Pricing and lot sizing of a decaying item under group dispatching with time-dependent demand and decay rates

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\textbf{Abstract.} Determining appropriate inventory and pricing policies is an important issue in scientific and industrial research. Here, an inventory control model of a decaying item with zero lead time is studied. Two mathematical models under different assumptions are developed. In the first model, deterioration rate is time-dependent and demand rate is price-sensitive while in the second model, deterioration rate is constant and demand rate is time- and price-dependent. The aim of this research is optimizing total cost by deriving decision variables such as dispatch cycle length, order quantity, and wholesale price. To optimize the total cost, a shipment group dispatching policy is used.

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1. Introduction

Determining optimal inventory control policy and selling price for different products is one of the main issues in industrial and scientific research, especially when the product is perishable. Recently, due to globalization flow, increasing costs, time-sensitivity occurrence of an action, and running out of resources, researchers have focused on supply chains coordination [1]. Here, we investigate shipment consolidation, pricing, and inventory strategies of a seller selling a decaying item. Thus, some research related to pricing, inventory, and shipment consolidation decisions for deteriorating products is reviewed from the literature.

Since price is one of the main factors for customers to decide about buying a product, jointly determination of inventory and pricing decisions is much important and, first, it was studied by Whithin [2]. Chen et al. [3] modeled a joint inventory-pricing problem for a periodic-review system. Ray et al. [4] analyzed the joint operation-marketing decision-making in a periodic review inventory system for a firm with stochastic and price-sensitive demand. Huang et al. [5] modeled the coordination and selection of suppliers such that pricing and replenishment decisions in a three-level chain as a dynamic non-cooperative game model were optimized. Polatoglou [6] developed a joint inventory-pricing model using a single-period problem for which demand rate was assumed a linear function. Zhu [7] formulated the integrated pricing and inventory control problem in a random demand condition and finite planning horizon with return and expediting. You et al. [8] developed a seasonal inventory model with trial periods during which customers could return the products without any penalty. Su and Genes [9] studied the price promotions effects on the total profit in a two-stage chain under deterministic demand.

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Muth and Cetinkaya [10] concentrated on a channel and compared the profits of both decentralized and centralized channels, where demand rate depended on selling price. Maddah and Bish [11] investigated a joint pricing-inventory problem in a newsboy system.

Recently, many researchers have focused on inventory control models of deteriorating products. Maity and Maiti [12] presented optimal production quantity and advertising expenditure of a multi-product inventory control model with inflation and time discounting under different constraints. Yu et al. [13] studied an inventory problem for a VMI system where both raw material and finished products were perishable. Hongjie et al. [14] extended an inventory control model for a decaying item in which vendor-managed inventory system was used. Mahata [15] formulated an inventory-production model for deteriorating products with delayed payment. Taleizadeh and Nematiollahi [16] extended an inventory control model in which the influences of inflation and time value of money on best strategies of deteriorating products were examined. Lee and Chung [17] used system dynamics to propose a new order system for deteriorating products and prepared a systematical simulation.

In shipment consolidation policy, the orders of customer are combined to make a larger batch to deliver to the customers. This policy is used to decrease the dispatching cost. Indeed, since several shipments are combined during a cycle, consolidation makes increase in carrying costs. Thus, replenishment and consolidation decisions must be made simultaneously. Time-Based Consolidation (TBC) and Quantity-Based Consolidation (QBC) are two types of this policy. In the first one, accumulated orders of customers are dispatched within each period. But in the second type, orders are distributed when the cumulative orders become larger than economic values.

