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A 'basic form'-focused modeling and a modified parameter estimation technique for grey prediction models

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Abstract. Grey modeling is an alternative approach to time series forecasting with growing popularity. There is no theoretical limitation to grey prediction models to adapt to almost every process by taking the appropriate order. However, deficiencies of traditional higher-order models have made researchers overlook flexibility and make use of first-order models by default. In order to bridge the mentioned gap, this paper makes two contributions. First, a novel discrete modeling is developed with the basic-form equation, reconciling estimation and prediction processes. Second, the traditional least-squares estimation technique is modified by shifting the focus from nominal parameters to parameters practically employed in the prediction process. The new approach named 'Basic Form'-focused Grey Model (BFGM) is applied to first-order, second-order, and Verhulst grey models. Then, it is validated through comparing its performance with the traditional approach's. Results show that, in most cases, BFGM makes considerable improvements in simulation and prediction accuracy, while it has reasonable computational complexity. Improvements are particularly dramatic when BFGM is applied to GM (2, 1). The resultant BFGM (2, 1) is superior in simulation and short-term prediction and, therefore, can be regarded as the basis for developing efficient higher-order grey formulations.

1. Introduction

Grey theory was introduced as a systems analysis approach by Deng in 1982, quickly turning him into the most-cited Chinese scholar [1]. It is called grey because it is especially aimed at modeling systems with limited information and partially unknown features. It contrasts with white systems which are completely known and black systems which the system modeler has no information about [2]. In other words, it focuses on information incompleteness (greyness) instead of information inaccuracy, contrary to the majority of approaches to uncertainty. Today, grey theory is among the most researched theories of uncertainty [1] and is well known to many system analyzers because of its successful implementation in different fields [2].

Grey prediction is a major component of grey system theory [3] with increasing popularity among forecasting researchers [2,4]. Such popularity may have been derived from its ability to analyze limited data [5-14], simple and efficient computations [5-7,10,11,15], and making no statistical assumptions [8,13]. In addition to their countless applications to limited data, grey prediction techniques have also proved applicable to many high-frequency time series [2,3,14,16-23].

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Traditional grey prediction offers an assortment of models. First-order models can be considered the first generation of grey prediction models characterized by their simplicity. However, this simplicity comes at the price of a rigid structure adaptive to a narrow range of real processes [21]. Higher-order models, on the other hand, are capable of overcoming many of these limitations. Nevertheless, most researches have missed their remarkable advantages, especially due to their complexity [19,21].

In lieu of utilizing higher-order models, most researchers have developed a variety of revisions to the first-order models, as summarized in Table 1. However, there are a couple of researches that utilized the second-order models, as shown in Table 2. Additionally, nonstandard grey prediction models are also available. Among these, Grey Verhulst Model (GVM) is an important one [2,24-28] with widespread applications [1,26]. It incorporates nonlinear features into the first-order models while keeping them first-order. Being well established, GVM is a current area of abundant researches as Table 3 outlines.

Table 1 shows that most researchers working on parameter estimation of first-order models have missed the focal point, least-squares estimation of primary parameters. A similar, yet bigger, research gap is identified in second-order models since the recent literature has totally neglected to improve parameter estimates therein, as implied by Table 2.

Most importantly, there is inherent inconsistency in traditional grey models due to their continuous prediction (simulation) process and discrete estimation process [1,10,14,20,26,29-33]. Xie and Liu [32] proposed a solution to this problem, called Discrete Grey Model (DGM). It discretizes the whole algorithm to avoid the error originating with skips from continuous to discrete computations, and vice versa. As a milestone, it has been applied to many further researches. Indeed, every research involving discretization of the first-order models in Table 1 has relied on DGM.

However, DGM does not adequately adhere to the original grey modeling framework (Subsection 2.1) as it omits the essential mean generation operation. Moreover, Xie and Liu [32] focused only on discretizing GM (1, 1). Notwithstanding theoretical feasibility of DGM (n, h), our extensive literature review detected no practical higher-order DGMs. Interestingly, the few instances of discretization in Table 2, i.e., Chiang and Tseng [17] and Chuang and Kao [7], are decade-old researches being conducted before DGM was ever introduced. Equally important, it is shown that discretization of higher-order grey models has been overlooked for years.

Furthermore, to the best of our knowledge, DGM has never been customized for nonstandard grey models. This has led researchers to inexact applications of DGM (1, 1)'s forecasting functions to GVM-based forecasting problems, e.g., Wang et al. [31] and Xiao and Qin [26]. It is noteworthy that Guo et al. [34] managed to discretize GVM without relying on DGM. Nonetheless, the lack of an exact discretization of GVM persists since they only focus on a completely transformed formulation of GVM.

Having targeted recognized research gaps, this paper develops a new formulation named 'Basic Form'-focused Grey Model (BFGM) consisting of two main contributions. First, BFGM introduces a novel modeling for resolving leaps between continuous and discrete computations, especially in higher-order and non-standard models. Our proposed modeling reconciles estimation and prediction processes based on the discrete equation known as basic form. Second, BFGM modifies traditional least-squares for optimizing parameters practically applied to the prediction (simulation) process instead of optimizing nominal parameters. Notice that BFGM focuses on primary (essential) parameters of each grey model shared among all of its formulations and not on its auxiliary (optional) parameters. Similar to Xu et al. [27] and Zhu [28], the triplet of simple first-order, second-order, and Verhulst grey models are analyzed. By applying BFGM, these models are respectively developed into BFGM (1, 1), BFGM (2, 1), and BFGVM in this research.

Despite the multitude of authors concerning complexity when deciding among grey prediction models [2,6,14,19,21,22,35], there are few authors actually measuring it [22,23,36,37]. We believe that quantitative analyses can be the first step in relieving the computational complexity of higher-order models. Therefore, our experimental analysis covers not only the standard accuracy criterion, but also the disregarded processing time statistics. In addition to comparing the new formulation with the traditional one, traditional grey models can be compared with each other. Accordingly, the reported time-inefficiency of the second-order grey model may be particularly investigated.

