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Efficient ratio-type estimators of finite population mean based on correlation coefficient

M. Irfan[a,b,*], **M. Javed**[a,b], **and Zh. Lin**[a]

a. *Department of Mathematics, Institute of Statistics, Zhejiang University, Hangzhou 310027, China.*
b. *Department of Statistics, Government College University, Faisalabad, Pakistan.*

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Abstract. We proposed efficient families of ratio-type estimators to estimate finite population mean using known correlation coefficient between study variable and auxiliary variable by adopting Singh and Tailor's [Singh, H.P., and Tailor, R. "Use of known correlation coefficient in estimating the finite population means", Statistics in Transition, 6(4), pp. 555-560 (2003)] estimator and Kadilar and Cingi's [Kadilar, C., and Cingi, H. "An improvement in estimating the population mean by using the correlation coefficient", Hacettepe Journal of Mathematics and Statistics, 35(1) pp. 103-109 (2006a)] class of estimators in simple random sampling without replacement. The newly proposed estimators behaved efficiently as compared to the common unbiased estimator, traditional ratio estimator, and the other competing estimators. Bias, mean squared error, and minimum mean squared error of the proposed ratio-type estimators were derived. Moreover, theoretical findings were proven with cooperation of two real data sets.

1. Introduction

The ratio estimator suggested by Cochran [1] plays an important role in the scenario of positive (high) correlation among study and auxiliary variables. One of the hottest issues in the theory of sample survey is the estimation of finite population mean with different sampling techniques. Extensive work has been done on this issue using auxiliary information, such as population median, mean, quartiles, deciles, coefficient of variation, coefficient of correlation, coefficient of skewness, coefficient of kurtosis, etc. Various estimators or classes of estimators have been proposed by several authors, including Sisodia and Dwivedi [2], Upadhyaya and Singh [3], Singh and Tailor [4], Kadilar and Cingi [5,6], Singh et al. [7,8], Gupta and Shabbir [9], Koyuncu and Kadilar [10,11], Yan and Tian [12], Haq and Shabbir [13], Singh and Solanki [14], Yadav and Kadilar [15], Subramani and Kumarapandiyan [16, 17], Subramani and Prabavathy [18], Yadav et al. [19], Khan et al. [20], Kumar [21], Irfan et al. [22], Walia et al. [23] etc., as solutions to this issue to get improved results.

Let "N" be the population size and "n" be the sample size such that $n < N$ is drawn under simple random sampling without replacement (SRSWOR) scheme. Assume y and x are the study and auxiliary variables, respectively. Here, y_i and x_i represent the values of y and x for the ith unit of the population.

Some important formulae used in this manuscript are given as follows:

Population mean of study variable:

$$\bar{Y} = N^{-1} \sum_{i=1}^{N} y_i.$$

*. *Corresponding author.*
E-mail addresses: mirfan@zju.edu.cn (M. Irfan); mariastat@zju.edu.cn (M. Javed); zlin@zju.edu.cn (Zh. Lin)

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Population mean of auxiliary variable:

$$\bar{X} = N^{-1}\sum_{i=1}^{N} x_i.$$

Sample mean of study variable:

$$\bar{y} = n^{-1}\sum_{i=1}^{n} y_i.$$

Sample mean of auxiliary variable:

$$\bar{x} = n^{-1}\sum_{i=1}^{n} x_i.$$

Sampling fraction:

$$f = \frac{n}{N}.$$

Ratio of population mean of study variable to population mean of auxiliary variable:

$$R = \frac{\bar{Y}}{\bar{X}}.$$

Constant term:

$$\theta = \left(\frac{1}{n} - \frac{1}{N}\right).$$

Relative error terms and their expectations have been defined in order to get the bias, Mean Squared Error (MSE), and minimum MSE for the existing and proposed ratio-type estimators in this way.

Let $\xi_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$ and $\xi_1 = \frac{\bar{x}-\bar{X}}{\bar{X}}$ such that $E(\xi_i) = 0$ for $i = 0$ and 1, where $E(.)$ represents the mathematical expectation and:

$$E(\xi_0^2) = \theta\frac{S_y^2}{\bar{Y}^2}, \quad E(\xi_1^2) = \theta\frac{S_x^2}{\bar{X}^2}, \quad E(\xi_0\xi_1) = \theta\frac{S_{yx}}{\bar{Y}\bar{X}},$$

where:

$$S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2,$$

$$S_x^2 = (N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})^2,$$

$$S_{yx} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X}).$$

The motivation behind this manuscript is to develop new efficient families of ratio-type estimators to estimate population mean using SRSWOR scheme with the help of known correlation coefficient between study variable and auxiliary variable. In the following, some well-known traditional and existing estimators in SRSWOR are provided.

Commonly used unbiased estimator of $\bar{Y}$ is:

$$\hat{\bar{Y}} = \bar{y}. \tag{1}$$

Estimator in Eq. (1) has the following mean squared error or variance:

$$\text{MSE}(\hat{\bar{Y}}) = V(\hat{\bar{Y}}) = \theta S_y^2. \tag{2}$$

The traditional ratio estimator of population mean suggested by Cochran [1] is defined as:

$$\hat{\bar{Y}}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right), \qquad \bar{x} \neq 0. \tag{3}$$

Bias and MSE of $\hat{\bar{Y}}_R$ up to the first degree of approximation are, respectively, given as:

$$\text{Bias }(\hat{\bar{Y}}_R) \approx \theta\bar{Y}\left[C_x^2 - \rho_{yx}C_yC_x\right], \tag{4}$$

and:

$$\text{MSE }(\hat{\bar{Y}}_R) \approx \theta\bar{Y}^2\left[C_y^2 + C_x^2(1-2\varphi)\right], \tag{5}$$

where:

$$\varphi = \rho_{yx}\frac{C_y}{C_x}, \quad C_y = \frac{S_y}{\bar{Y}},$$

$$C_x = \frac{S_x}{\bar{X}}, \quad \rho_{yx} = (S_yS_x)^{-1}S_{yx}.$$

Given below is the introduction of some other competing ratio-type estimators that take known correlation coefficient between study and auxiliary variables into account.

1.1. Singh and tailor estimator

In order to estimate finite population mean, Singh and Tailor [4] proposed ratio estimator by utilizing the known correlation coefficient ρ_{yx}:

$$\hat{\bar{Y}}_{ST} = \bar{y}\left(\frac{\bar{X}+\rho_{yx}}{\bar{x}+\rho_{yx}}\right). \tag{6}$$

Bias and MSE up to the first order of approximation of $\hat{\bar{Y}}_{ST}$ in Eq. (6) are given by:

$$\text{Bias }(\hat{\bar{Y}}_{ST}) \approx \theta\bar{Y}\lambda\left[\lambda C_x^2 - \rho_{yx}C_yC_x\right], \tag{7}$$

and:

$$\text{MSE}(\hat{\bar{Y}}_{ST}) \approx \theta\bar{Y}^2\left[C_y^2 + C_x^2\lambda(\lambda-2\varphi)\right], \tag{8}$$

where:

$$\lambda = \frac{\bar{X}}{\bar{X}+\rho_{yx}}.$$

1.2. Kadilar and Cingi estimators

Kadilar and Cingi [24] proposed a class of ratio estimators to estimate population mean under SRSWOR in the light of the work done by Upadhyaya and Singh [3] and Singh and Tailor [4]. Their proposed class of estimators is:

$$\hat{\bar{Y}}_{KC1} = \bar{y}\left(\frac{\bar{X}C_x + \rho_{yx}}{\bar{x}C_x + \rho_{yx}}\right), \tag{9}$$

$$\hat{\bar{Y}}_{KC2} = \bar{y}\left(\frac{\bar{X}\rho_{yx} + C_x}{\bar{x}\rho_{yx} + C_x}\right), \tag{10}$$

$$\hat{\bar{Y}}_{KC3} = \bar{y}\left(\frac{\bar{X}\beta_2(x) + \rho_{yx}}{\bar{x}\beta_2(x) + \rho_{yx}}\right), \tag{11}$$

$$\hat{\bar{Y}}_{KC4} = \bar{y}\left(\frac{\bar{X}\rho_{yx} + \beta_2(x)}{\bar{x}\rho_{yx} + \beta_2(x)}\right), \tag{12}$$

where ρ_{yx}, C_x, and $\beta_2(x)$ are the correlation coefficient between study and auxiliary variables, coefficient of variation, and coefficient of kurtosis of the auxiliary variable, respectively.