Cetinkaya and Bookbinder [18] determined optimal (QBC) policy and related optimal cycle length. Cetinkaya et al. [19] analyzed both quantity- and time-based consolidation policies comparatively. Wong et al. [20] extended a shipment consolidation policy and the effects of consolidation were studied in their research. Marklund [21] extended a model to examine the effects of consolidation and replenishment. Howard and Marklund [22] evaluated the effects of time-based consolidation and stock allocation in a chain. Taleizadeh et al. [23] extended a joint replenishment problem under prepayment strategy for imported raw material with several operating limitations. Taleizadeh et al. [24,25] extended a multi-product single-machine imperfect production system without and with shortage. Ulku and Bookbinder [26] optimized the vendor’s profit when the selling price depended on arrival times of orders. Sajadieh and Jobar [27] focused on a two-echelon chain and extended a joint production-marketing-inventory problem to optimize total profit. Olson [28] developed a based-stock model for perishable items. Also, demands were considered as Poisson random variable. On the contrary, lifetime and lead-time were assumed to be fixed. Herbon et al. [29] extended an inventory management problem with perishable products. Maximization of retailer’s profit was the goal by considering customer’s satisfaction. Taleizadeh [30,31] developed a lot-sizing model for evaporating and deteriorating products with partial backordering. Diabat et al. [32] considered integrated inventory and routing problems for perishable products. Lu et al. [33] considered an inventory system with limited replenishment capacity for perishable goods. Also, the demand rate depended on the stock quantity. Gallego and Hu [34] studied dynamic pricing of complementary and substitutable perishable assets in an oligopolistic market. An integrated production-distribution model was developed by Tayal et al. [35] in a two-echelon supply chain for perishable goods. Taleizadeh et al. [36] studied optimal quantity and multi-discount price for perishable items. They assumed a time-dependent demand function under two scenarios. A new multi-product economic order quantity problem was considered by Maleki Vishkaei et al. [37]. They assumed that defective items were screened out 100% throughout screen process and were sold after screening period. Also, other related research was performed by Taleizadeh et al. [38–48], and Teimouri and Kazemi [49].

Generally, up to now, many outstanding studies about pricing, inventory control, and shipment consolidation for decaying items have been separately developed, but none of them has considered optimal shipment consolidation, inventory, and pricing policies together. The above-mentioned demand is an important gap in this context, and a motivation for this research. Here, a joint inventory-pricing model of a decaying product under shipment consolidation policy is extended.

In the next section, the problem description is provided.

2. Problem description

Consider an inventory system selling deteriorating product for which deterioration rate is linear time-proportional, \( \theta(t) = bt \). The seller wants to apply a TBC policy using which orders are consolidated and distributed in every period of time \( T \). The dispatch cycle length is a period within two deliveries and ordering cycle includes at least one distribution cycle and vendor goes to replenish whenever the on-hand inventory reaches zero. The lead time is zero and shortage is not allowed. Two scenarios under different assumptions are studied. In the first scenario, deterioration rate is time-dependent and demand rate
is price-sensitive while in the second one, deterioration rate is constant and demand rate is time- and price-dependent. The main aim is to extend two models to optimize the dispatch cycle length, selling price, and replenishment quantity for the explained two scenarios such that the total cost is minimized or total profit is maximized. The proposed models in this paper are applicable for every deteriorating product, such as dairy products, vegetables, and whatever is being perished as time passes.

Figure 1 indicates vendor’s inventory level in which $Q$ is the replenishment quantity, $Q_{pi}$ is the deteriorated quantity in each dispatch cycle ($i = 1 \ldots k$), and $D_i(T)$ shows the demand rate in each dispatch cycle. The following notations are used:

**Variables:**

$T$ Dispatch cycle length  
$Q$ Order quantity

**Parameters:**

$I_i(t)$ The level of inventory in the $i$th dispatch cycle at time $t$  
$\theta$ The decaying rate  
$\theta(t)$ Time-dependent decaying rate  
$D(p)$ The price-sensitive demand rate  
$D_i(T)$ The demand during the $i$th dispatch cycle  
$F_D$ The fixed dispatch cost  
$F_R$ The fixed order cost  
$C_D$ Dispatch cost per unit  
$C_R$ Purchasing cost per unit  

\[ w \] The waiting cost of product (unit/time)  
\[ h \] The carrying cost of product (unit/time)  
\[ k \] The number of dispatching periods during a replenishment cycle  
\[ v \] Time-dependent deterioration rate  
\[ S_j \] The $j$th demand arrival time in a dispatch cycle  
\[ Z(t) \] Deteriorated quantity in the first model  
\[ Q_p \] Deteriorated quantity in the second model

### 3. The first scenario

In this scenario, we assume that decaying rate depends on time and is a continuous function of time, $\theta(t) = bt$, $b \in [0, 1]$, and demand rate is price-sensitive and a linear function of wholesale price, $D(p) = (a - Bp)T$. Decision variables that should be determined are dispatch cycle length, wholesale price, and order quantity.

#### 3.1. Mathematical model

Here, a mathematical model of a joint pricing-inventory problem for a time-dependent decaying product using a time-based group dispatching policy is developed. From Figure 1, the changes of inventory level are presented as follows:

\[
\begin{aligned}
\frac{dI_i(t)}{dt} &= -\theta(t)I_i(t) - (i - 1)D(p); \quad t \in [(i - 1)T, iT]. \\
I((iT)) &= 0
\end{aligned}
\]  

(1)

By solving this equation, we have:

\[
I_i(t) = -(i - 1)D(p) \left( t + \frac{bt^3}{6} \right) e^{-\frac{bt^2}{2}} + I(0)e^{-\frac{bt^2}{2}}.  
\]

(2)

To obtain the optimum values of variables, first, the cost function should be modeled. Since each replenishment period includes $k$ (random variable) distributing cycles with length $T$, the expected value of cycle length is $E(k)T$.

