$- D_0 \sum_{n=2}^{\infty} B_n(s)n(n - 1)(x - 1)^{n-2} = 0, \tag{15}$$

$$\beta_1 s \sum_{n=1}^{\infty} A_n(s)n(x - 1)^{n-1} + \tau_1 s \sum_{n=0}^{\infty} B_n(s)(x - 1)^n$$

$$+ \frac{A}{T_0}s \sum_{n=0}^{\infty} D_n(s)(x-1)^n + \frac{\hat{c}\hat{\omega}}{T_0}s^2 \sum_{n=0}^{\infty} D_n(s)(x-1)^n$$

$$- \frac{k}{T_0} \sum_{n=2}^{\infty} D_n(s)n(n - 1)(x - 1)^{n-2} = 0. \tag{16}$$

Eqs. (14)-(16) may be rewritten in the new forms of:

$$\sum_{n=0}^{\infty} [(\lambda + 2\mu)(n + 1)(n + 2)A_{n+2}(s)$$

$$- \alpha(n + 1)B_{n+1}(s) - \beta_1(n + 1)D_{n+1}(s)$$

$$- \rho s^2 A_n(s)](x - 1)^n = 0, \tag{17}$$

$$\sum_{n=0}^{\infty} \left[\frac{\alpha}{\beta}s(n + 1)A_{n+1}(s) + sB_n(s) + \tau_0 s^2 B_n(s) \right.$$

$$\left. + \frac{\tau_1}{\beta}sD_n(s) - D_0(n+1)(n+2)B_{n+2}(s) \right](x-1)^n$$

$$= 0, \tag{18}$$

$$\sum_{n=0}^{\infty} \left[\beta_1 s(n + 1)A_{n+1}(s) + \tau_1 sB_n(s) + \frac{A}{T_0}sD_n(s) \right.$$

$$\left. + \frac{\hat{c}\hat{\omega}}{T_0}s^2 D_n(s) - \frac{k}{T_0}(n+1)(n+2)D_{n+2}(s) \right](x - 1)^n$$

$$= 0. \tag{19}$$

To find the unknown coefficients, the following recurrence relations can be derived as follows:

$$A_{n+2}(s) = \frac{\alpha}{(n + 2)(\lambda + 2\mu)}B_{n+1}(s)$$

$$+ \frac{\rho s^2}{(n + 1)(n + 2)(\lambda + 2\mu)}A_n(s)$$

$$+ \frac{\beta_1}{(n + 2)(\lambda + 2\mu)}D_{n+1}(s), \tag{20}$$

$$B_{n+2}(s) = \frac{\alpha s}{\beta D_0(n+2)} A_{n+1}(s)$$

$$+ \frac{s + \tau_0 s^2}{D_0(n+1)(n+2)} B_n(s)$$

$$+ \frac{\tau_1 s}{\beta D_0(n+1)(n+2)} D_n(s), \tag{21}$$

$$D_{n+2}(s) = \frac{T_0 \beta_1 s}{k(n+2)} A_{n+1}(s)$$

$$+ \frac{T_0 \tau_1 s}{k(n+1)(n+2)} B_n(s)$$

$$+ \frac{As + \hat{c}\hat{\omega}s^2}{k(n+1)(n+2)} D_n(s). \tag{22}$$

It can be seen in Eqs. (21), (22), and (23) that all coefficients $A_n(s)$, $B_n(s)$, and $D_n(s)$ can be expressed in terms of "$A_0(s)$", "$A_1(s)$", "$B_0(s)$", "$B_1(s)$", "$D_0(s)$", and "$D_1(s)$" when "$n > 1$". The terms "$A_0(s)$", "$A_1(s)$", "$B_0(s)$", "$B_1(s)$, "$D_0(s)$", and "$D_1(s)$" should be determined using boundary conditions. Therefore, the solutions (11), (12), and (13) can be rewritten in new forms as:

$$\bar{u}(x,s) = \sum_{n=0}^{\infty} \{ g_1^n(s) A_0(s) + g_2^n(s) A_1(s)$$

$$+ g_3^n(s) B_0(s) + g_4^n(s) B_1(s) + g_5^n(s) D_0(s)$$

$$+ g_6^n(s) D_1(s)\}(x-1)^n, \tag{23}$$

$$\bar{c}(x,s) = \sum_{n=0}^{\infty} \{ K_1^n(s) A_0(s) + K_2^n(s) A_1(s)$$

$$+ K_3^n(s) B_0(s) + K_4^n(s) B_1(s) + K_5^n(s) D_0(s)$$

$$+ K_6^n(s) D_1(s)\}(x-1)^n, \tag{24}$$

$$\bar{\theta}(x,s) = \sum_{n=0}^{\infty} \{ L_1^n(s) A_0(s) + L_2^n(s) A_1(s)$$

$$+ L_3^n(s) B_0(s) + L_4^n(s) B_1(s) + L_5^n(s) D_0(s)$$

$$+ L_6^n(s) D_1(s)\}(x-1)^n. \tag{25}$$

The coefficients "$g_i^n(s)$", "$K_i^n(s)$", and "$L_i^n(s)$" when "$i = 1$ to 6" may be calculated using the following equations:

$$g_i^{n+2}(s) = \frac{\alpha}{(n+2)(\lambda+2\mu)} K_i^{n+1}(s)$$

$$+ \frac{\rho s^2}{(n+1)(n+2)(\lambda+2\mu)} g_i^n(s)$$

$$+ \frac{\beta_1}{(n+2)(\lambda+2\mu)} L_i^{n+1}(s), \tag{26}$$

$$k_i^{n+2}(s) = \frac{s + \tau_0 s^2}{D_0(n+1)(n+2)} K_i^n(s)$$

$$+ \frac{\alpha s}{\beta D_0(n+2)} g_i^{n+1}(s)$$

$$+ \frac{\tau_1 s}{\beta D_0(n+1)(n+2)} L_i^n(s), \tag{27}$$

$$L_i^{n+2}(s) = \frac{As + \hat{c}\hat{\omega}s^2}{k(n+1)(n+2)} L_i^n(s) + \frac{T_0 \beta_1 s}{k(n+2)} g_i^{n+1}(s)$$

$$+ \frac{T_0 \tau_1 s}{k(n+1)(n+2)} k_i^n(s). \tag{28}$$

Also:

$$g_1^0(s) = 1, \qquad g_2^0(s) = 0, \qquad g_3^0(s) = 0,$$

$$g_4^0(s) = 0, \qquad g_5^0(s) = 0, \qquad g_6^0(s) = 0,$$

$$g_1^1(s) = 0, \qquad g_2^1(s) = 1, \qquad g_3^1(s) = 0,$$

$$g_4^1(s) = 0, \qquad g_5^1(s) = 0, \qquad g_6^1(s) = 0,$$

$$g_1^2(s) = \frac{\rho s^2}{2(\lambda+2\mu)}, \qquad g_2^2(s) = 0, \qquad g_3^2(s) = 0,$$

$$g_4^2(s) = \frac{\alpha}{2(\lambda+2\mu)}, \qquad g_5^2(s) = 0,$$

$$g_6^2(s) = \frac{\beta_1}{2(\lambda+2\mu)}, \tag{29}$$

$$K_1^0(s) = 0, \qquad K_2^0(s) = 0, \qquad K_3^0(s) = 1,$$

$$K_4^0(s) = 0, \qquad K_5^0(s) = 0, \qquad K_6^0(s) = 0,$$

$$K_1^1(s) = 0, \qquad K_2^1(s) = 0, \qquad K_3^1(s) = 0,$$

$$K_4^1(s) = 1, \qquad K_5^1(s) = 0, \qquad K_6^1(s) = 0,$$

$$K_1^2(s) = 0, \qquad K_2^2(s) = \frac{\alpha s}{2\beta D_0},$$

$$K_3^2(s) = \frac{s + \tau_0 s^2}{2D_0}, \qquad K_4^2(s) = 0,$$

$$K_5^2(s) = \frac{\tau_1 s}{2\beta D_0}, K_6^2(s) = 0, \tag{30}$$

as well as:

$$L_1^0(s) = 0, \qquad L_2^0(s) = 0, \qquad L_3^0(s) = 0,$$

$$L_4^0(s) = 0, \qquad L_5^0(s) = 1, \qquad L_6^0(s) = 0,$$

$$L_1^1(s) = 0, \qquad L_2^1(s) = 0, \qquad L_3^1(s) = 0,$$

$$L_4^1(s) = 0, \qquad L_5^1(s) = 0, \qquad L_6^1(s) = 1,$$

$$L_1^2(s) = 0, \qquad L_2^2(s) = \frac{\beta_1 T_0 s}{2k},$$

$$L_3^2(s) = \frac{\tau_1 s T_0}{2k}, \qquad\qquad L_4^2(s) = 0,$$

$$L_5^2(s) = \frac{As + \hat{c}\hat{\omega} s^2}{2k}, \qquad L_6^2(s) = 0. \tag{31}$$

Eqs. (23)-(25) are the analytical solutions for displacement, molar concentration, and temperature fields in Laplace domain. To determine the solutions in time domain, the present work uses the Talbot algorithm [37], which is based on deforming the contour in the Bromwich inversion integral to reduce numerical error. This formulation yields relations in time domain as follows:

$$u(x,t) = \frac{2}{5t} \sum_{k=0}^{M-1} \mathrm{Re}(\gamma_k \bar{u}(x, s_k)),$$

$$c(x,t) = \frac{2}{5t} \sum_{k=0}^{M-1} \mathrm{Re}(\gamma_k \bar{c}(x, s_k)),$$

$$\theta(x,t) = \frac{2}{5t} \sum_{k=0}^{M-1} \mathrm{Re}(\gamma_k \bar{\theta}(x, s_k)), \tag{32}$$

where:

$$s_k = \frac{\delta_k}{t}, \qquad \delta_0 = \frac{2M}{5}, \qquad \gamma_0 = 0.5 e^{\delta_0},$$

$$\delta_k = \frac{2k\pi}{5} \left(\cot\left(\frac{k\pi}{M}\right) + i \right),$$

$$\gamma_k = \left[1 + i\left(\frac{k\pi}{M}\right)\left(1 + \left[\cot\left(\frac{k\pi}{M}\right)\right]^2 \right) - i\cot\left(\frac{k\pi}{M}\right) \right] e^{\delta_k},$$

$$0 < k < M. \tag{33}$$

## 4. Numerical results and discussion

In this section, some numerical results are presented for the problem. For the sake of comparison, some of the material specifications of the problem are given as in [34]. However, material specifications and other

Table 1. Material specifications and other parameters of the problem.