Bias and MSE of the above-mentioned estimators in Eqs. (9) to (12) are listed below:

$$\text{Bias }(\hat{\bar{Y}}_{KCi}) \approx \theta\bar{Y}C_x^2\lambda_i(\lambda_i - \varphi), \quad i = 1,2,3,4. \tag{13}$$

and:

$$\text{MSE }(\hat{\bar{Y}}_{KCi}) \approx \theta\bar{Y}^2\left[C_y^2 + C_x^2\lambda_i(\lambda_i - 2\varphi)\right],$$

$$i = 1,2,3,4, \tag{14}$$

where:

$$\lambda_1 = \frac{\bar{X}C_x}{\bar{X}C_x + \rho_{yx}}, \qquad \lambda_2 = \frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx} + C_x},$$

$$\lambda_3 = \frac{\bar{X}\beta_2(x)}{\bar{X}\beta_2(x) + \rho_{yx}}, \qquad \lambda_4 = \frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx} + \beta_2(x)}.$$

Kadilar and Cingi [25] defined another class of ratio-type estimators as:

$$\hat{\bar{Y}}^*_{KCi} = \left[\bar{y} + b(\bar{X} - \bar{x})\right]\alpha_i, \quad i = 1,2,3,4,5, \tag{15}$$

where:

$$\alpha_1 = \frac{\bar{X} + \rho_{yx}}{\bar{x} + \rho_{yx}}, \qquad \alpha_2 = \frac{\bar{X}C_x + \rho_{yx}}{\bar{x}C_x + \rho_{yx}},$$

$$\alpha_3 = \frac{\bar{X}\rho_{yx} + C_x}{\bar{x}\rho_{yx} + C_x}, \qquad \alpha_4 = \frac{\bar{X}\beta_2(x) + \rho_{yx}}{\bar{x}\beta_2(x) + \rho_{yx}},$$

$$\alpha_5 = \frac{\bar{X}\rho_{yx} + \beta_2(x)}{\bar{x}\rho_{yx} + \beta_2(x)}.$$

The following are the expressions for bias and MSE of the $\hat{\bar{Y}}^*_{KCi}$ estimators:

$$\text{Bias }(\hat{\bar{Y}}^*_{KCi}) \approx \theta\bar{Y}C_x^2\alpha_i^{*2}, \qquad i = 1,2,3,4,5, \tag{16}$$

and:

$$\text{MSE }(\hat{\bar{Y}}^*_{KCi}) \approx \theta\bar{Y}^2\left[\alpha_i^{*2}C_x^2 + C_y^2(1 - \rho_{yx}^2)\right],$$

$$i = 1,2,3,4,5, \tag{17}$$

where:

$$\alpha_1^* = \frac{\bar{X}}{\bar{X} + \rho_{yx}}, \qquad \alpha_2^* = \frac{\bar{X}C_x}{\bar{X}C_x + \rho_{yx}},$$

$$\alpha_3^* = \frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx} + C_x}, \qquad \alpha_4^* = \frac{\bar{X}\beta_2(x)}{\bar{X}\beta_2(x) + \rho_{yx}},$$

$$\alpha_5^* = \frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx} + \beta_2(x)}.$$

2. Proposed families of ratio-type estimators

Inspired by Singh and Tailor [4] and Kadilar and Cingi [24,25], we propose efficient families of ratio-type estimators, i.e., ($\hat{\bar{Y}}_{Pi}$, i.e., $i = 1,3,5,...,11$) and ($\hat{\bar{Y}}_{Pj}$, i.e., $j = 2,4,6,...,16$), to estimate population mean using the correlation coefficient between study and auxiliary variables under SRSWOR scheme.

2.1. The first proposed family of ratio-type estimators

$$\hat{\bar{Y}}_{P1} = k\bar{y}\left(\frac{\bar{X}C_x + \rho_{yx}}{\bar{x}C_x + \rho_{yx}}\right)^{\frac{\bar{X}C_x}{\bar{X}C_x + \rho_{yx}}}, \tag{18}$$

$$\hat{\bar{Y}}_{P3} = k\bar{y}\left(\frac{\bar{X}\beta_2(x) + \rho_{yx}}{\bar{x}\beta_2(x) + \rho_{yx}}\right)^{\frac{\bar{X}\beta_2(x)}{\bar{X}\beta_2(x) + \rho_{yx}}}, \tag{19}$$

$$\hat{\bar{Y}}_{P5} = k\bar{y}\left(\frac{\bar{X}\beta_2(x) + \rho_{yx}C_x}{\bar{x}\beta_2(x) + \rho_{yx}C_x}\right)^{\frac{\bar{X}\beta_2(x)}{\bar{X}\beta_2(x) + \rho_{yx}C_x}}, \tag{20}$$

$$\hat{\bar{Y}}_{P7} = k\bar{y}\left(\frac{\bar{X}\rho_{yx}\beta_2(x) + C_x}{\bar{x}\rho_{yx}\beta_2(x) + C_x}\right)^{\frac{\bar{X}\rho_{yx}\beta_2(x)}{\bar{X}\rho_{yx}\beta_2(x) + C_x}}, \tag{21}$$

$$\hat{\bar{Y}}_{P9} = k\bar{y}\left(\frac{\bar{X} + \rho_{yx}}{\bar{x} + \rho_{yx}}\right)^{\frac{\bar{X}}{\bar{X} + \rho_{yx}}}, \tag{22}$$

$$\hat{\bar{Y}}_{P11} = k\bar{y}\left(\frac{\bar{X}C_x\beta_2(x) + \rho_{yx}}{\bar{x}C_x\beta_2(x) + \rho_{yx}}\right)^{\frac{\bar{X}C_x\beta_2(x)}{\bar{X}C_x\beta_2(x) + \rho_{yx}}}, \tag{23}$$

where k is the selected appropriate constant whose value is determined later in Theorem 2.1 and $\bar{X}$ population mean, C_x population coefficient of variation, ρ_{yx} population correlation coefficient, and $\beta_2(x)$ population coefficient of kurtosis are the known parameters.

Theorem 2.1

The properties of $\hat{\hat{Y}}_{Pi}$, i.e., $i=1,3,5,...,11$ are:

1) Bias $(\hat{\hat{Y}}_{Pi}) \approx \bar{Y}(k-1)$

$$+\theta\bar{Y}k\left[\frac{1}{2}(\psi_i^4+\psi_i^3)C_x^2-\psi_i^2\rho_{yx}C_yC_x\right], \qquad (24)$$

2) MSE $(\hat{\hat{Y}}_{Pi}) \approx \bar{Y}^2\Bigg[\left(\frac{A_{1i}}{A_{2i}}-1\right)^2+\left(\frac{A_{1i}}{A_{2i}}\right)^2\Big(E(\xi_0^2)$

$$+(2\psi_i^4+\psi_i^3)E(\xi_1^2)-4\psi_i^2E(\xi_0\xi_1)\Big)$$

$$-\frac{A_{1i}}{A_{2i}}\big((\psi_i^4+\psi_i^3)E(\xi_1^2)-2\psi_i^2E(\xi_0\xi_1)\big)\Bigg], \qquad (25)$$

3) $\text{MSE}_{\min}(\hat{\hat{Y}}_{Pi}) \approx \bar{Y}^2\left[1-\frac{A_{1i}^2}{2A_{2i}}\right], \qquad (26)$

where:

$$\psi_1=\frac{\bar{X}C_x}{\bar{X}C_x+\rho_{yx}}, \qquad \psi_3=\frac{\bar{X}\beta_2(x)}{\bar{X}\beta_2(x)+\rho_{yx}},$$

$$\psi_5=\frac{\bar{X}\beta_2(x)}{\bar{X}\beta_2(x)+\rho_{yx}C_x}, \qquad \psi_7=\frac{\bar{X}\rho_{yx}\beta_2(x)}{\bar{X}\rho_{yx}\beta_2(x)+C_x},$$

$$\psi_9=\frac{\bar{X}}{\bar{X}+\rho_{yx}}, \qquad \psi_{11}=\frac{\bar{X}C_x\beta_2(x)}{\bar{X}C_x\beta_2(x)+\rho_{yx}}.$$