The remainder of this paper is organized as follows. Section 2 introduces computational procedures of selected traditional grey prediction models. Section 3 describes our new formulation for these models. Section 4 introduces the practical experimentation of all of the models on some low-frequency and high-frequency time series, making provisions for a detailed comparative analysis. Eventually, Section 5 concludes results and recommends some future research guidelines.

2. Traditional grey prediction models

2.1. The general grey modeling framework

GM (M, N) stands for standard Grey Model with an underlying differential equation (Whitenization equa-

Table 1. Major research trends on first-order grey models compared with this research.

	Preprocessing	Rolling mechanism	Initial conditions	Adapting to non-equidistant data	Hybridizing by other forecasting models		Sequence operators		Transforming main equations		Parameter estimation			Discretization
					Simple additive (Error compensation)	Sophisticated	AGO/IAGO	Mean generation	Whitenization equation	Forecasting functions	Auxiliary parameters	Primary parameters: Alternative techniques	Primary parameters: Modeling least-squares	
Lin et al. [22]	√				√									
Chang [6]		√						√				√		
Li and Chen [21]		√								√				
Tsaur [42]						√						√		
Tien [43]									√		√	√		
Wang et al. [25]	√	√				√								
Xie and Liu [32]			√											√
Kayacan et al. [2]	√	√		√	√	√								
Zhu [28]						√								
Hsu [8]								√		√				
Kayacan and Kaynak [19]		√												
Li et al. [9]	√			√			√							
Shih et al. [38]		√	√					√						
Wang et al. [29]		√	√								√			√
Wang et al. [15]	√											√		
Wang et al. [30]	√	√	√							√	√			√
Xu et al. [27]			√											
Jin et al. [4]	√							√						
Cui et al. [18]			√						√					
Xie, et al. [33]			√							√				√
Zhou and He [14]										√	√			√
Chang et al. [16]		√		√				√			√			
Choi et al. [36]					√									
Evans [24]		√								√			√	
Li and Xiao [11]				√	√			√	√					
Liu et al. [37]		√						√			√			
Wu et al. [44]							√			√				√
Xia and Wong [3]			√				√			√				√
Li et al. [20]		√			√									
Luo and Chang [23]		√				√								
Tsai [40]							√	√			√			
Wu et al. [12]							√							
Xia, et al. [45]		√						√						
Wu et al. [13]							√			√				
Wu et al. [41]	√					√								
This research													√	√

Table 2. Major research trends on second-order grey models compared with this research.

	Preprocessing	Rolling mechanism	Initial conditions	Hybridizing (sophisticated)	Sequence operators (AGO/IAGO)	Sequence operators (mean generation)	Numerical methods for differentiation	Parameter estimation	Discretization
Chiang and Tseng [17]	√								√
Chuang and Kao [7]	√			√					√
Li et al. [35]		√	√				√		
Zhu [28]				√					
Li et al. [10]		√				√	√		
Xu et al. [27]			√						
Wu et al. [12]			√		√				
This research								√	√

Table 3. Major research trends on grey Verhulst model compared with this research.

	Preprocessing	Rolling mechanism	Initial conditions	Direct modeling	Adapting to non-equidistant data	Hybridizing (simple additive)	Hybridizing (sophisticated)	Sequence operators (mean generation)	Transforming main equations (forecasting functions)	Parameter estimation	Discretization (inexact)	Discretization (exact)
Wang et al. [25]	√	√		√			√					
Wang et al. [31]	√		√								√	
Kayacan et al. [2]	√	√				√	√					
Zhu [28]				√			√					
Xu et al. [27]			√	√								
Xiao and Qin [26]	√										√	
Guo et al. [34]		√					√	√	√		√	
Evans [24]		√							√	√		
Bezuglov and Comert [5]		√				√						
Hashem Nazari et al. [46]				√	√							√
This research												√

tion) of order M holding 1 dependent plus $N-1$ independent variables.

A defining characteristic of grey prediction is preprocessing/post-processing by sequence operators, e.g., accumulated generation, its inverse, and mean generation.

Accumulation adapts grey models to quasi-smooth sequences [1]. By applying the 1st-order Accumulated Generation Operation (1-AGO) to original sequence (dependent) $Y^{(0)}$, the 1st-order accumulated sequence $Y^{(1)}$ is formed as follows:

$$Y^{(1)}(t) = Y^{(1)}(t-1) + Y^{(0)}(t) = \sum_{j=1}^{t} Y^{(0)}(j);$$

$$t = 1, 2, \ldots, n. \tag{1}$$

Inverse AGO (IAGO) complements AGO as its post-processing. It is usually represented by operator α. 1-IAGO, i.e., the 1st-order IAGO, is formulated as subtraction of two consecutive sequence values.

Mean generation is averaging on two successive values (Eq. (2)) to smoothen data [24,30] or fill missing

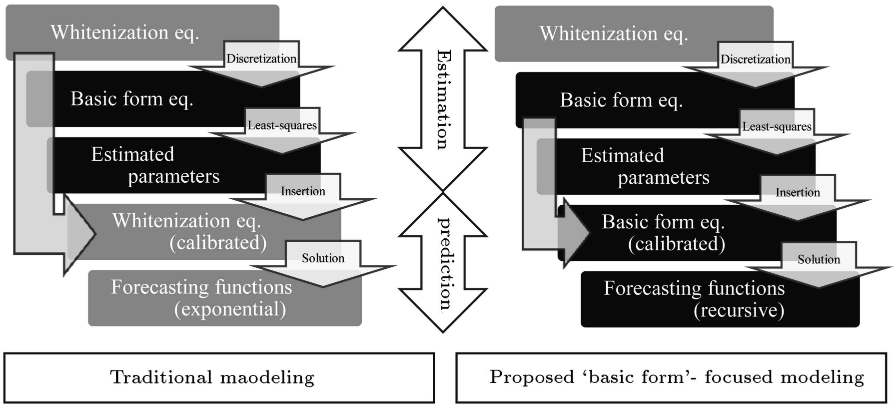

Figure 1. Schematic comparison between the traditional and the proposed modeling. The figure illustrates how the proposed modeling avoids unnecessary leaps between continuous computations and discrete computations, which can be identified by light and dark gray boxes, respectively.