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$\lambda = 1.326 \times 10^9 \left(\frac{N}{m^2}\right)$	$\mu = 0.884 \times 10^9 \left(\frac{N}{m^2}\right)$
$\rho = 2000 \left(\frac{kg}{m^3}\right)$	$D_0 = 10000 \left(\frac{m^2}{s}\right)$
$\beta = 5832 \left(\frac{N.m^4}{mol^2}\right)$	$\alpha = 1.87082 \left(\frac{N.m}{mol}\right)$
$\tau_0 = 3.086 \times 10^{-1}$ (s)	$\alpha_t = 1.78 \times 10^{-5} \left(\frac{1}{\circ K}\right)$
$k = 1000 \left(\frac{N}{\circ K.s}\right)$	$A = 1.8 \left(\frac{N}{m^2.\circ K}\right)$
$\tau_1 = 0.0001 \left(\frac{N.m}{mol.\circ K}\right)$	$\hat{c}\hat{\omega} = 24.5 \left(\frac{N.s}{m^2.\circ K}\right)$
$c_1 = 93540 \left(\frac{mol}{m^3}\right)$	$T_0 = 300$ ($\circ$K)
$\beta_1 = (3\lambda + 2\mu)\alpha_t$	$\theta_1 = 450$ ($\circ$K)
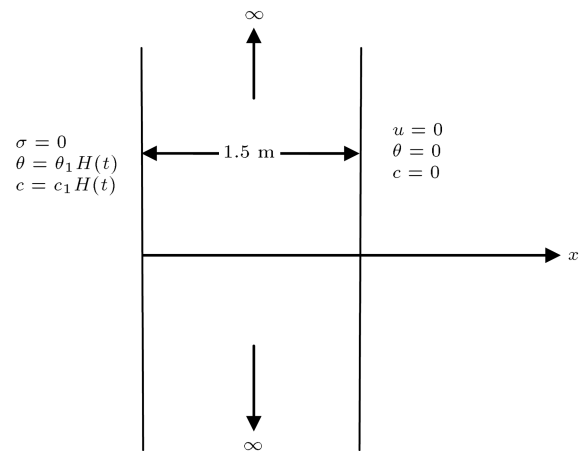
**Figure 1.** Schematic of the boundary conditions for the strip.
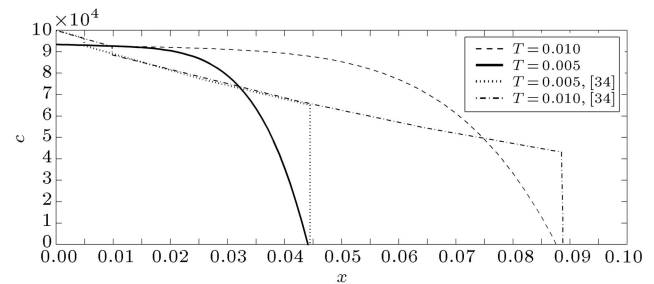
Figure 2. The comparison between obtained results and those of Ref. [34] for molar concentration along the x direction.

parameters are presented in Table 1. The boundary conditions are illustrated in Figure 1.

For validation, a comparison between the molar concentration obtained by the proposed analytical method without considering temperature and those obtained in [33] is presented in Figure 2. The comparison reveals a good agreement. Therefore, the proposed analytical method can be considered as an effective analytical method with a high capability for solving coupled governing equations of this type and similar types. The variations of displacement, temperature, and molar concentration in time domain
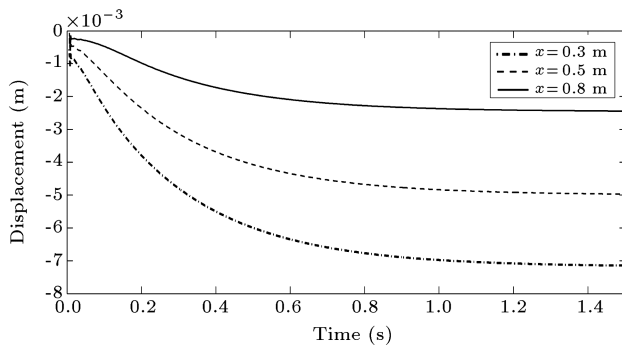
Figure 3. The dynamic behavior of the displacement at different positions in time domain.
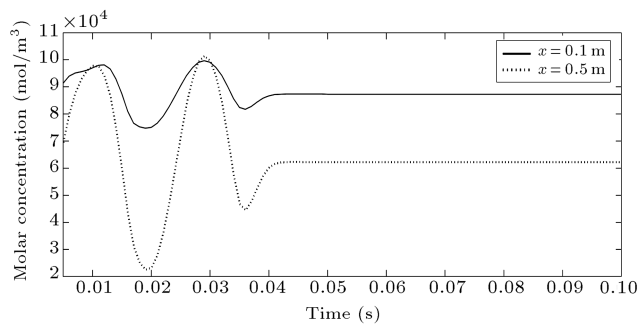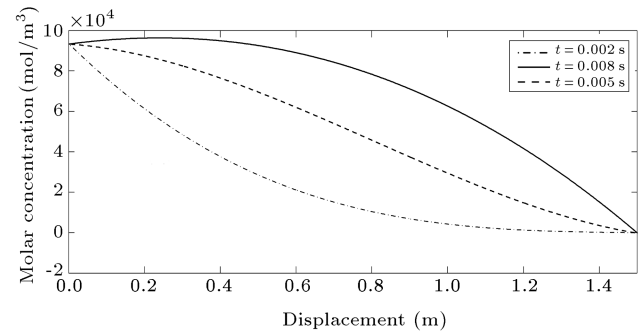