Proof

To prove Eqs. (24)-(26), the proposed family of ratio-type estimators ($\hat{\hat{Y}}_{Pi}$, i.e., $i=1,3,5,...,11$) can be written in terms of ξ_i's as:

$$\hat{\hat{Y}}_{Pi}=k\bar{Y}(1+\xi_0)[1+\psi_i\xi_1]^{-\psi_i}, \qquad (27)$$

or:

$$\hat{\hat{Y}}_{Pi}=k\bar{Y}(1+\xi_0)\left[1-\psi_i^2\xi_1+\frac{1}{2}\psi_i^3(\psi_i+1)\xi_1^2\right]. \qquad (28)$$

Simplifying the R.H.S. of Eq. (28) up to the first degree of approximation and, then, subtracting $\bar{Y}$ from both sides, we get:

$$\hat{\hat{Y}}_{Pi}-\bar{Y}=\bar{Y}\Bigg[k+k\xi_0-k\psi_i^2\xi_1+\frac{1}{2}k(\psi_i^4+\psi_i^3)\xi_1^2$$

$$-k\psi_i^2\xi_0\xi_1-1\Bigg]. \qquad (29)$$

Bias and the MSE of the proposed family of ratio-type estimators ($\hat{\hat{Y}}_{Pi}$, i.e., $i=1,3,5,...,11$) are, respectively, given as:

$$\text{Bias}(\hat{\hat{Y}}_{Pi})=E(\hat{\hat{Y}}_{Pi}-\bar{Y})\approx\bar{Y}(k-1)$$

$$+\theta\bar{Y}k\left[\frac{1}{2}(\psi_i^4+\psi_i^3)C_x^2-\psi_i^2\rho_{yx}C_yC_x\right], \qquad (30)$$

and:

$$\text{MSE}(\hat{\hat{Y}}_{Pi})=(\hat{\hat{Y}}_{Pi}-\bar{Y})^2\approx\bar{Y}^2\Bigg[(k-1)^2$$

$$+k^2\Big(E(\xi_0^2)+(2\psi_i^4+\psi_i^3)E(\xi_1^2)$$

$$-4\psi_i^2E(\xi_0\xi_1)\Big)-k\Big((\psi_i^4+\psi_i^3)E(\xi_1^2)$$

$$-2\psi_i^2E(\xi_0\xi_1)\Big)\Bigg]. \qquad (31)$$

To get the optimal value of k, differentiating Eq. (31) with respect to k and equating to zero, we have:

$$k=\frac{2+(\psi_i^4+\psi_i^3)E(\xi_1^2)-2\psi_i^2E(\xi_0\xi_1)}{2+2\left[E(\xi_0^2)+(2\psi_i^4+\psi_i^3)E(\xi_1^2)-4\psi_i^2E(\xi_0\xi_1)\right]},$$

$$k=\frac{1+\frac{\theta}{2}\left[(\psi_i^4+\psi_i^3)C_x^2-2\psi_i^2\rho_{yx}C_yC_x\right]}{1+\theta\left[C_y^2+(2\psi_i^4+\psi_i^3)C_x^2-4\psi_i^2\rho_{yx}C_yC_x\right]},$$

$$k=\frac{A_{1i}}{A_{2i}},$$

where:

$$A_{1i}=1+\frac{\theta}{2}\left[(\psi_i^4+\psi_i^3)C_x^2-2\psi_i^2\rho_{yx}C_yC_x\right],$$

and:

$$A_{2i}=1+\theta\left[C_y^2+(2\psi_i^4+\psi_i^3)C_x^2-4\psi_i^2\rho_{yx}C_yC_x\right].$$

Putting the optimal values of k in Relation (31), we have MSE as:

$$\text{MSE}(\hat{\hat{Y}}_{Pi})\approx\bar{Y}^2\Bigg[\left(\frac{A_{1i}}{A_{2i}}-1\right)^2+\left(\frac{A_{1i}}{A_{2i}}\right)^2\Big(E(\xi_0^2)$$

$$+(2\psi_i^4+\psi_i^3)E(\xi_1^2)-4\psi_i^2E(\xi_0\xi_1)\Big)$$

$$-\frac{A_{1i}}{A_{2i}}\big((\psi_i^4+\psi_i^3)E(\xi_1^2)-2\psi_i^2E(\xi_0\xi_1)\big)\Bigg]. \qquad (32)$$

Simplifying Relation (32), we have the minimum MSE of the proposed ratio-type estimators:

$$\text{MSE}_{\min}(\hat{\hat{Y}}_{Pi})\approx\bar{Y}^2\left[1-\frac{A_{1i}^2}{2A_{2i}}\right]. \qquad (33)$$

2.2. Second proposed family of ratio-type estimators

$$\hat{\bar{Y}}_{P2} = t\left[\bar{y}\left(\frac{\bar{X}\rho_{yx}+R}{\bar{x}\rho_{yx}+R}\right)^{\frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx}+R}} + b(\bar{X}-\bar{x})\right], \quad (34)$$

$$\hat{\bar{Y}}_{P4} = t\left[\bar{y}\left(\frac{\bar{X}\rho_{yx}+RC_x}{\bar{x}\rho_{yx}+RC_x}\right)^{\frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx}+RC_x}} + b(\bar{X}-\bar{x})\right], \quad (35)$$

$$\hat{\bar{Y}}_{P6} = t\left[\bar{y}\left(\frac{\bar{X}\rho_{yx}+R\beta_2(x)}{\bar{x}\rho_{yx}+R\beta_2(x)}\right)^{\frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx}+R\beta_2(x)}} + b(\bar{X}-\bar{x})\right], \quad (36)$$

$$\hat{\bar{Y}}_{P8} = t\left[\bar{y}\left(\frac{\bar{X}\rho_{yx}+RC_x\beta_2(x)}{\bar{x}\rho_{yx}+RC_x\beta_2(x)}\right)^{\frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx}+RC_x\beta_2(x)}} + b(\bar{X}-\bar{x})\right], \quad (37)$$

$$\hat{\bar{Y}}_{P10} = t\left[\bar{y}\left(\frac{\bar{X}+R\rho_{yx}}{\bar{x}+R\rho_{yx}}\right)^{\frac{\bar{X}}{\bar{X}+R\rho_{yx}}} + b(\bar{X}-\bar{x})\right], \quad (38)$$

$$\hat{\bar{Y}}_{P12} = t\left[\bar{y}\left(\frac{\bar{X}+R\rho_{yx}C_x}{\bar{x}+R\rho_{yx}C_x}\right)^{\frac{\bar{X}}{\bar{X}+R\rho_{yx}C_x}} + b(\bar{X}-\bar{x})\right], \quad (39)$$

$$\hat{\bar{Y}}_{P14} = t\left[\bar{y}\left(\frac{\bar{X}+R\rho_{yx}\beta_2(x)}{\bar{x}+R\rho_{yx}\beta_2(x)}\right)^{\frac{\bar{X}}{\bar{X}+R\rho_{yx}\beta_2(x)}} + b(\bar{X}-\bar{x})\right], \quad (40)$$

$$\hat{\bar{Y}}_{P16} = t\left[\bar{y}\left(\frac{\bar{X}+R\rho_{yx}C_x\beta_2(x)}{\bar{x}+R\rho_{yx}C_x\beta_2(x)}\right)^{\frac{\bar{X}}{\bar{X}+R\rho_{yx}C_x\beta_2(x)}} + b(\bar{X}-\bar{x})\right], \quad (41)$$

where t is the unknown constant, which is to be determined later on, and $\bar{X}$, C_x, ρ_{yx}, and $\beta_2(x)$ are the known population parameters.