#### Ordering cost:

The order quantity is demand plus deteriorated product as shown in Eq. (3):

\[
Q = Z(t) + \sum_{i=1}^{k-1} D_i(T).  
\]

(3)

where the quantity deteriorated during each cycle is:

\[
Z(t) = I(t)\text{without deterioration} - I(t)\text{with deterioration}.
\]

(4)
Inventory level without deterioration is obtained using Eq. (5):

\[ I(t)_{\text{without deterioration}} = \lim_{t \to 0} I(t)_{\text{with deterioration}} = 1 - t(i - 1)D(p) + I(0). \]

(5)

Thus, using Eqs. (2) and (5) and using \( I(0) = D(P)(t + \frac{b}{n}) + I(t)e^{\frac{t^2}{T}} \), Eq. (4) changes to:

\[ Z(t) = I(t)(e^{\frac{t^2}{T}} - 1) + (i - 1)D(p)\frac{b^3}{6}. \]

(6)

Since \( I(kT) = 0 \), deteriorated quantity at the end of each replenishment cycle is:

\[ Z(KT) = (i - 1)D(p)\frac{b^3T^3}{6}. \]

(7)

Therefore, replenishment quantity is:

\[ Q = Z(KT) + KD(p)T = (i - 1)D(p)\frac{b^3T^3}{6} + KD(p)T. \]

(8)

Finally, the expected replenishment quantity according to Eq. (8) is equal to:

\[ E(Q) = (i - 1)D(p)\frac{bE(K^3)}{6} + E(k)D(p)T. \]

(9)

Thus, the expected related cost is:

\[ E(R_c) = FR + CR \cdot E(Q) = FR + CR\left(a - Bp\right)e^{\frac{K^3T}{6}} + C_{E}(E(k)(a - Bp)T. \]

(10)

**Dispatch cost:**

Based on existing \( k \) dispatch cycles and dispatch quantity, which is \( E(D_i(T)) \), the dispatch cost can be determined as follows:

\[ \text{dispatch cost} = F_D E(k) + C_D E(k)E(D_i(T)) = F_D E(k) + C_D E(k)(a - b)pT. \]

(11)

**Holding cost:**

The level of inventory at time \( t \), after substitution of Eq. (8) in Eq. (2), is:

\[ I(t) = D(p) \left[ (kT - t) + \frac{b}{6}(k^3T^3 - t^3) \right] e^{\frac{-t^2}{T}}. \]

Utilizing the Taylor series expansion, \( e^{-\frac{t^2}{T}} = 1 - \frac{t^2}{2} + \frac{b^2}{4} \), the cyclic inventory carrying cost is given by:

\[ hE \left( \frac{\sum_{i=1}^{k} \int_{i(T-1)}^{iT} I(t) \, dt}{2} \right) = hE \left( \frac{k}{2} \left( \int_{0}^{1} D(p) \left( kT - t + \frac{b}{6}(k^3T^3 - t^3) \right) e^{-\frac{t^2}{T}} \right) \right) = \frac{hE \left( \frac{k^2D(P)^2T^2}{2} + \frac{D(p)B^4T^4}{12} \right)}{2} \]

(12)

**The cost of waiting:**

Using the definition of \( S_j \), the waiting time of customer is \( T - S_j \) (see Figure 2). Therefore, the cost of waiting for customer is:

\[ wE \left[ \left( T - S_1 \right) + \left( T - S_2 \right) + \ldots + \left( T - S_{N(T)} \right) \right] = wE \left[ D_{i}(T)T - \sum_{n=1}^{N(T)} S_n \right] = wE(k)\frac{\lambda T^2}{2}. \]

(13)

Using Eqs. (10) to (13), the total cost function is:

\[ \text{Total cost function} = FR + CR\lambda T E(k) + \frac{C_{R}B^3E(k^3)}{6} + F_{D} E(k) + C_{D} E(k)\lambda T + \frac{hE \left( \frac{k^2D(P)^2T^2}{2} + \frac{D(p)B^4T^4}{12} \right)}{2} + wE(k)\frac{\lambda T^2}{2}. \]

(14)
Dividing Eq. (16) over \( E(k)T \) yields:

\[
TC(p, T, Q) = \frac{F_R}{E(k)T} + C_R \lambda + \frac{C_R \lambda^2 T^2 E(k^3)}{6E(k)} + \frac{F_D}{T} + C_D \lambda + \frac{h \lambda T E(k^2)}{2E(k)} + \frac{h \lambda^3 T^3 E(k^4)}{12E(k)} + \frac{w_a \lambda T}{2}.
\]

\[
(15)
\]

**3.2. Solution method**

The following Lemma is presented to derive a solution method.