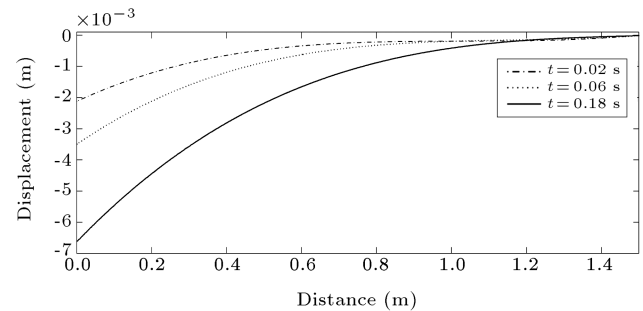**Figure 4.** The dynamic behavior of the molar concentration at different positions in time domain.
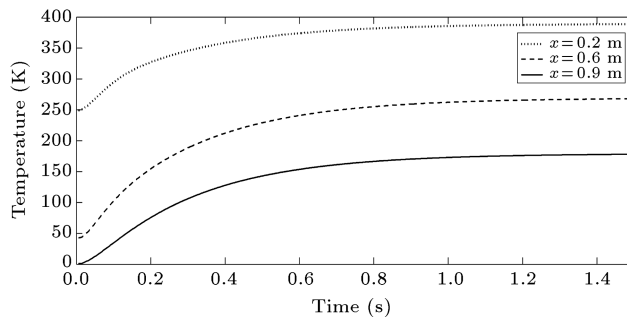
Figure 5. The dynamic behavior of the temperature at different positions in time domain.

Figure 6. The wave fronts of molar concentration wave propagation along the x direction for various time instants.

Figure 7. The wave fronts of displacement wave propagation along the x direction for various time instants.

Figure 8. The wave fronts of temperature wave propagation along the x direction for various time instants.
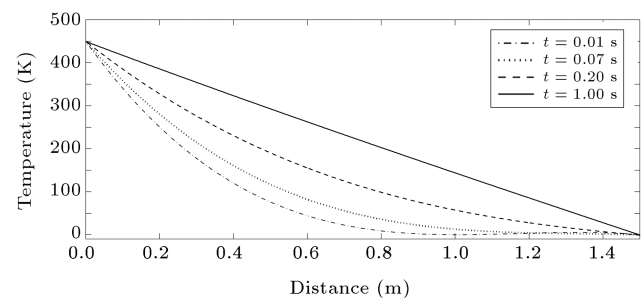
are respectively presented in Figures 3, 4, and 5 for different positions. The transient as well as steady state behaviors are shown in these figures. The conversion between thermal and mechanical energy is the reason of disturbances in the transient parts of these diagrams. From time histories of displacement, temperature, and molar concentration, it is clearly concluded that the presented analytical method has an acceptable convergence rate. The distributions of molar concentration along the x direction are shown in Figure 6. The propagation of wave front can be observed with finite speed of propagation. Also, as shown in this figure, with increasing the diffusion time, the diffusion distance gradually increases and, at the same time, concentration decreases. The distributions