Theorem 2.2

The properties of $\hat{\bar{Y}}_{Pj}$, i.e., $j = 2, 4, 6, ..., 16$, are:

1) $\text{Bias}(\hat{\bar{Y}}_{Pj}) \approx \bar{Y}(t-1)$

$$+ \theta\bar{Y}t\left[\frac{1}{2}(\varphi_j^4+\varphi_j^3)C_x^2 - \varphi_j^2\rho_{yx}C_yC_x\right], \quad (42)$$

2) $$\text{MSE}(\hat{\bar{Y}}_{Pj}) \approx \Bigg[\left(\frac{A_{3j}}{A_{4j}}\bar{Y}-\bar{Y}\right)^2 + \left(\frac{A_{3j}}{A_{4j}}\right)^2\Bigg(\bar{Y}^2E(\xi_1^2) + \Big\{2\varphi_j^4\bar{Y}^2 + b^2\bar{X}^2 + \varphi_j^3\bar{Y}^2 + 2\varphi_j^2 b\bar{Y}\bar{X}\Big\}E(\xi_1^2) - \Big\{4\varphi_j^2\bar{Y}^2 + 2b\bar{Y}\bar{X}\Big\}E(\xi_0\xi_1)\Bigg) - \frac{A_{3j}}{A_{4j}}\Bigg(\Big\{\varphi_j^4\bar{Y}^2 + \varphi_j^3\bar{Y}^2\Big\}E(\xi_1^2) - 2\varphi_j^2\bar{Y}^2E(\xi_0\xi_1)\Bigg)\Bigg]. \quad (43)$$

3) $$\text{MSE}_{\min}(\hat{\bar{Y}}_{Pj}) \approx \left[\bar{Y}^2 - \frac{A_{3j}^2}{2A_{4j}}\right], \quad (44)$$

where:

$$\varphi_2 = \frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx}+R}, \qquad \varphi_4 = \frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx}+RC_x},$$

$$\varphi_6 = \frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx}+R\beta_2(x)}, \qquad \varphi_8 = \frac{\bar{X}\rho_{yx}}{\bar{X}\rho_{yx}+RC_x\beta_2(x)},$$

$$\varphi_{10} = \frac{\bar{X}}{\bar{X}+R\rho_{yx}}, \qquad \varphi_{12} = \frac{\bar{X}}{\bar{X}+R\rho_{yx}C_x},$$

$$\varphi_{14} = \frac{\bar{X}}{\bar{X}+R\rho_{yx}\beta_2(x)}, \quad \varphi_{16} = \frac{\bar{X}}{\bar{X}+R\rho_{yx}C_x\beta_2(x)}.$$

Proof

To prove Relations (42) to (44), the proposed estimators ($\hat{\bar{Y}}_{Pj}$, i.e., $j = 2, 4, 6, ..., 16$) may be written in terms of ξ_i's as follows:

$$\hat{\bar{Y}}_{Pj} = t\left[(\bar{Y}+\bar{Y}\xi_0)(1+\varphi_j\xi_1)^{-\varphi_j} - b\bar{X}\xi_1\right],$$

or:

$$\hat{\bar{Y}}_{Pj} = t\left[(\bar{Y}+\bar{Y}\xi_0)\left\{1-\varphi_j^2\xi_1+\frac{1}{2}\varphi_j^3(\varphi_j+1)\xi_1^2\right\} - b\bar{X}\xi_1\right], \quad (45)$$

where:

$$b = \frac{\rho_{yx}S_y}{S_x}.$$

Subtracting $\bar{Y}$ from Eq. (45) and expanding it up to the first degree of approximation, we obtain:

$$\hat{\bar{Y}}_{Pj} - \bar{Y} = \left[t\bar{Y} + t\bar{Y}\xi_0 - t\bar{Y}\varphi_j^2\xi_1 - tb\bar{X}\xi_1 + \frac{1}{2}t\bar{Y}(\varphi_j^4+\varphi_j^3)\xi_1^2 - t\bar{Y}\varphi_j^2\xi_0\xi_1 - \bar{Y}\right]. \quad (46)$$

The bias and the MSE of $\hat{\hat{Y}}_{Pj}$, i.e., $j = 2, 4, 6, ..., 16$, up to the first degree of approximation are given as:

$$\text{Bias } (\hat{\hat{Y}}_{Pj}) \approx \bar{Y}(t-1) + \theta\bar{Y}t\Bigg[\frac{1}{2}(\varphi_j^4 + \varphi_j^3)C_x^2 - \varphi_j^2\rho_{yx}C_yC_x\Bigg], \tag{47}$$

$$\text{MSE}(\hat{\hat{Y}}_{Pj}) \approx \Bigg[(t\bar{Y} - \bar{Y})^2 + t^2\Bigg(\bar{Y}^2E(\xi_0^2) + \Big\{2\varphi_j^4\bar{Y}^2 + b^2\bar{X}^2 + \varphi_j^3\bar{Y}^2 + 2\varphi_j^2 b\bar{Y}\bar{X}\Big\}E(\xi_1^2) - \big\{4\varphi_j^2\bar{Y}^2 + 2b\bar{Y}\bar{X}\big\}E(\xi_0\xi_1)\Bigg) - t\Bigg(\{\varphi_j^4\bar{Y}^2 + \varphi_j^3\bar{Y}^2\}E(\xi_1^2) - 2\varphi_j^2\bar{Y}^2E(\xi_0\xi_1)\Bigg)\Bigg]. \tag{48}$$

Differentiating Eq. (48) with respect to t and equating to zero, we have t as shown in Box I, where:

$$A_{3j} = \bar{Y}^2 + \frac{\theta}{2}\left[(\varphi_j^4 + \varphi_j^3)R^2S_x^2 - 2\varphi_j^2 RS_{yx}\right],$$

and:

$$A_{4j} = \bar{Y}^2 + \theta\Bigg[S_y^2 + \Big\{2\varphi_j^4R^2 + b^2 + \varphi_j^3R^2 + 2\varphi_j^2 bR\Big\}S_x^2 - \{4\varphi_j^2R + 2b\}S_{yx}\Bigg].$$

Inserting the value of t in Eq. (48), we obtain:

$$\text{MSE}(\hat{\hat{Y}}_{Pj}) \approx \Bigg[\left(\frac{A_{3j}}{A_{4j}}\bar{Y} - \bar{Y}\right)^2 + \left(\frac{A_{3j}}{A_{4j}}\right)^2\Bigg(\bar{Y}^2E(\xi_0^2) + \big\{2\varphi_j^4\bar{Y}^2 + b^2\bar{X}^2 + \varphi_j^3\bar{Y}^2 + 2\varphi_j^2 b\bar{Y}\bar{X}\big\}E(\xi_1^2) - \big\{4\varphi_j^2\bar{Y}^2 + 2b\bar{Y}\bar{X}\big\}E(\xi_0\xi_1)\Bigg) - \frac{A_{3j}}{A_{4j}}\Bigg(\{\varphi_j^4\bar{Y}^2 + \varphi_j^3\bar{Y}^2\}E(\xi_1^2) - 2\varphi_j^2\bar{Y}^2E(\xi_0\xi_1)\Bigg)\Bigg]. \tag{49}$$

Thus, the minimum MSE is as follows:

$$\text{MSE}_{\min}(\hat{\hat{Y}}_{Pj}) \approx \left[\bar{Y}^2 - \frac{A_{3j}^2}{2A_{4j}}\right]. \tag{50}$$

Interesting note

Many more ratio-type estimators based on correlation coefficient can be formulated by taking different measures of ψ_i and φ_j.

3. Efficiency of the proposed estimators

This section deals with the derivation of algebraic situations under which the proposed estimators will have minimum MSE as compared to unbiased estimator (sample mean), traditional ratio estimator, Singh and Tailor's [4] estimator, and Kadilar and Cingi's [24,25] classes of estimators.