**Lemma 1.** The following equations can be used to derive the optimal solutions:

\[
E(k) = \frac{\bar{Q}}{\lambda T}, \quad \bar{Q}(k+1) = \frac{\bar{Q}(k) + 1}{\lambda^2 T^2}, \quad \bar{Q}(k+1) = \frac{\bar{Q}(k) + 1(\bar{Q} + 2)}{\lambda^3 T^3}, \quad \bar{Q}(k+1) = \frac{\bar{Q}(k) + 1(\bar{Q} + 2)(\bar{Q} + 3)}{\lambda^4 T^4}.
\]

\[
(16) \quad (17) \quad (18) \quad (19)
\]

**Proof.** Let \( f(0) \) denote the distribution function of \( D_1(T) \), and \( f^{(k)}(0) \) denote the \( k \)-fold convolution of \( f(0) \). From \( k = \inf \{ k : \sum_{i=1}^{k} D_1(T) \geq Q \} \), we have

\[
P[k \geq k + 1] = f^{(k)}(Q) \text{ and, thus, } P[k \leq k + 1] = 1 - f^{(k)}(Q).
\]

Since \( f(0) \) is a Poisson distribution with parameter \( \lambda T \), \( k \)-fold convolution of \( f(0) \) is a Poisson distribution with parameter \( \lambda T k \), where:

\[
f^{(k)}(Q) = \sum_{i=0}^{Q} (\lambda T)^i e^{-\lambda T} \frac{i!}{i!}.
\]

Then, for \( k = 1, 2, \ldots \), we have:

\[
p[k \leq k] = 1 - \sum_{i=0}^{Q} (\lambda T)^i e^{-\lambda T} \frac{i!}{i!},
\]

of which the right side is a \( Q \)-stage P.D.F. with parameter \( \lambda T \) and expected value of \( \frac{Q+1}{\lambda T} \).

For simplicity, by using Lemma 1 and substituting \( \bar{Q} = Q + 1 \) and \( \lambda = a - b p \), the expected long-run average cost changes to:

\[
TC(P, T, Q) = \frac{F_R(a - b p)}{Q} + C_R(a - b p) + \frac{b C_R (\bar{Q} + 1)(\bar{Q} + 2)}{6(a - b p)} + \frac{F_D}{T} + C_D(a - b p) + \frac{h (\bar{Q} + 1)}{2} + \frac{h b (\bar{Q} + 1)(\bar{Q} + 2)(\bar{Q} + 3)}{12(a - b p^2)} + \frac{w(a - b p) T}{2}.
\]

\[
(20)
\]

**Lemma 2.** For each couple of \( T \) and \( p \), the optimum quantity of \( \bar{Q} \) should satisfy Inequality (21):

\[
\bar{Q}^*(\bar{Q}^*-1) + \frac{b \bar{Q}^*(\bar{Q}^*+1)(\bar{Q}^*-1)}{(a - b p)} \leq \frac{2C_R}{3h} \left( \frac{(\bar{Q}^*+2)}{2(a - b p)} \right) \leq \frac{2F_R(a - b p)}{h}
\]

\[
\leq \bar{Q}^*(\bar{Q}^*+1) + \frac{b \bar{Q}^*(\bar{Q}^*+1)(\bar{Q}^*+2)}{(a - b p)} \left( \frac{2C_R}{3h} + \frac{(\bar{Q}^*+3)}{2(a - b p)} \right) \quad \text{(21)}
\]

**Proof.** For each couple of \( p \) and \( T \), the optimum quantity of \( \bar{Q} \) should satisfy \( TC(\bar{Q}^* - 1) \geq TC(\bar{Q}^*) \) and \( TC(\bar{Q}^* + 1) \geq TC(\bar{Q}^*) \). Using Eq. (20), the optimality condition for \( \bar{Q} \) is:

\[
\bar{Q}^*(\bar{Q}^* - 1) + \frac{b \bar{Q}^*(\bar{Q}^* + 1)(\bar{Q}^* - 1)}{(a - b p)} \left( \frac{2C_R}{3h} + \frac{(\bar{Q}^*+2)}{2(a - b p)} \right) \leq \frac{2F_R(a - b p)}{h} \leq \bar{Q}^*(\bar{Q}^* + 1) + \frac{b \bar{Q}^*(\bar{Q}^* + 1)(\bar{Q}^* + 2)}{(a - b p)} \left( \frac{2C_R}{3h} + \frac{(\bar{Q}^* + 3)}{2(a - b p)} \right).
\]