values [1]. It is usually formulated as follows:

$$Z^{(1)}(t) = \frac{Y^{(1)}(t) + Y^{(1)}(t-1)}{2}. \qquad (2)$$

Combined with Eq. (1), it can be transformed as follows:

$$Z^{(1)}(t) = \frac{Y^{(0)}(t) + 2Y^{(1)}(t-1)}{2}. \qquad (3)$$

Traditional forecasting procedure comprises a bipartite solution of whitenization equation, as depicted on the left side of Figure 1. The first part is parameter estimation which requires to approximate continuous whitenization equation into discrete basic-form equation compliant with least-squares. The second is prediction through calibrated whitenization equation. Its solution called time response function provides accumulated forecasts. Finally, 1-IAGO post-processing creates the function that provides restored values, i.e., non-accumulated forecasts.

2.2. Formulating traditional GM (1, 1)

GM (1, 1) is the basic grey model with whitenization Eq. (4). It is also the most widely used mainly due to its simplicity and computational efficiency [2, 12,19,24,36,37]:

$$\frac{dY^{(1)}(t)}{dt} + aY^{(1)}(t) = b, \qquad t = 1, 2, \ldots . \qquad (4)$$

The basic-form equation approximates integration and differentiation to summation, i.e., AGO, and difference, i.e., IAGO, respectively. In addition, it substitutes $Z^{(1)}$ for $Y^{(1)}$ After simple manipulation, we have the rearranged basic form:

$$Y^{(0)}(t) = -aZ^{(1)}(t) + b; \qquad t = 2, 3, \ldots, n, \qquad (5)$$

in which $Z^{(1)}$ is quantified by Eq. (2). Formulated in matrix term, it turns into:

$$Y_0 = B.A; \qquad Y_0 = \begin{bmatrix} Y^{(0)}(2) \\ Y^{(0)}(3) \\ \vdots \\ Y^{(0)}(n) \end{bmatrix},$$

$$B = \begin{bmatrix} -Z^{(1)}(2) & 1 \\ -Z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -Z^{(1)}(n) & 1 \end{bmatrix}, \quad A = \begin{bmatrix} a \\ b \end{bmatrix}. \qquad (6)$$

Then, Eq. (7) derives least-squares parameters [1]:

$$\hat{A} = (B^T.B)^{-1}.B^T.Y_0. \qquad (7)$$

Now, the whitenization Eq. (4) can be solved to establish the time response function. Tuned by traditional first-datum-based initial condition, it derives accumulated forecasts $\hat{Y}^{(1)}(t)$. Then, 1-IAGO post-processing gives restored values through Eq. (8):

$$\hat{Y}^{(0)}(t) = (1 - e^a)(Y^{(0)}(1) - b/a)e^{-a(t-1)};$$

$$t = 2, 3, \ldots . \qquad (8)$$

2.3. Formulating traditional GM (2, 1)

Instead of exploiting vast flexibility of GM (M, N), the literature has usually applied GM (1, 1) by default [2]. Despite its advantages, GM (1, 1) is inappropriate for complicated time series, e.g., non-monotonic trends [4,6,10,38]. Such deficiency is best addressed by GM (2, 1), a standard grey model with whitenization Eq. (9):

$$\frac{d^2Y^{(1)}(t)}{dt^2} + a_1\frac{dY^{(1)}(t)}{dt} + a_2Y^{(1)}(t) = b;$$

$$t = 1, 2, \ldots, n. \qquad (9)$$

Approximating the first and second derivatives of $Y^{(1)}(t)$ to $Y^{(0)}(t)$ and $\alpha^{(1)}Y^{(0)}(t)$, respectively, followed by replacing $Y^{(1)}(t)$ itself with $Z^{(1)}(t)$ provides the basic form. Afterwards, we may readily sort out some terms to obtain the rearranged basic form:

$$\alpha^{(1)}Y^{(0)}(t) = -a_1Y^{(0)}(t) - a_2Z^{(1)}(t) + b;$$
$$t = 2, 3, \ldots, n, \tag{10}$$

where $Z^{(1)}$ is calculated through Eq. (2). Translated to matrix terms, Eq. (10) becomes:

$$\alpha Y_0 = B.A; \qquad \alpha Y_0 = \begin{bmatrix} \alpha^{(1)}Y^{(0)}(2) \\ \alpha^{(1)}Y^{(0)}(3) \\ \vdots \\ \alpha^{(1)}Y^{(0)}(n) \end{bmatrix},$$

$$B = \begin{bmatrix} -Y^{(0)}(2) & -Z^{(1)}(2) & 1 \\ -Y^{(0)}(3) & -Z^{(1)}(3) & 1 \\ \vdots & \vdots & \vdots \\ -Y^{(0)}(n) & -Z^{(1)}(n) & 1 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix}. \tag{11}$$

Least-squares parameters are determined by Eq. (12) [1].

$$\hat{A} = (B^T.B)^{-1}.B^T.\alpha Y_0. \tag{12}$$

Before solving Eq. (9), one should analyze its characteristic by Eq. (13) [39]:

$$m^2 + a_1m + a_2 = 0. \tag{13}$$

The solution procedure depends on the sign of discriminant $\Delta = a_1^2 - 4a_2$.