Theorem 3.1

1. $\hat{\hat{Y}}_{Pi}$ (i.e., $i = 1, 3, 5, ..., 11$) perform better than $\hat{\bar{Y}}$ if:
$$\left[\theta C_y^2 + \frac{A_{1i}^2}{2A_{2i}}\right] > 1. \tag{51}$$
2. $\hat{\hat{Y}}_{Pj}$ (i.e., $j = 2, 4, 6, ..., 16$) perform better than $\hat{\bar{Y}}$ if:
$$\left[\theta C_y^2 + \frac{A_{3j}^2}{2\bar{Y}^2A_{4j}}\right] > 1. \tag{52}$$

Proof

1. By comparing Relations (2) and (33):
$$\text{MSE}_{\min}(\hat{\hat{Y}}_{Pi}) < \text{MSE}(\hat{\bar{Y}}),$$
$$\bar{Y}^2\left[1 - \frac{A_{1i}^2}{2A_{2i}}\right] < \theta S_y^2,$$

$$t = \frac{2\bar{Y}^2 + \{\varphi_j^4\bar{Y}^2 + \varphi_j^3\bar{Y}^2\}E(\xi_1^2) - 2\varphi_j^2\bar{Y}^2E(\xi_0\xi_1)}{2\bar{Y}^2 + 2\left[\bar{Y}^2E(\xi_0^2) + \left\{2\varphi_j^4\bar{Y}^2 + b^2\bar{X}^2 + \varphi_j^3\bar{Y}^2 + 2\varphi_j^2 b\bar{Y}\bar{X}\right\}E(\xi_1^2) - \left\{4\varphi_j^2\bar{Y}^2 + 2b\bar{Y}\bar{X}\right\}E(\xi_0\xi_1)\right]},$$

$$t = \frac{\bar{Y}^2 + \frac{\theta}{2}\left[(\varphi_j^4 + \varphi_j^3)R^2S_x^2 - 2\varphi_j^2RS_{yx}\right]}{\bar{Y}^2 + \theta\left[S_y^2 + \left\{2\varphi_j^4R^2 + b^2 + \varphi_j^3R^2 + 2\varphi_j^2bR\right\}S_x^2 - \left\{4\varphi_j^2R + 2b\right\}S_{yx}\right]},$$

$$t = \frac{A_{3j}}{A_{4j}},$$

Box I

$$\left[\theta C_y^2 + \frac{A_{1i}^2}{2A_{2i}}\right] > 1.$$

2. By comparing Relations (2) and (50):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pj}) < \mathrm{MSE}(\hat{\bar{Y}}),$$

$$\left[\bar{Y}^2 - \frac{A_{3j}^2}{2A_{4j}}\right] < \theta S_y^2,$$

$$\left[\theta C_y^2 + \frac{A_{3j}^2}{2\bar{Y}^2 A_{4j}}\right] > 1.$$

Theorem 3.2

1. $\hat{\hat{Y}}_{Pi}$ (i.e., $i = 1, 3, 5, ..., 11$) perform better than traditional ratio estimator, $\hat{\bar{Y}}_R$, if:

$$\theta\left[C_y^2 + C_x^2(1 - 2\varphi)\right] + \frac{A_{1i}^2}{2A_{2i}} > 1. \tag{53}$$

2. $\hat{\hat{Y}}_{Pj}$ (i.e., $j = 2, 4, 6, ..., 16$) perform better than traditional ratio estimator, $\hat{\bar{Y}}_R$, if:

$$\theta\left[C_y^2 + C_x^2(1 - 2\varphi)\right] + \frac{A_{3j}^2}{2\bar{Y}^2 A_{4j}} > 1. \tag{54}$$

Proof

1. By comparing Relations (5) and (33):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pi}) < \mathrm{MSE}(\hat{\bar{Y}}_R),$$

$$\bar{Y}^2\left[1 - \frac{A_{1i}^2}{2A_{2i}}\right] < \theta\bar{Y}^2\left[C_y^2 + C_x^2(1 - 2\varphi)\right],$$

$$\theta\left[C_y^2 + C_x^2(1 - 2\varphi)\right] + \frac{A_{1i}^2}{2A_{2i}} > 1.$$

2. By comparing Relations (5) and (50):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pj}) < \mathrm{MSE}(\hat{\bar{Y}}_R),$$

$$\left[\bar{Y}^2 - \frac{A_{3j}^2}{2A_{4j}}\right] < \theta\bar{Y}^2\left[C_y^2 + C_x^2(1 - 2\varphi)\right],$$

$$\theta\left[C_y^2 + C_x^2(1 - 2\varphi)\right] + \frac{A_{3j}^2}{2\bar{Y}^2 A_{4j}} > 1.$$

Theorem 3.3

1. $\hat{\hat{Y}}_{Pi}$ (i.e., $i = 1, 3, 5, ..., 11$) perform better than Singh and Tailor's [4] estimator, $\hat{\bar{Y}}_{ST}$, if:

$$\theta\left[C_y^2 + C_x^2\lambda(\lambda - 2\varphi)\right] + \frac{A_{1i}^2}{2A_{2i}} > 1. \tag{55}$$

2. $\hat{\hat{Y}}_{Pj}$ (i.e., $j = 2, 4, 6, ..., 16$) perform better than Singh and Tailor's [4] estimator, $\hat{\bar{Y}}_{ST}$, if:

$$\theta\left[C_y^2 + C_x^2\lambda(\lambda - 2\varphi)\right] + \frac{A_{3j}^2}{2\bar{Y}^2 A_{4j}} > 1. \tag{56}$$

Proof

1. By comparing Relations (8) and (33):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pi}) < \mathrm{MSE}(\hat{\bar{Y}}_{ST}),$$

$$\bar{Y}^2\left[1 - \frac{A_{1i}^2}{2A_{2i}}\right] < \theta\bar{Y}^2\left[C_y^2 + C_x^2\lambda(\lambda - 2\varphi)\right],$$

$$\theta\left[C_y^2 + C_x^2\lambda(\lambda - 2\varphi)\right] + \frac{A_{1i}^2}{2A_{2i}} > 1.$$

2. By comparing Relations (8) and (50):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pj}) < \mathrm{MSE}(\hat{\bar{Y}}_{ST}),$$

$$\left[\bar{Y}^2 - \frac{A_{3j}^2}{2A_{4j}}\right] < \theta\bar{Y}^2\left[C_y^2 + C_x^2\lambda(\lambda - 2\varphi)\right],$$

$$\theta\left[C_y^2 + C_x^2\lambda(\lambda - 2\varphi)\right] + \frac{A_{3j}^2}{2\bar{Y}^2 A_{4j}} > 1.$$

Theorem 3.4

1. $\hat{\hat{Y}}_{Pi}$ (i.e., $i = 1, 3, 5, ..., 11$) perform better than Kadilar and Cingi's [24] estimators, $\hat{\bar{Y}}_{KCi}$, if:

$$\theta\left[C_y^2 + C_x^2\lambda_i(\lambda_i - 2\varphi)\right] + \frac{A_{1i}^2}{2A_{2i}} > 1. \tag{57}$$

2. $\hat{\hat{Y}}_{Pj}$ (i.e., $j = 2, 4, 6, ..., 16$) perform better than Kadilar and Cingi's [24] estimators, $\hat{\bar{Y}}_{KCi}$, if:

$$\theta\left[C_y^2 + C_x^2\lambda_i(\lambda_i - 2\varphi)\right] + \frac{A_{3j}^2}{2\bar{Y}^2 A_{4j}} > 1. \tag{58}$$

Proof

1. By comparing Relations (14) and (33):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pi}) < \mathrm{MSE}(\hat{\bar{Y}}_{KCi}),$$

$$\bar{Y}^2\left[1 - \frac{A_{1i}^2}{2A_{2i}}\right] < \theta\bar{Y}^2\left[C_y^2 + C_x^2\lambda_i(\lambda_i - 2\varphi)\right],$$

$$\theta\left[C_y^2 + C_x^2\lambda_i(\lambda_i - 2\varphi)\right] + \frac{A_{1i}^2}{2A_{2i}} > 1.$$

2. By comparing Relations (14) and (50):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pj}) < \mathrm{MSE}(\hat{\bar{Y}}_{KCi}),$$

$$\left[\bar{Y}^2 - \frac{A_{3j}^2}{2A_{4j}}\right] < \theta\bar{Y}^2\left[C_y^2 + C_x^2\lambda_i(\lambda_i - 2\varphi)\right],$$

$$\theta\left[C_y^2 + C_x^2\lambda_i(\lambda_i - 2\varphi)\right] + \frac{A_{3j}^2}{2\bar{Y}^2 A_{4j}} > 1.$$