**Lemma 3.** The following condition should be satisfied by the upper bound of \( \bar{Q} \):

\[
\bar{Q}^*(\bar{Q}^* - 1) + \frac{b \bar{Q}^*(\bar{Q}^* + 1)(\bar{Q}^* - 1)}{a} \left( \frac{2C_R}{3h} + \frac{(\bar{Q}^* + 2)}{2a} \right) \leq \frac{2F_R a}{h} \leq \bar{Q}^*(\bar{Q}^* + 1) + \frac{b \bar{Q}^*(\bar{Q}^* + 1)(\bar{Q}^* + 2)}{a} \left( \frac{2C_R}{3h} + \frac{(\bar{Q}^* + 3)}{2a} \right). \quad \text{(22)}
\]

**Proof.** \( \frac{2F_R(a - b p)}{h} \) in Eq. (21) is decreasing with respect to \( p \). Therefore, the maximum \( \bar{Q} \) is derived at \( p = 0 \).
Theorem 1. The cost function becomes convex if
\[ w \leq \frac{2F_D}{B_T^2} + \frac{b\rho(\bar{Q} + 1)(\bar{Q} + 2)}{4\lambda^3} \left( \frac{C_R}{3} + \frac{h(\bar{Q} + 3)}{2\lambda} \right). \]

Proof. The total cost function is convex if \( XHXT = [P \ T] \times H \times [P \ T]^T \geq 0 \), where:
\[
H = \begin{bmatrix}
\frac{\gamma T P}{\rho P} & \frac{\gamma T P}{\rho P} \\
\frac{\gamma T P}{\rho P} & \frac{\gamma T P}{\rho P} \\
\frac{\gamma T P}{\rho P} & \frac{\gamma T P}{\rho P} \\
\frac{\gamma T P}{\rho P} & \frac{\gamma T P}{\rho P}
\end{bmatrix}
= \begin{bmatrix}
\frac{B^2\rho^2(\bar{Q} + 1)(\bar{Q} + 2)}{\lambda^3} \\
\lambda^3 \left( \frac{C_R}{3} + \frac{h(\bar{Q} + 3)}{2\lambda} \right) - \frac{Bw}{2} \\
\lambda^3 \left( \frac{C_R}{3} + \frac{h(\bar{Q} + 3)}{2\lambda} \right) + \frac{2F_D}{T} - B\rho T w,
\end{bmatrix}
(23)

Therefore, we have:
\[ XH 	imes X^T = \frac{B^2\rho^2(\bar{Q} + 1)(\bar{Q} + 2)}{\lambda^3} \]
\[ \left( \frac{C_R}{3} + \frac{h(\bar{Q} + 3)}{2\lambda} \right) + \frac{2F_D}{T} - B\rho T w, \]
and we should show that:
\[ XH 	imes X^T = \frac{B^2\rho^2(\bar{Q} + 1)(\bar{Q} + 2)}{\lambda^3} \]
\[ \left( \frac{C_R}{3} + \frac{h(\bar{Q} + 3)}{2\lambda} \right) + \frac{2F_D}{T} - B\rho T w \geq 0. \]
Thus, the cost function is convex if and only if:
\[ w \leq \frac{2F_D}{B_T^2} + \frac{b\rho(\bar{Q} + 1)(\bar{Q} + 2)}{T\lambda^3} \left( \frac{C_R}{3} + \frac{h(\bar{Q} + 3)}{2\lambda} \right). \]
(26)

Using the first derivative of \( TC(p, T, Q) \) with respect to \( p \) yields:
\[
\frac{\partial TC(p, T, Q)}{\partial P} = -B \left( \frac{F_R}{Q} + \frac{wT}{2} + C_R + C_D \right)
- \frac{b\rho(\bar{Q} + 1)(\bar{Q} + 2)}{6\lambda^2}
- \frac{b\rho((\bar{Q} + 1)(\bar{Q} + 2)(\bar{Q} + 3))}{6\lambda^3}.
\]
(27)