of displacement and temperature along the x direction are illustrated in Figures 7 and 8 at various time instants. Obviously, as the diffusion distance increases, the displacement gradually approaches zero. It is also seen that the assumed mechanical boundary conditions are satisfied at each side of domain. It means that the presented method has a high convergence rate for satisfying the boundary conditions. As it can be observed in Figures 6-8, there are not any disturbances in the vicinity of wave fronts or other situations in diagrams. The reason may be the superiority of the convergence rate and high accuracy of the presented analytical method. The concentration field from the coupled non-Fick/non-classic diffusion-thermoelasticity theory and that from the coupled classical diffusion-thermoelasticity are compared in Figure 9. It is obviously seen that at shorter times,
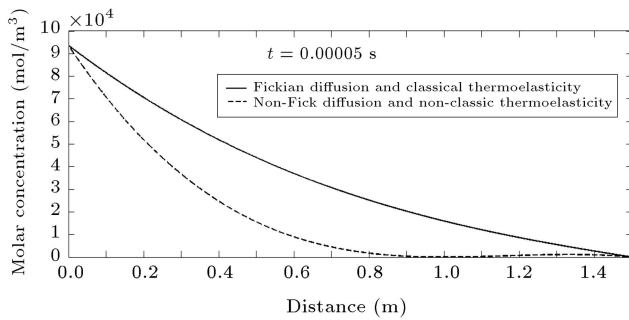
Figure 9. The comparison between classical and non-classical form of concentration along the x direction.
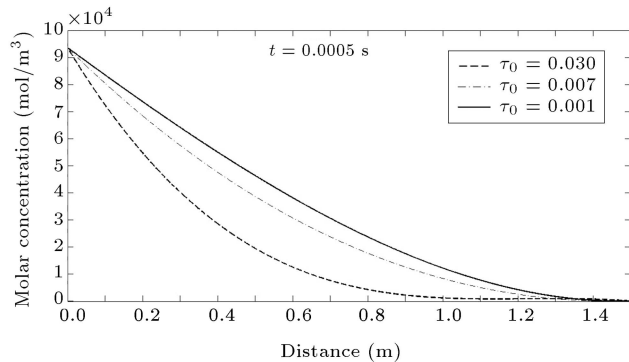
Figure 10. The effect of relaxation time on concentration along the x direction.

the difference between the two solutions is significant. The effect of the diffusion relaxation time, τ_0, on the velocity of wave propagation in molar concentration field can be observed in Figure 10. As it is shown, when the relaxation time decreases, the wave propagation velocities decrease.

5. Conclusions

In this article, an analytical solution was presented for the transient analysis of coupled non-Fickian diffusion-thermo-elasticity based on the classical theory of coupled thermoelasticity. The coupled governing equations were transferred to Laplace domain and, then, the unknown parameters including temperature, molar concentration, and displacement were obtained in closed form in Laplace domain. Using the Talbot inversion Laplace technique, the dynamic behaviors were illustrated in time domain. The main conclusions of this study can be outlined as follows:

- The presented solution may be effectively used for studying the wave propagations of displacement, molar concentration, and temperature fields. The molar concentration and displacement wave fronts can be tracked using the presented analytical method at every arbitrary time instant;
- The time history and dynamic behaviors of displacement, molar concentration, and temperature

are analyzed using an analytical method. Also, the effects of shock loadings of both molar concentration and temperature fields on the dynamic behaviors are obtained;

- The presented analytical solution and series forms for molar concentration and displacement furnish a suitable framework for optimization of coupled non-Fickian diffusion-thermoelasticity problems.

Nomenclature

A	Temperature constant
$A_n(s)$, $B_n(s)$, $D_n(s)$	Unknown coefficients
c	Mass concentration
$\hat{c}$	Specific heat
c_0	Reference concentration
c_1	Shock concentration
C_{ijkl}	The elastic constants
D_0	Diffusion coefficient
f_i	Body force
I	Diffusion source
J_i	Diffusion flux
k	Thermal conductivity
q_i	Heat flow
r	Body heat source strength
R	Universal gas constant
$\tilde{s}$	Entropy
$\tilde{s}^{(a)}$	Proportional to the rate of temperature
$\hat{\omega}$	Constant coefficient
t	Time
T	Absolute temperature
T_0	Reference temperature
u_i	Components of displacement vector
x	Position
α_{ij}, α	Mechanical diffusion coefficient
α_t	Coefficient of linear thermal expansion
β_{1ij}, β_1	Mechanical-thermo coefficient
β	Chemical potential constant
γ	Constant coefficient
λ, μ	Lame's constants
$\tilde{\mu}$	Chemical potential
$\tilde{\mu}^{(a)}$	Proportional to the rate of concentration
ρ	Density
σ_{ij}	Components of stress tensor
τ_0	Relaxation time

τ_1 Chemical-thermo coefficient

ε_{ij} Components of strain tensor

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