Theorem 3.5

1. $\hat{\hat{Y}}_{Pi}$ (i.e., $i = 1,3,5,...,11$) perform better than Kadilar and Cingi's [25] estimators, $\hat{\hat{Y}}^*_{KCi}$, if:

$$\theta\left[\alpha_i^{*2}C_x^2 + C_y^2(1-\rho_{yx}^2)\right] + \frac{A_{1i}^2}{2A_{2i}} > 1. \qquad (59)$$

2. $\hat{\hat{Y}}_{Pj}$ (i.e., $j = 2,4,6,...,16$) perform better than Kadilar and Cingi's [25] estimators, $\hat{\hat{Y}}^*_{KCi}$, if:

$$\theta\left[\alpha_i^{*2}C_x^2 + C_y^2(1-\rho_{yx}^2)\right] + \frac{A_{3j}^2}{2\bar{Y}^2A_{4j}} > 1. \qquad (60)$$

Proof

1. By comparing Relations (17) and (33):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pi}) < \mathrm{MSE}(\hat{\hat{Y}}^*_{KCi}),$$

$$\bar{Y}^2\left[1-\frac{A_{1i}^2}{2A_{2i}}\right] < \theta\bar{Y}^2\left[\alpha_i^{*2}C_x^2 + C_y^2(1-\rho_{yx}^2)\right],$$

$$\theta\left[\alpha_i^{*2}C_x^2 + C_y^2(1-\rho_{yx}^2)\right] + \frac{A_{1i}^2}{2A_{2i}} > 1.$$

2. By comparing Relations (17) and (50):

$$\mathrm{MSE}_{\min}(\hat{\hat{Y}}_{Pj}) < \mathrm{MSE}(\hat{\hat{Y}}^*_{KCi}),$$

$$\left[\bar{Y}^2-\frac{A_{3j}^2}{2A_{4j}}\right] < \theta\bar{Y}^2\left[\alpha_i^{*2}C_x^2 + C_y^2(1-\rho_{yx}^2)\right],$$

$$\theta\left[\alpha_i^{*2}C_x^2 + C_y^2(1-\rho_{yx}^2)\right] + \frac{A_{3j}^2}{2\bar{Y}^2A_{4j}} > 1.$$

4. Empirical study

This section evaluates the performance of the proposed families of ratio-type estimators, i.e., ($\hat{\hat{Y}}_{Pi}$ i.e., $i = 1,3,5,...,11$) and ($\hat{\hat{Y}}_{Pj}$ i.e., $j = 2,4,6,...,16$), with competitive estimators. For this purpose, we consider two natural populations described below.

Population 1 (source: Kadilar and Cingi [24]):

The data relates to 104 villages of East Anatolia Region of Turkey in 1999. The following variables are taken into consideration:

y The level of apple production (in 1000 tones);

x The number of apple trees.

Values of different required parameters are: $N = 104$, $n = 20$, $\rho_{yx} = 0.865$, $\bar{Y} = 625.37$, $\bar{X} = 13.931$, $C_y = 1.866$, $C_x = 1.653$, and $\beta_2(x) = 17.52$.

Population 2 (source: Murthy [26]):

The variables are defined as follows:

y The output for 80 factories;

x The number of workers.

Values of different required parameters are: $N = 80$, $n = 20$, $\rho_{yx} = 0.9150$, $\bar{Y} = 51.8264$, $\bar{X} = 2.8513$, $C_y = 0.3542$, $C_x = 0.9484$, $S_y = 18.3569$, $S_x = 2.7042$, and $\beta_2(x) = 1.3005$.

Consider the MSE given in Section 1 and derived in Section 2; the values for existing and proposed families of estimators are computed and placed in Tables 1 and 2. The following are some important findings observed in Tables 1 and 2:

- It is important to note that the traditional ratio estimator is more efficient than the suggested class of estimators by Kadilar and Cingi [25], i.e., $\hat{\hat{Y}}^*_{KC1}$, $\hat{\hat{Y}}^*_{KC2}$, $\hat{\hat{Y}}^*_{KC3}$, $\hat{\hat{Y}}^*_{KC4}$, and $\hat{\hat{Y}}^*_{KC5}$, for both real populations discussed in this manuscript;
- It can be concluded that proposed families of ratio-type estimators are more efficient as they have lesser values of MSE as compared to the usual unbiased estimator, traditional ratio estimator, Singh and Tailor's [4] estimator, and Kadilar and Cingi's [24,25] classes of estimators;
- It is perceived in both populations that the two proposed estimators $\hat{\hat{Y}}_{P11}$ and $\hat{\hat{Y}}_{P6}$ have the least

Table 1. MSEs of the competing estimators for Population 1.

Estimators	MSE	Estimators	MSE	Estimators	MSE	Estimators	MSE
$\hat{\hat{Y}}$	54993.750	$\hat{\hat{Y}}^*_{KC1}$	52102.800	$\hat{\hat{Y}}_{P5}$	13335.710	$\hat{\hat{Y}}_{P8}$	13372.840
$\hat{\hat{Y}}_R$	13869.960	$\hat{\hat{Y}}^*_{KC2}$	53933.140	$\hat{\hat{Y}}_{P7}$	13341.230	$\hat{\hat{Y}}_{P10}$	13718.360
$\hat{\hat{Y}}_{ST}$	13898.700	$\hat{\hat{Y}}^*_{KC3}$	47217.400	$\hat{\hat{Y}}_{P9}$	13847.380	$\hat{\hat{Y}}_{P12}$	13489.420
$\hat{\hat{Y}}_{KC1}$	13852.970	$\hat{\hat{Y}}^*_{KC4}$	56697.220	$\hat{\hat{Y}}_{P11}$	**13328.150***	$\hat{\hat{Y}}_{P14}$	13373.720
$\hat{\hat{Y}}_{KC2}$	14252.910	$\hat{\hat{Y}}^*_{KC5}$	21012.160	$\hat{\hat{Y}}_{P2}$	13556.390	$\hat{\hat{Y}}_{P16}$	13373.020
$\hat{\hat{Y}}_{KC3}$	13863.330	$\hat{\hat{Y}}_{P1}$	13540.760	$\hat{\hat{Y}}_{P4}$	13436.650		
$\hat{\hat{Y}}_{KC4}$	27816.360	$\hat{\hat{Y}}_{P3}$	13330.560	$\hat{\hat{Y}}_{P6}$	13373.240		

*Bold value indicates minimum MSE.

Table 2. MSEs of the competing estimators for Population 2.

Estimators	MSE	Estimators	MSE	Estimators	MSE	Estimators	MSE
$\hat{\hat{Y}}$	12.6366	$\hat{\hat{Y}}^*_{KC1}$	53.9814	$\hat{\hat{Y}}_{P5}$	10.6136	$\hat{\hat{Y}}_{P8}$	2.0664
$\hat{\hat{Y}}_R$	41.3150	$\hat{\hat{Y}}^*_{KC2}$	52.6355	$\hat{\hat{Y}}_{P7}$	8.3250	$\hat{\hat{Y}}_{P10}$	2.0975
$\hat{\hat{Y}}_{ST}$	17.6849	$\hat{\hat{Y}}^*_{KC3}$	50.7866	$\hat{\hat{Y}}_{P9}$	6.7023	$\hat{\hat{Y}}_{P12}$	2.1057
$\hat{\hat{Y}}_{KC1}$	16.9505	$\hat{\hat{Y}}^*_{KC4}$	60.3414	$\hat{\hat{Y}}_{P11}$	9.2264	$\hat{\hat{Y}}_{P14}$	2.0724
$\hat{\hat{Y}}_{KC2}$	15.9550	$\hat{\hat{Y}}^*_{KC5}$	42.4043	$\hat{\hat{Y}}_{P2}$	2.0783	$\hat{\hat{Y}}_{P16}$	2.0759
$\hat{\hat{Y}}_{KC3}$	21.2570	$\hat{\hat{Y}}_{P1}$	6.1271	$\hat{\hat{Y}}_{P4}$	2.0829		
$\hat{\hat{Y}}_{KC4}$	11.6626	$\hat{\hat{Y}}_{P3}$	9.9128	$\hat{\hat{Y}}_{P6}$	**2.0645***		

*Bold value indicates minimum MSE.