Moreover, the first derivative of \( TC(p, T, Q) \) with respect to \( T \) yields:
\[
\frac{\partial TC(p, T, Q)}{\partial T} = \frac{w\lambda}{2} - \frac{F_D}{T^2}.
\]
(28)

Setting Eq. (28) equal to zero gives:
\[ p^* = \frac{a}{B} - \frac{2F_D}{BwT^2}. \]
(29)

After some algebraic calculations, the following equation is derived:
\[ AT^4 + ET^3 + CT + D = 0, \]
where:
\[ A = \frac{-b\rho(\bar{Q} + 1)(\bar{Q} + 2)w^2}{24F_D^2}, \]
\[ E = \frac{-b\rho(\bar{Q} + 1)(\bar{Q} + 2)(\bar{Q} + 3)w^2}{48F_D^2}, \]
\[ C = \frac{w}{2}, \]
\[ D = \frac{F_R}{Q} + C_D + C_R. \]
(34)

Now, we can use the following algorithm to solve the problem.

Step 1. Compute \( \bar{Q}_{\text{max}} \) using Eq. (22);
Step 2. For \( (\bar{Q} = 1, \bar{Q}_{\text{max}}) \), determine the coefficients of the polynomial shown in Eq. (30);
Step 3. Determine all acceptable roots for \( T \) using Eq. (30) and MATLAB software;
Step 4. Calculate \( p \) for all acceptable roots of \( T \) from the previous step;
Step 5. Now, total cost should be calculated using Eq. (20) and convexity should be examined using Theorem 1;
Step 6. In the comparison of the derived total costs, the lowest cost shows the related optimal solutions.

4. The second scenario

In this scenario, we assume that decaying rate is constant and demand rate is time- and price-sensitive and a function of time and selling price \( D(p, t) = (a - bp)T e^{-\gamma T} = \lambda T e^{-\gamma T} \). Decision variables, namely, dispatch cycle length, order quantity, and selling price should be determined to optimize the total profit.

4.1. Mathematical modeling

In this scenario, Eqs. (1) and (2) will change to Eqs. (35) and (36) as follows:
\[
\frac{dI_i(t)}{dt} = -\theta I_i(t) - (i - 1)D(p, t) \quad t \in [i - 1, i - 1],
\]
(35)
\[
I_i(t) = \frac{(i - 1)(a - bp)e^{\gamma t}}{(v + \theta)} \left[ e^{(v + \theta)(\frac{kT}{\lambda} - 1)} - 1 \right].
\]
(36)

Therefore, the order quantity is:
\[
Q = I(0) = \frac{(a - bp)}{(v + \theta)} \left[ e^{(v + \theta)(\frac{kT}{\lambda})} - 1 \right].
\]
(37)
According to the description provided for the previous case, \( E(D_i(T)) = \lambda e^{\theta T} = (a - bp)e^{\theta T} \) and the expected income is:

\[
PE \left( \sum_{i=1}^{k-1} \left( \int_{(i-1)T}^{iT} D_i(T)dt \right) \right) = PE \left( \sum_{i=1}^{k-1} \left( \int_{(i-1)T}^{iT} (a - bp)e^{\theta T}dt \right) \right) = P(a - bp)T^2E(k) + \frac{P(a - bp)T^2vE(k^2)}{2}. \tag{38}
\]

To derive the cost function, we act as follows.

**Ordering cost:**

The ordering cost of this case, using the logic of the previous case, is:

\[
Q = Q_p + \sum_{i=1}^{k-1} D_i(T). \tag{39}
\]

Moreover, quantity deteriorated during each cycle is:

\[
Q_p = I(0) - I(KT) - \sum_{i=1}^{k-1} D_i(t), \tag{40}
\]

where:

\[
Q = I(0) = \frac{(a - bp)}{(v + \theta)} \left[ e^{(v + \theta)kT} - 1 \right],
\]

\[
I(KT) = 0, \quad \sum_{i=1}^{k-1} \left( \int_{(i-1)T}^{iT} (a - bp)e^{\theta T}dt \right),
\]

and the expected deteriorated quantity is:

\[
E(Q_p) = E(I(0)) - E(I(KT)) = E \left( \sum_{i=1}^{k-1} \left( \int_{(i-1)T}^{iT} (a - bp)e^{\theta T}dt \right) \right) = \lambda E(k)T + \frac{\lambda e^{(v+\theta)kT}E(k^2)}{2} + \lambda E(k)T^2 + \frac{\lambda \theta T^3E(k^2)}{2}. \tag{41}
\]