When the discriminant is a positive number ($\Delta > 0$), is zero ($\Delta = 0$), and is negative number ($\Delta < 0$), the following procedures are applied:

- **Case I:** $\Delta > 0$. By acknowledging $m_1 = \frac{1}{2}(\sqrt{\Delta} - a_1)$ and $m_2 = \frac{-1}{2}(\sqrt{\Delta} + a_1)$ as distinct real roots of Eq. (13), the general solution to Eq. (9) is formulated as the former time response function. Evidently, applying 1-IAGO gives the former function its restored values through Eq. (14):

$$\hat{Y}^{(0)}(t) = e^{m_2(t-2)}\Bigg(c_1e^{\sqrt{\Delta}(t-2)}\Big(e^{\sqrt{\Delta}+m_2-1)}$$
$$+ c_2\Big(e^{m_2} - 1\Big)\Bigg). \tag{14}$$

Eq. (14) together with traditional first-datum-based initial conditions determines c_1 and c_2 thus creating the final restored values.

- **Case II:** $\Delta = 0$. Eq. (13) has a repeated real root $m = -a_1/2$; accordingly, Eq. (9) is solved to derive former time responses and, then, former restored values as follows:

$$\hat{Y}^{(0)}(t) = e^{m(t-2)}\Bigg(c_1(e^m - 1)$$
$$+c_2\Big(e^m(t-1) - (t-2)\Big)\Bigg). \tag{15}$$

Once again, traditional initial conditions are established and solved to quantify c_1 and c_2 and build final forecasting functions.

- **Case III:** $\Delta < 0$. Eq. (13) has no real roots. Similar computational procedure derives restored values as follows:

$$\hat{Y}^{(0)}(t) = e^{-\frac{a_1}{2}(t-1)}$$
$$\left(\begin{matrix} c_1\left(\cos\left(\frac{\sqrt{|\Delta|}(t-1)}{2}\right) - e^{\frac{a_1}{2}}\cos\left(\frac{\sqrt{|\Delta|}(t-2)}{2}\right)\right) + \\ c_2\left(\sin\left(\frac{\sqrt{|\Delta|}(t-1)}{2}\right) - e^{\frac{a_1}{2}}\sin\left(\frac{\sqrt{|\Delta|}(t-2)}{2}\right)\right)\end{matrix}\right), \tag{16}$$

which are, yet again, finalized by applying first-datum-based initial conditions.

2.4. Formulating traditional GVM

Grey Verhulst Model (GVM) is a nonstandard grey model as it does not comply with GM (M, N). Its whitenization equation and basic form are represented in Eqs. (17) and (18), respectively:

$$\frac{dY^{(1)}(t)}{dt} + aY^{(1)}(t) = b(Y^{(1)}(t))^2;$$
$$t = 1, 2, \ldots, n. \tag{17}$$

$$Y^{(0)}(t) + aZ^{(1)}(t) = b(Z^{(1)}(t))^2;$$
$$t = 1, 2, \ldots, n. \tag{18}$$

The basic-form Eq. (18) can be easily transformed into:

$$Y_0 = B.A; \quad Y_0 = \begin{bmatrix} Y^{(0)}(2) \\ Y^{(0)}(3) \\ \vdots \\ Y^{(0)}(n) \end{bmatrix},$$

$$B = \begin{bmatrix} -Z^{(1)}(2) & \left(Z^{(1)}(2)\right)^2 \\ -Z^{(1)}(3) & \left(Z^{(1)}(3)\right)^2 \\ \vdots & \vdots \\ -Z^{(1)}(n) & \left(Z^{(1)}(n)\right)^2 \end{bmatrix}, \quad A = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{19}$$

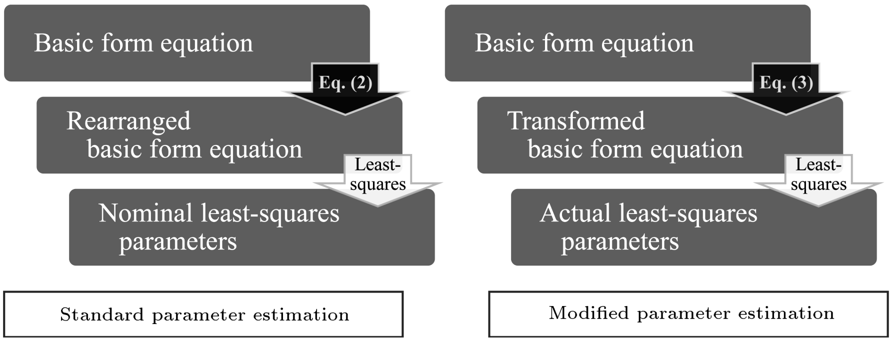

Figure 2. Schematic comparison between the standard and the modified parameter estimation techniques.

GM (1, 1)'s least squares formula in Eq. (7) applies herein, too. Considering traditional initial conditions, time responses and, then, restored values are formed as follows [2]:

$$\hat{Y}^{(0)}(t) = \alpha^{(1)}\hat{Y}^{(1)}(t) = \left(\frac{aY^{(0)}(1)\left(a - bY^{(0)}(1)\right)}{bY^{(0)}(1) + \left(a - bY^{(0)}(1)\right)e^{a(t-1)}}\right) \left(\frac{(1-e^{a})\,e^{a(t-2)}}{bY^{(0)}(1) + \left(a - bY^{(0)}(1)\right)e^{a(t-2)}}\right) \quad t = 2, 3, \ldots \tag{20}$$

3. Developing a new formulation for grey prediction models

Within traditional grey models, the estimation process relies on the discrete basic-form equation, while the prediction process is based on the continuous whitenization equation. Moreover, the actual forecasting applies to a discrete set of points. Such inconsistency has been considered a major drawback to grey models causing several instances of unsatisfactory outputs [10,14,29,30,32,33]. This paper contributes to the literature firstly by proposing a novel discrete modeling, which is cohesively reliant upon the basic-form equation.

More specifically, both modeling methods discretize the whitenization equation to establish the basic-form equation, which is the basis of subsequent least-squares computations. Differences emerge when making use of estimated parameters. Traditional modeling applies estimated parameters to calibrate the whitenization equation, which is then solved to derive forecasting functions. On the contrary, the proposed 'basic form'-focused modeling utilizes estimated parameters to calibrate the basic form itself, which is then solved to build forecasting functions. Accordingly, the proposed modeling is cohesive in that both its estimation and prediction processes depend on basic form. Additionally, it avoids the unnecessary leap from discrete computations back to continuous computations as Figure 1 illustrates.