Table 3. PREs of the competing estimators for Population 1.

Proposed estimators	Existing estimators											
	$\hat{\hat{Y}}$	$\hat{\hat{Y}}_R$	$\hat{\hat{Y}}_{ST}$	$\hat{\hat{Y}}_{KC1}$	$\hat{\hat{Y}}_{KC2}$	$\hat{\hat{Y}}_{KC3}$	$\hat{\hat{Y}}_{KC4}$	$\hat{\hat{Y}}^*_{KC1}$	$\hat{\hat{Y}}^*_{KC2}$	$\hat{\hat{Y}}^*_{KC3}$	$\hat{\hat{Y}}^*_{KC4}$	$\hat{\hat{Y}}^*_{KC5}$
$\hat{\hat{Y}}_{P1}$	406.1349	102.4312	102.6434	102.3057	105.2593	102.3822	20 5.4269	384.7849	398.3022	348.7057	418.7152	155.1771
$\hat{\hat{Y}}_{P3}$	412.5389	104.0463	104.2619	103.9189	106.9191	103.9966	208.6661	390.8523	404.5827	354.2042	425.3176	157.6240
$\hat{\hat{Y}}_{P5}$	412.3796	104.0062	104.2217	103.8788	106.8778	103.9564	208.5855	390.7014	404.4265	354.0674	425.1534	157.5631
$\hat{\hat{Y}}_{P7}$	412.2090	103.9631	104.1786	103.8358	106.8336	103.9134	208.4992	390.5397	404.2591	353.9209	424.9775	157.4979
$\hat{\hat{Y}}_{P9}$	397.1419	100.1631	100.3706	100.0404	102.9286	100.1152	200.8781	376.2647	389.4826	340.9844	409.4437	151.7411
$\hat{\hat{Y}}_{P11}$	412.6135	104.0652	104.2808	103.9377	106.9384	104.0154	208.7038	390.9230	404.6559	354.2682	425.3945	157.6525
$\hat{\hat{Y}}_{P2}$	405.6666	102.3131	102.5251	102.1878	105.1379	102.2642	205.1900	384.3413	397.8429	348.3036	418.2324	154.9982
$\hat{\hat{Y}}_{P4}$	409.2817	103.2248	103.4387	103.0984	106.0749	103.1755	207.0186	387.7663	401.3883	351.4075	421.9595	156.3795
$\hat{\hat{Y}}_{P6}$	411.2223	103.7143	103.9292	103.5872	106.5778	103.6647	208.0002	389.6049	403.2915	353.0738	423.9602	157.1209
$\hat{\hat{Y}}_{P8}$	411.2346	103.7174	103.9323	103.5903	106.5810	103.6678	208.0064	389.6166	403.3036	353.0843	423.9729	157.1256
$\hat{\hat{Y}}_{P10}$	400.8770	101.1051	101.3146	100.9812	103.8966	101.0568	202.7674	379.8034	393.1457	344.1913	413.2944	153.1682
$\hat{\hat{Y}}_{P12}$	407.6806	102.8210	103.0341	102.6951	105.6599	102.7719	206.2087	386.2494	399.8181	350.0328	420.3088	155.7677
$\hat{\hat{Y}}_{P14}$	411.2076	103.7106	103.9255	103.5835	106.5740	103.6610	207.9927	389.5909	403.2770	353.0611	423.9450	157.1153
$\hat{\hat{Y}}_{P16}$	411.2291	103.7160	103.9309	103.5889	106.5796	103.6664	208.0036	389.6113	403.2981	353.0796	423.9672	157.1235

MSE values (13328.150 and 2.0645) among all the proposed estimators;

- It is interesting to note that the Kadilar and Cingi's [24] estimators, $\hat{\hat{Y}}_{KC1}$ and $\hat{\hat{Y}}_{KC3}$, are the special cases of the proposed estimators, $\hat{\hat{Y}}_{P1}$ and $\hat{\hat{Y}}_{P3}$. When we put the values of $k = 1$ and $\psi_i = 1$, where $i = 1$ and 2, in $\hat{\hat{Y}}_{P1}$ and $\hat{\hat{Y}}_{P3}$, these estimators give the same result as the estimators $\hat{\hat{Y}}_{KC1}$ and $\hat{\hat{Y}}_{KC3}$.

This study calculated the Percentage Relative Efficiencies (PREs) for both populations under consideration to prove the dominance of the proposed families of ratio-type estimators, i.e., ($\hat{\hat{Y}}_{Pi}$, i.e., $i = 1, 3, 5, ..., 11$) and ($\hat{\hat{Y}}_{Pj}$, i.e., $j = 2, 4, 6, ..., 16$), over the existing estimators, i.e., $\hat{\hat{Y}}$, $\hat{\hat{Y}}_R$, $\hat{\hat{Y}}_{ST}$, $\hat{\hat{Y}}_{KC1}$, $\hat{\hat{Y}}_{KC2}$, $\hat{\hat{Y}}_{KC3}$, $\hat{\hat{Y}}_{KC4}$, $\hat{\hat{Y}}^*_{KC1}$, $\hat{\hat{Y}}^*_{KC2}$, $\hat{\hat{Y}}^*_{KC3}$, $\hat{\hat{Y}}^*_{KC4}$, and $\hat{\hat{Y}}^*_{KC5}$. Here, the Percentage Relative Efficiency (PRE) is the ratio of MSE of the existing estimators (e) to the MSE of the proposed ratio-type estimators (p) and is given by:

$$\text{PRE}(e, p) = \frac{\text{MSE}(e)}{\text{MSE}(p)} \times 100. \tag{61}$$

Tables 3 and 4 provide the PREs calculated for both populations. It is revealed in Tables 3 and 4 that the proposed families of ratio-type estimators are more competent than the existing estimators for both populations as they have higher values.

To get deeper insight, another important tool, i.e., Relative Root Mean Square Error (RRMSE), is computed in this study. It is the most useful measure to compare the precision of the estimators (see, e.g., [12,22,27,28]). The RRMSE of an estimator can be calculated by the relation given below:

$$\text{RRMSE} = \frac{\sqrt{\text{MSE}(\hat{\phi})}}{\phi}, \tag{62}$$

where MSE may be defined as:

$$\text{MSE}(\hat{\phi}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\phi} - \phi)^2, \tag{63}$$

where $\hat{\phi}$ is the estimate of ϕ on the ith sample.

It is quite obvious from the values of RRMSE shown in Tables 5 and 6 that the proposed families of ratio-type estimators perform better than the existing ones discussed in this article.

Table 4. PREs of the competing estimators for Population 2.