Finally, using the Taylor series expansion, \( e^{\theta Tk} = 1 + \theta Tk + \frac{(\theta Tk)^2}{2} \), Eq. (41) changes to:

\[
E(Q) = E(I(0)) = (a - bp)E(k)T + \frac{(a - bp)(v + \theta)T^2E(k^2)}{2}. \tag{42}
\]

And the expected ordering cost is:

\[
E(R_c) = F_R + C_R E(Q) = F_R + C_R (a - bp)TE(k) + \frac{C_R(a - bp)T^2(v + \theta)E(k^2)}{2}. \tag{43}
\]

**Dispatch cost:**

For the distribution cost, similar to the previous case, we have:

\[
E(D_C) = F_D E(k) + C_D E(k)E(D_i(T)) = F_D E(k) + \frac{C_D E \left( \sum_{i=1}^{k} \left( \int_{(i-1)T}^{iT} (a - bp)e^{\theta T}dt \right) \right)}{2} = F_D E(k) + C_D (a - bp)T^2E(k) + \frac{C_D (a - bp)T^2vE(k^2)}{2}. \tag{44}
\]

**Carrying cost:**

From Eq. (36), the level of inventory at time \( t \) is \( I(t) = \frac{a - bp}{(v + \theta)} \left[ e^{(v + \theta)(kT-t)} - 1 \right] \). Utilizing the Taylor series expansion, \( e^{\theta Tk} = 1 + \theta Tk + \frac{(\theta Tk)^2}{2} \), the cyclic inventory carrying cost is:

\[
hE \left( \sum_{i=1}^{k} \left( \int_{(i-1)T}^{iT} I(t)dt \right) \right) = hE \left( \sum_{i=1}^{k} \left( \int_{(i-1)T}^{iT} \frac{D(p)e^{\theta t}}{(v + \theta)}(e^{(v + \theta)(kT-t)} - 1) dt \right) \right) = -\frac{h(a - bp)e^{\theta T^2}E(k^2)}{2(v + \theta)} + \frac{h(a - bp)vT^2E(k^2)}{2(v + \theta)} - \frac{h(a - bp)vT^2E(k^2)}{2(v + \theta)}. \tag{45}
\]

Using Eqs. (38), (43), (44), and (45), the total profit function is:

\[
\text{Total Profit Function} = E(k)\lambda T^2p + \frac{p\lambda \theta T^3E(k^2)}{2} \left\{ F_R + C_R \lambda \theta T E(k) + \frac{C_R \lambda T^2(v + \theta)E(k^2)}{2} \right\}
\]

\[
+ F_D E(k) + C_D E(k)\lambda T^2 + \frac{C_D E(k^2)vT^3}{2} - \frac{\lambda \theta T^2E(k^2)}{2(v + \theta)} + \frac{\lambda \theta T^2E(k^2)}{2} \right\} \tag{46}
\]
Dividing the above total profit function by $E(k)T$, we have:

$$\text{Total Profit Function} = \lambda T p + \frac{p \lambda v T^2 E(k^2)}{2E(k)}$$

$$\left\{ \frac{F_n}{Q} + C_R \lambda + \frac{C_n \lambda T (v+\delta) E(k^2)}{2E(k)} + \frac{F_n p}{Q} \right\} - \frac{C_D \lambda T + C_D v T (v+\delta) E(k^2)}{2e^{\alpha T}} - h \frac{(Q+1) (v+\delta) E(k^2)}{2e^{\alpha T}} \right\}.$$ (47)

By substituting $E(k) = \frac{Q+1}{(a-bp)e^{\alpha T}}$ and $\lambda = a - bp$, and assuming $\bar{Q} = Q + 1$, the expected long-run average profit changes to:

$$\text{Total Profit Function} = \lambda T p + \frac{p v T (\bar{Q} + 1)}{2e^{\alpha T}}$$

$$\left\{ \frac{F_n}{Q} + C_R \lambda + \frac{C_n \lambda T (v+\delta) E(k^2)}{2E(k)} + \frac{F_n p}{Q} \right\} - \frac{C_D \lambda T + C_D v T (v+\delta) E(k^2)}{2e^{\alpha T}} - h \frac{(\bar{Q}+1) (v+\delta) E(k^2)}{2e^{\alpha T}} \right\}.$$ (48)