The other contribution of this paper is modifying the least-squares estimation technique. Such modification is made through shifting the focus from nominal parameters to the parameters practically applied to the prediction process.

According to the schematic demonstration given in Figure 2, the standard estimation process makes use of Eq. (2). Thus, it derives the rearranged basic-form equation with a linear function of $Z^{(1)}(t)$ and maybe $Y^{(0)}(t)$ on its right side–Eqs. (5) and (10). The modified parameter estimation technique, on the other hand, employs Eq. (3). Hence, it establishes the transformed basic–form equation with a linear function of $Y^{(1)}(t-1)$ and maybe $Y^{(0)}(t-1)$ on its right side–Eqs. (21) and (28). Therefore, B is formulated based on current point t in the rearranged basic-form equation, while it relies upon previous point $t-1$ in the transformed basic-form equation.

Of note, regardless of the estimation technique, the prediction process should treat preceding points of data to generate the forecast at a specific point, e.g., Eqs. (8), (14), (15), and (16) as well as Eqs. (24), (25), and (30). Accordingly, the advantage of the modified parameter estimation technique is found since the formulation of its B is consistent with its subsequent application in the prediction process. Consequently, the set of estimated parameters $\hat{A}$ is directly applied to prediction without any rearrangement.

3.1. Formulating the new GM (1, 1): BFGM (1, 1)

An appropriate manipulation after introducing Eq. (3) into basic-form of Eq. (5) leads to the following transformed basic-form equation:

$$Y^{(0)}(t) = \frac{b}{\frac{a}{2}+1} - \frac{a}{\frac{a}{2}+1}Y^{(1)}(t-1); \quad t = 2, 3, \ldots, n. \tag{21}$$

It can also be outlined in matrix terms as follows:

$$Y_0 = B.A; \quad Y_0 = \begin{bmatrix} Y^{(0)}(2) \\ Y^{(0)}(3) \\ \vdots \\ Y^{(0)}(n) \end{bmatrix},$$

$$B = \begin{bmatrix} Y^{(1)}(1) & 1 \\ Y^{(1)}(2) & 1 \\ \vdots & \vdots \\ Y^{(1)}(n-1) & 1 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 = \frac{-a}{\frac{a}{2}+1} \\ A_2 = \frac{b}{\frac{a}{2}+1} \end{bmatrix}. \tag{22}$$

The modified parameter estimation technique keeps the fundamental equation (Eq. (7)), yet switches the formulation of its A from Eq. (6) to Eq. (22). We expect it to result in the least square error due to its focus on parameters directly applicable to the prediction process. In other words, the modified parameter estimation technique optimizes actual parameters of predictions A_1 and A_2 instead of its nominal parameters a and b.

In addition to the modified parameter estimation technique, the novel discrete modeling of GM (1, 1) also relies on the transformed basic-form equation (Eq. (21)). The new function for restored values is as follows:

$$\hat{Y}^{(0)}(t) = A_2 + A_1 Y^{(1)}(t-1), \quad t = 2, 3, \ldots, n+1. \tag{23}$$

Eq. (23) may be applied to forecast within the one-period horizon and farther horizons via Eqs. (24) and (25), respectively:

$$\hat{Y}^{(0)}(n+1) = A_2 + A_1 Y^{(1)}(n). \tag{24}$$

$$\hat{Y}^{(0)}(t) = A_2 + A_1 \hat{Y}^{(1)}(t-1); \quad t = n+2, n+3, \ldots. \tag{25}$$

$\hat{Y}^{(1)}$ is computed recursively as Eqs. (26) and (27):

$$\hat{Y}^{(1)}(n+1) = \hat{Y}^{(0)}(n+1) + Y^{(1)}(n). \tag{26}$$

$$\hat{Y}^{(1)}(t) = \hat{Y}^{(0)}(t) + \hat{Y}^{(1)}(t-1); \quad t = n+2,\ n+3,\ \ldots. \tag{27}$$

3.2. Formulating the new GM (2, 1): BFGM (2, 1)

BFGM (2, 1) shares its central idea with BFGM (1, 1). By transforming basic-form equation (Eq. (10)), we obtain:

$$Y^{(0)}(t) = \frac{b}{1+a_1+\frac{a_2}{2}} + \frac{1}{1+a_1+\frac{a_2}{2}}, \quad Y^{(0)}(t-1) + \frac{-a_2}{1+a_1+\frac{a_2}{2}}, \quad Y^{(1)}(t-1) \qquad t = 2, 3 \ldots\ n. \tag{28}$$

The transformed basic-form Eq. (28) presents a recursive formula underlying BFGM (2, 1). If we translate it into matrix operations, we will have:

$$Y_0 = B.A; \quad Y_0 = \begin{bmatrix} Y^{(0)}(2) \\ Y^{(0)}(3) \\ \vdots \\ Y^{(0)}(n) \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & Y^{(0)}(1) & Y^{(1)}(1) \\ 1 & Y^{(0)}(2) & Y^{(1)}(2) \\ \vdots & \vdots & \vdots \\ 1 & Y^{(0)}(n-1) & Y^{(1)}(n-1) \end{bmatrix},$$

$$A = \begin{bmatrix} A_1 = \frac{b}{1+a_1+\frac{a_2}{2}} \\ A_2 = \frac{1}{1+a_1+\frac{a_2}{2}} \\ A_3 = \frac{-a_2}{1+a_1+\frac{a_2}{2}} \end{bmatrix}. \tag{29}$$

Similarly, Eq. (7) yields the least-squares estimate for A.