Proposed estimators	Existing estimators											
	$\hat{\bar{Y}}$	$\hat{\bar{Y}}_R$	$\hat{\bar{Y}}_{ST}$	$\hat{\bar{Y}}_{KC1}$	$\hat{\bar{Y}}_{KC2}$	$\hat{\bar{Y}}_{KC3}$	$\hat{\bar{Y}}_{KC4}$	$\hat{\bar{Y}}^*_{KC1}$	$\hat{\bar{Y}}^*_{KC2}$	$\hat{\bar{Y}}^*_{KC3}$	$\hat{\bar{Y}}^*_{KC4}$	$\hat{\bar{Y}}^*_{KC5}$
$\hat{\bar{Y}}_{P1}$	206.2411	674.2994	288.6341	276.6480	260.4005	346.9341	19 0.3445	881.0269	859.0606	828.8848	984.8281	692.0778
$\hat{\bar{Y}}_{P3}$	127.4776	416.7844	178.4047	170.9961	160.9535	214.4399	117.6519	544.5626	530.9852	512.3335	608.7221	427.7732
$\hat{\bar{Y}}_{P5}$	119.0605	389.2647	166.6249	159.7055	150.3260	200.2808	109.8835	508.6059	495.9250	478.5049	568.5291	399.5280
$\hat{\bar{Y}}_{P7}$	151.7910	496.2763	212.4312	203.6096	191.6517	255.3393	140.0913	648.4252	632.2583	610.0492	724.8216	509.3610
$\hat{\bar{Y}}_{P9}$	188.5412	616.4302	263.8632	252.9057	238.0526	317.1598	174.0089	805.4161	785.3349	757.7488	900.3088	632.6828
$\hat{\bar{Y}}_{P11}$	136.9613	447.7911	191.6771	183.7174	172.9277	230.3932	126.4047	585.0754	570.4879	550.4487	654.0081	459.5975
$\hat{\bar{Y}}_{P2}$	608.0258	1987.9228	850.9310	815.5945	767.6948	1022.8071	561.1606	2597.3825	2532.6228	2443.6607	2903.4018	2040.3359
$\hat{\bar{Y}}_{P4}$	606.6830	1983.5326	849.0518	813.7933	765.9993	1020.5483	559.9213	2591.6463	2527.0296	2438.2640	2896.9898	2035.8299
$\hat{\bar{Y}}_{P6}$	612.0901	2001.2109	856.6190	821.0463	772.8264	1029.6440	564.9116	2614.7445	2549.5519	2459.9952	2922.8094	2053.9743
$\hat{\bar{Y}}_{P8}$	611.5273	1999.3709	855.8314	820.2913	772.1158	1028.6973	564.3922	2612.3403	2547.2077	2457.7333	2920.1220	2052.0858
$\hat{\bar{Y}}_{P10}$	602.4601	1969.7259	843.1418	808.1287	760.6675	1013.4446	556.0238	2573.6067	2509.4398	2421.2920	2876.8248	2021.6591
$\hat{\bar{Y}}_{P12}$	600.1140	1962.0554	839.8585	804.9817	757.7053	1009.4980	553.8586	2563.5846	2499.6676	2411.8630	2865.6219	2013.7864
$\hat{\bar{Y}}_{P14}$	609.7568	1993.5823	853.3536	817.9164	769.8803	1025.7190	562.7582	2604.7771	2539.8330	2450.6176	2911.6676	2046.1446
$\hat{\bar{Y}}_{P16}$	608.7287	1990.2211	851.9148	816.5374	768.5823	1023.9896	561.8093	2600.3854	2535.5508	2446.4859	2906.7585	2042.6947

Table 5. RRMSEs of the competing estimators for Population 1.

Estimators	RRMSE	Estimators	RRMSE	Estimators	RRMSE	Estimators	RRMSE
$\hat{\bar{Y}}$	0.3750	$\hat{\bar{Y}}^*_{KC1}$	0.3650	$\hat{\bar{Y}}_{P5}$	0.1846	$\hat{\bar{Y}}_{P8}$	0.1849
$\hat{\bar{Y}}_R$	0.1883	$\hat{\bar{Y}}^*_{KC2}$	0.3714	$\hat{\bar{Y}}_{P7}$	0.1847	$\hat{\bar{Y}}_{P10}$	0.1872
$\hat{\bar{Y}}_{ST}$	0.1885	$\hat{\bar{Y}}^*_{KC3}$	0.3475	$\hat{\bar{Y}}_{P9}$	0.1881	$\hat{\bar{Y}}_{P12}$	0.1857
$\hat{\bar{Y}}_{KC1}$	0.1882	$\hat{\bar{Y}}^*_{KC4}$	0.3808	$\hat{\bar{Y}}_{P11}$	0.1846	$\hat{\bar{Y}}_{P14}$	0.1849
$\hat{\bar{Y}}_{KC2}$	0.1909	$\hat{\bar{Y}}^*_{KC5}$	0.2318	$\hat{\bar{Y}}_{P2}$	0.1861	$\hat{\bar{Y}}_{P16}$	0.1849
$\hat{\bar{Y}}_{KC3}$	0.1883	$\hat{\bar{Y}}_{P1}$	0.1860	$\hat{\bar{Y}}_{P4}$	0.1853		
$\hat{\bar{Y}}_{KC4}$	0.2667	$\hat{\bar{Y}}_{P3}$	0.1846	$\hat{\bar{Y}}_{P6}$	0.1849		

Table 6. RRMSEs of the competing estimators for Population 2.

Estimators	RRMSE	Estimators	RRMSE	Estimators	RRMSE	Estimators	RRMSE
$\hat{\bar{Y}}$	0.0686	$\hat{\bar{Y}}^*_{KC1}$	0.1418	$\hat{\bar{Y}}_{P5}$	0.0628	$\hat{\bar{Y}}_{P8}$	0.0277
$\hat{\bar{Y}}_R$	0.1240	$\hat{\bar{Y}}^*_{KC2}$	0.1400	$\hat{\bar{Y}}_{P7}$	0.0556	$\hat{\bar{Y}}_{P10}$	0.0279
$\hat{\bar{Y}}_{ST}$	0.0811	$\hat{\bar{Y}}^*_{KC3}$	0.1375	$\hat{\bar{Y}}_{P9}$	0.0499	$\hat{\bar{Y}}_{P12}$	0.0280
$\hat{\bar{Y}}_{KC1}$	0.0794	$\hat{\bar{Y}}^*_{KC4}$	0.1499	$\hat{\bar{Y}}_{P11}$	0.0586	$\hat{\bar{Y}}_{P14}$	0.0277
$\hat{\bar{Y}}_{KC2}$	0.0771	$\hat{\bar{Y}}^*_{KC5}$	0.1256	$\hat{\bar{Y}}_{P2}$	0.0278	$\hat{\bar{Y}}_{P16}$	0.0278
$\hat{\bar{Y}}_{KC3}$	0.0890	$\hat{\bar{Y}}_{P1}$	0.0477	$\hat{\bar{Y}}_{P4}$	0.0278		
$\hat{\bar{Y}}_{KC4}$	0.0659	$\hat{\bar{Y}}_{P3}$	0.0607	$\hat{\bar{Y}}_{P6}$	0.0277		

5. Conclusion

In this article, we proposed improved families of ratio-type estimators of population mean under simple random sampling without replacement (SRSWOR) scheme using correlation coefficient between study and auxiliary variables. We obtained bias, Mean Square Error (MSE), and minimum MSE formulae of the proposed families of ratio-type estimators and compared them theoretically with those of the traditional and exiting modified ratio estimators in the literature. It was found that the newly proposed estimators were more efficient than the traditional estimators, such as usual unbiased and ratio, i.e., $\hat{\bar{Y}}$ and $\hat{\bar{Y}}_R$, and existing modified ratio estimators, i.e., $\hat{\bar{Y}}_{ST}$ (Singh and Tailor [4]), $\hat{\bar{Y}}_{KCi}$ (Kadilar and Cingi [24]), and $\hat{\bar{Y}}^*_{KCi}$ (Kadilar and Cingi [25]), in terms of Mean Squared Error (MSE), Percentage Relative Efficiency (PRE), and Relative Root Mean Square Error (RRMSE). It was also empirically observed that the proposed families of ratio-type estimators performed better than the traditional and existing modified ratio estimators by using two natural population data sets. Hence, we strongly suggest the use of our newly proposed ratio-type estimators over the existing ratio-type estimators used in this study for future work.

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Biographies

Muhammad Irfan is a doctoral student of Statistics at Zhejiang University, China. He has been working as lecturer in the Department of Statistics, Government College University, Faisalabad, Pakistan, since 2004. He has 11 research papers published in reputed journals. His areas of interest are survey sampling, probability distributions, and time series analysis.

Maria Javed is a doctoral student of Statistics at Zhejiang University, China. She has been working as lecturer in the Department of Statistics, Government College University, Faisalabad, Pakistan, since 2004. She has more than 8 research publications in international research journals. Her fields of work are probability distributions and survey sampling.

Zhengyan Lin is a Professor of Statistics at Zhejiang University, China. He has over 230 refereed publications and 8 edited book volumes, and has guided research students at all levels of the curriculum, including undergraduate and PhD students. He has received external funding from National Natural Science.