Eq. (28) is the basis of BFGM (2, 1)'s prediction process, too. The new function deriving restored values is:

$$\hat{Y}^{(0)}(t) = A_1 + A_2 Y^{(0)}(t-1) + A_3 Y^{(1)}(t-1); \quad t = 2, 3, \ldots \quad n+1. \tag{30}$$

Practical one-period and multi-period forecasts extracted from this function are expressed in Eqs. (31) and (32), respectively.

$$\hat{Y}^{(0)}(n+1) = A_1 + A_2 Y^{(0)}(n) + A_3 Y^{(1)}(n). \tag{31}$$

$$\hat{Y}^{(0)}(t) = A_1 + A_2 \hat{Y}^{(0)}(t-1) + A_3 \hat{Y}^{(1)}(t-1); \quad t = n+2, n+3, \ldots. \tag{32}$$

$Y(\hat{1})$ is computed in a recursive manner similar to Eqs. (26) and (27).

3.3. Formulating the new GVM: BFGVM

Developing the 'Basic Form'-focused Grey Verhulst Model (BFGVM) consists of a method similar to Subsections 3.1 and 3.2. Accordingly, Eq. (3) is used to expand and, then, rearrange traditional GVM's basic-form Eq. (18) to obtain:

$$-\left(\frac{b}{4}\right)\left(Y^{(0)}(t)\right)^2 + \left(1 + \frac{a}{2} - bY^{(1)}(t-1)\right)Y^{(0)}(t) + \left(-b\left(Y^{(1)}(t-1)\right)^2 + aY^{(1)}(t-1)\right) = 0. \tag{33}$$

The nonlinear right side of GVM's whitenization Eq. (17) is the model's distinguishing characteristic. Notwithstanding its advantages, it leads to a second-order Eq. (33) underlying BFGVM which imposes specific complications. For instance, the proposed matrix formulation for estimating parameters is inapplicable herein. Moreover, solving Eq. (33) yields dual functions (34) and (35) for restored values:

$$\hat{Y}_1^{(0)}(t) = \frac{-2bY^{(1)}(t-1)+(a+2)+\sqrt{(a+2)^2-8bY^{(1)}(t-1)}}{b}, \tag{34}$$

$$\hat{Y}_2^{(0)}(t) = \frac{-2bY^{(1)}(t-1)+(a+2)+\sqrt{(a+2)^2-8bY^{(1)}(t-1)}}{b}. \tag{35}$$

Final restored values may not be determined unless such duality is resolved. Restored values of both functions can be computed for every point in the dataset. Negativity of a single restored value for one function implies that the other (all-positive) one is appropriate. In the case both sets of restored values are totally non-negative, we will consider accuracy criterion to identify the appropriate BFGVM forecasting function.

4. Results and discussion

This section comprises comparative performance analyses of models developed in Section 3. Consistent with the recent key researches, e.g., Tsai [40], Wu et al. [13], Wu et al. [41], Xia and Wong [3], and Zhou and He [14], data splitting is applied. We analyze simulation accuracy, i.e., in-sample errors, and prediction accuracy, i.e., out-of-sample errors.

More than half of the data at the beginning of each sequence are introduced into the sample. Considering the huge difference in sample sizes of test problems 2 and 3, common percentage of out-of-sample data can hardly be selected. Hence, we set it to be 40% for test problem 2 and 20% for test problem 3. In other words, the last 40 points of data are kept out of the sample for both problems which suffice to analyze predictions in the short and long terms. Similar to researchers such as Hsu [8], Xie and Liu [32], and Zhou and He [14], the three forthcoming periods and farther forecast horizons are considered as short-term and long-term periods.

In order to prepare the comparison of the proposed modified parameter estimation technique with traditional least-squares, Sum of Squares Error (SSE) is employed. This is also in accordance with grey prediction researches, e.g., Chang [6] and Kayacan and Kaynak [19], exclusively relying on square error measures.

However, the performance analysis in this research extends beyond accuracy criterion. In fact, processing time statistics are also evaluated as a quantitative proxy for computational complexity. MATLAB® 7.8 is utilized to code and implement grey models on an AMD® A10 4600M processor with 6 Gigabytes of RAM. Processing times are based on 1000 runs excluding graphical operations. Certainly, there are outstanding outliers in almost all of the cases. Therefore, we prefer medians due to their robustness to outliers and higher levels of reliability according to our results. Nevertheless, averages are generally consistent with these results.

We apply some time series existing in the literature to facilitate further comparisons as Figure 3 depicts. LCD-TV sales data [42] are used as test problem 1 for low-frequency time series. Besides, numerically-simulated exponential-sine and sine-cosine sequences [21] are used respectively as test problems 2 and 3 for high-frequency time series.

Table 4 provides an overview of results in terms of forecasting SSEs as well as the processing time. Forecasting errors are illustrated in Figure 4, too. Since test problem 1 lacks the sample size required for data splitting, the performance analysis is limited to simulation errors therein.

Regarding the fact that grey prediction models make no statistical assumptions about data, they hardly allow statistical tests. Nevertheless, inconsistent interpretations should be avoided; hence, common sense tells us to regard 10% as the threshold of significant difference.

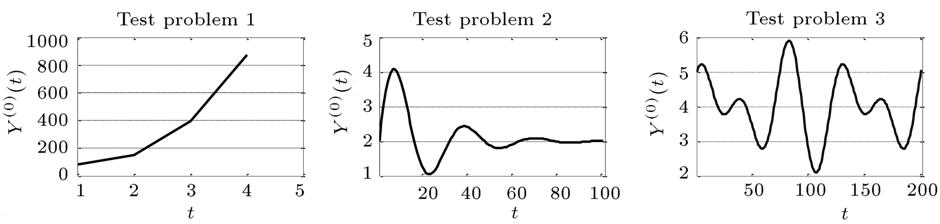

Figure 3. The original time series. Horizontal axes represent time, while vertical axes show original non-accumulated time series values.

Table 4. Experimental results.

	Class of models *	Modeling	Parameter estimation	Model ‡	Simulation SSE	Short-term prediction SSE §	Long-term prediction SSE §	Processing time (millisecond)
Test problem 1	GM (1, 1)	Traditional	Standard	GM (1, 1)	15009.373	–	–	0.110
			Modified	GM (1, 1)+	14644.140	–	–	0.128
		'Basic form'-focused	Standard	BFGM (1, 1)	1216.540	–	–	0.118
			Modified	BFGM (1, 1)+	1215.678	–	–	0.117
	GM (2, 1)	Traditional	Standard	GM (2, 1)	79702.465	–	–	0.163
			Modified	GM (2, 1)+	79702.465	–	–	0.168
		'Basic form'-focused	Standard	BFGM (2, 1)	9.51E-20	–	–	0.128
			Modified	BFGM (2, 1)+	5.37E-21	–	–	0.128
	GVM	Traditional	Standard	GVM	25678.693	–	–	0.124
		'Basic form'-focused	Standard	BFGVM	0.003	–	–	0.281
Test problem 2	GM (1, 1)	Traditional	Standard	GM (1, 1)	28.050	0.708	25.999	0.437
			Modified	GM (1, 1)+	28.045	0.737	26.750	0.438
		'Basic form'-focused	Standard	BFGM (1, 1)	27.910	0.724	26.273	1.085
			Modified	BFGM (1, 1)+	27.907	0.757	27.069	1.008
	GM (2, 1)	Traditional	Standard	GM (2, 1)	84.211	1.255	322.718	0.541
			Modified	GM (2, 1)+	52.603	3.196	106.567	0.560
		'Basic form'-focused	Standard	BFGM (2, 1)	1.719	0.222	143.053	0.498
			Modified	BFGM (2, 1)+	1.619	0.303	76.860	1.063
	GVM	Traditional	Standard	GVM	195.666	0.282	5.934	0.495
		'Basic form'-focused	Standard	BFGVM	96.486	2.272	92.194	2.436
Test problem 3	GM (1, 1)	Traditional	Standard	GM (1, 1)	138.66	0.18	18.37	0.554
			Modified	GM (1, 1)+	138.66	0.19	18.26	0.537
		'Basic form'-focused	Standard	BFGM (1, 1)	138.60	0.18	18.37	1.654
			Modified	BFGM (1, 1)+	138.60	0.19	18.26	1.567
	GM (2, 1)	Traditional	Standard	GM (2, 1)	202.51	2.41	143.19	0.715
			Modified	GM (2, 1)+	139.64	0.03	27.44	0.699
		'Basic form'-focused	Standard	BFGM (2, 1)	2.20	0.02	31.34	0.615
			Modified	BFGM (2, 1)+	2.17	0.03	27.31	1.735
	GVM	Traditional	Standard	GVM	1715.02	1.97	31.32	0.652
		'Basic form'-focused	Standard	BFGVM	467.31	6.19	118.27	3.457

*: Standard grey models are represented by GM (M, N) in which M is the order of the underlying differential equation and N is the total number of variables. Besides, GVM stands for Grey Verhulst Model.

‡: BF at the beginning of acronyms shows 'Basic Form'-focused modeling, while the plus sign at the end indicates the modified parameter estimation.

§: Prediction performance is left blank for test problem 1 since its low frequency allows no data splitting.

Finally, we should note that the two contributions regarding the modeling and the parameter estimation technique are analyzed separately to demonstrate their individual advantages. Furthermore, each model is represented by an appropriate acronym (Table 4) in which the added plus sign indicates the use of the modified parameter estimation technique.

4.1. The 'basic form'-focused versus the traditional modeling

In GM (1, 1) class for low-frequency time series, 'basic form'-focused modeling is preferred because of its obviously higher accuracy without imposing additional complexity. For high-frequency time series, there is no significant difference among the four models;

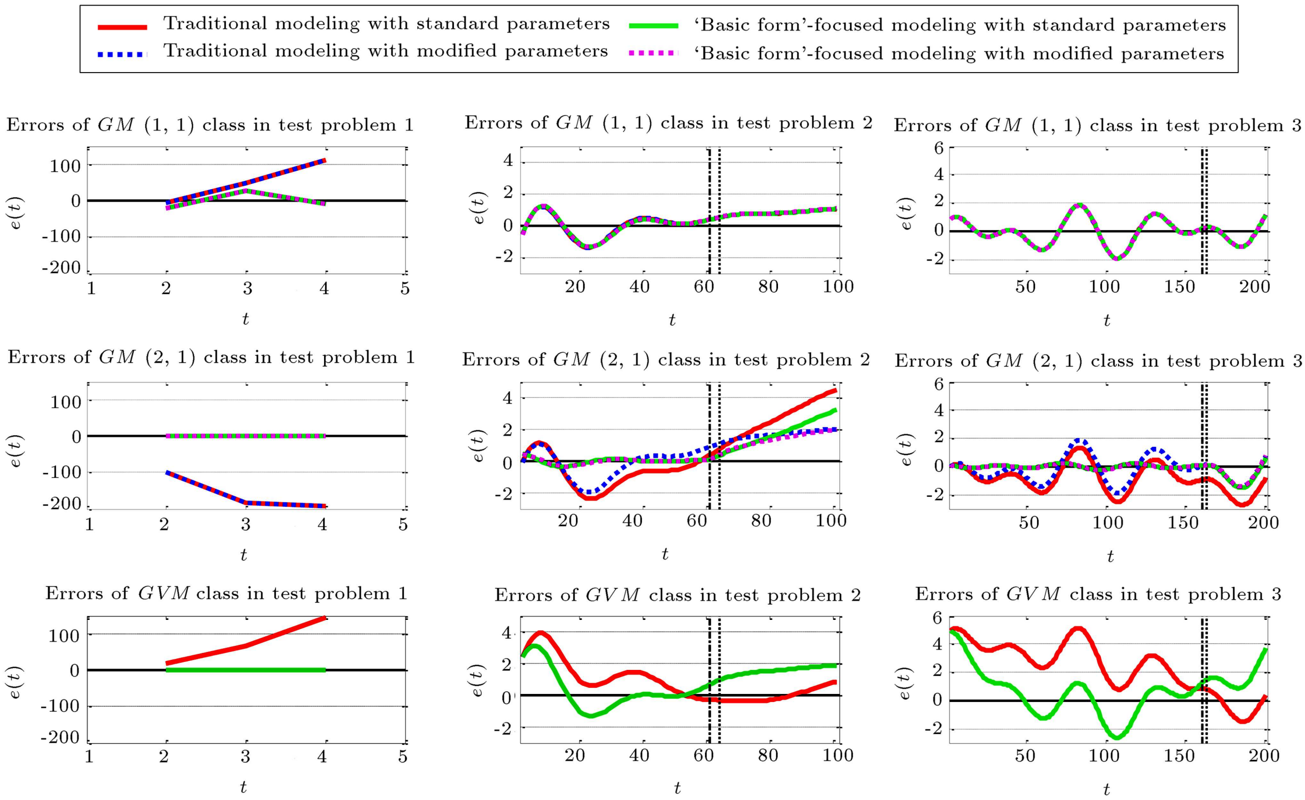

Figure 4. Forecasting errors of the traditional model versus the proposed grey models. Each chart portrays performance of a class of models in a test problem. Top, middle, and bottom rows represent first-order, second-order, and Verhulst classes of grey models, respectively, while columns are sorted according to test problems. Charts on the bottom row comprise only two curves since the modified parameter estimation is not applicable to GVM class. Vertical dash-dot and dotted lines divide charts with data splitting into simulation, short-term prediction, and long-term prediction, respectively, from left to right.

thus, we can prefer the lower complexity of traditional continuous modeling.

In GM (2, 1) class, 'basic form'-focused modeling is the choice because of its considerably higher accuracy, while it has comparable computational complexity. The two 'basic form'-focused models are carried out rather similarly in simulation; however, prediction results recommend BFGM (2, 1) for the short-term and BFGM (2, 1)$^+$ for the long-term period. Between the two traditional models, there is not much difference concerning short-term prediction and computational complexity, even though one can confidently prefer GM (2, 1)$^+$ to GM (2, 1) due to its superior simulation and long-term prediction.

In GVM class, 'basic form'-focused modeling has greatly improved every simulation result; its SSE is 9×10^6 times smaller in test problem 1. However, regarding the unsatisfactory performance of BFGVM in predicting test problems 2 and 3, we recommend applying it just to low-frequency time series. Furthermore, BFGVM is not computationally efficient, which is an apparent consequence of the time-consuming selection process between its dual forecasting functions.

4.2. The modified versus the standard parameter estimation technique

The modified parameter estimation technique almost always improves simulation SSE, either slightly or remarkably. These results confirm our postulation about this technique to be the actual least squares, as compared with the traditional one, which is claimed to be least-squares. This technique is specially recommended for GM (2, 1) class as it meaningfully improves simulation and long-term prediction often without considerable computational cost. Meanwhile, it is not suggested for GM (1, 1) class since it does not make significant improvements.

4.3. Comparison of GM (1, 1), GM1 (2, 1), and GVM

Each of the three classes of grey prediction models has its own range of applications. However, there are cases in which all of them are applicable. Hence, making the following comparison among different classes can be valuable.

Among three classical models, GM (2, 1) is neither accurate nor efficient. For the low-frequency time series, GM (1, 1) is absolutely superior. For high-frequency time series, GM (1, 1) is also dominant except in test problem 3 where GVM is a better predictor. Therefore, one should consider both GM (1, 1) and GVM for the specific high-frequency time series to determine the appropriate traditional model. Such results can explain opposite traditional attitudes toward GM (1, 1) versus GM (2, 1) in grey prediction community.

Eventually, the results are summarized to perform a comprehensive comparison among all of traditional and proposed models. BFGM (2, 1) and BFGM (2, 1)$^+$ are superior simulators, while BFGM (2, 1) is also the best short-term predictor. Nonetheless, long-term results suggest all of GM (1, 1)-classed models as well as the traditional GVM to be preferable. In high-frequency time series, where computational complexity is an issue, BFGM (2, 1) is recommended for simulation and short-term prediction. Yet, one has to choose between GM (1, 1), GM (1, 1)$^+$, and GVM for long-term prediction.

5. Conclusions and recommendations

The 'basic form'-focused modeling, i.e., the discrete modeling, developed in this paper was superior to the traditional continuous modeling for limited data. Such superiority was inherent to each class of models-GM (1, 1), GM (2, 1), and Grey Verhulst Model (GVM). For adequate data, 'basic form'-focused modeling excelled in GM (2, 1)-classed models.

The modified parameter estimation technique was confirmed to be the actual least squares due to its flawless superiority in simulation SSE. It also improves long-term predictions, especially when applied to GM (2, 1)-classed models.

Employing the proposed 'basic form'-focused formulation, we turned the inferior GM (2, 1) into the superior BFGM (2, 1) with unrivalled simulation and short-term prediction. Interestingly, it has no excess computational complexity. The new BFGM (2, 1) may be acknowledged as a critical revision, which reveals higher-order grey models' potential. The developed BFGVM also has some accuracy advantages over the traditional GVM, but it has much room for improvement in computational complexity.

Herein, BFGM was successfully applied to GM (1, 1), GM (2, 1), and GVM. Nevertheless, it is a general approach which can also prove effective when applied to many other grey prediction models. The computational complexity of BFGVM may be relieved by developing rules to decide between its dual forecasting functions. Working on our modified estimation technique to handle GVM will further reinforce BFGVM. A variety of initiatives originally developed to improve traditional formulations may be readily incorporated into our BFGMs for additional improvements. Finally, further investigation will ensure researchers where BFGM can be utilized best, e.g., high-frequency or low-frequency time series- smooth, quasi-smooth, or fluctuating time series- and short-term, medium-term, or long-term predictions.

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