### 4.2. Solution method

**Lemma 4.** For each couple of $p$ and $T$, the optimum quantity of $\bar{Q}$ should satisfy inequality (49):

$$\left\{ \left\{ \frac{(v+\delta) \bar{Q}^* \bar{Q}^* - 1}{e^{\alpha T}} + \frac{(v+\delta) C n \bar{Q}^* \bar{Q}^* - 1}{e^{\alpha T}} - \frac{v^2 \bar{Q}^* \bar{Q}^* - 1}{e^{\alpha T}} \right\} \right\}$$

$$\leq \frac{2 F_n (a - bp)e^{\alpha T}}{h}$$

$$\leq \left\{ \left\{ \frac{(v+\delta) \bar{Q}^* \bar{Q}^* - 1}{e^{\alpha T}} + \frac{(v+\delta) C n \bar{Q}^* \bar{Q}^* - 1}{e^{\alpha T}} - \frac{v^2 \bar{Q}^* \bar{Q}^* - 1}{e^{\alpha T}} \right\} \right\}$$

**Proof.** For each couple of $p$ and $T$, the optimum quantity of $\bar{Q}$ should satisfy $TP(\bar{Q}^* - 1) \leq TP(\bar{Q}^*)$ and $TP(\bar{Q}^* + 1) \leq TP(\bar{Q}^*)$. Using Eq. (48), the optimality condition for $\bar{Q}$ is obtained as Eq. (49).

**Lemma 5.** The upper bound of $\bar{Q}$ satisfies the following condition:

$$\frac{\theta C_R \bar{Q}^* (\bar{Q}^* - 1)}{\bar{h}} \leq \frac{2 F_n a}{h}$$

$$\leq \frac{\theta C_R \bar{Q}^* (\bar{Q}^* + 1)}{\bar{h}}.$$ (50)

**Proof.** The maximum amount of $\frac{2 F_n (a - bp)e^{\alpha T}}{h}$ or the maximum amount of replenishment is obtained when the demand is maximum. Thus, the maximum quantity of $\bar{Q}$ is derived at $p = v = 0$.

**Theorem 2.** The profit function is concave if and only if $F_D \geq \left[ a + b C_D - 3 b p + \frac{v (\bar{Q}+1)}{2e^{\alpha T}} \right] p T^2$.

**Proof.** The total profit function is concave if

$$X \times H \times X^T = [P \ T] \times H \times [P \ T]^T \leq 0,$$ where:

$$H = \left[ \begin{array}{c} \frac{\alpha T p}{a T p} \frac{\alpha T p}{a T p} \frac{\alpha T p}{a T p} \frac{\alpha T p}{a T p} \\ \end{array} \right]$$

$$= \left[ \begin{array}{c} -2 b T \ a - 2 b p + b C_D + \frac{v (\bar{Q}+1)}{2e^{\alpha T}} - \frac{2 F_n}{p T^2} \\ \end{array} \right]$$

then we have:

$$X \times H \times X^T = -3 b p + a + b C_D + \frac{v (\bar{Q}+1)}{2e^{\alpha T}}$$

$$- \frac{F_D}{p T^2} \leq 0.$$ (53)

In order to be sure that the profit function is concave, the following inequality should be held:

$$F_D \geq \left[ a + b C_D - 3 b p + \frac{v (\bar{Q}+1)}{2e^{\alpha T}} \right] p T^2.$$ (54)

Now, the first derivative of total profit function with respect to $p$ is:

$$\frac{\partial T P(p, T, Q)}{\partial P} = a T - 2 b p T + \frac{b F_n e^{\alpha T}}{\bar{Q}} + b C_D T$$

$$+ b T C_D + \frac{\alpha T (\bar{Q} + 1)}{2e^{\alpha T}}.$$ (55)

Moreover, the first derivative with respect to $T$ becomes:

$$\frac{\partial T P(p, T, Q)}{\partial T} = \frac{F_D}{T^2} - \frac{C_D \lambda + \alpha}{2e^{\alpha T}} + \frac{p v (\bar{Q}+1)}{2e^{\alpha T}}.$$ (56)

By setting Eq. (55) equal to zero, we have:

$$p^* = \frac{a}{2b} + \frac{F_n e^{\alpha T}}{2T \bar{Q}} + \frac{C_R}{2T} + \frac{C_D}{2} + \frac{v (\bar{Q}+1)}{2e^{\alpha T}}.$$ (57)
Substitution of Eq. (57) in Eq. (56) yields the following equation:

\[ AT^2 + BT + C = 0, \tag{58} \]

where

\[ A = -\frac{bF_R R^2 e^{2\theta t}}{4Q^2} - \frac{bF_R C R e^{\theta t}}{2Q} - \frac{b C R^2}{4} + F_D, \tag{59} \]

\[ B = -\frac{v(\bar{Q} + 1) C R}{4e^{\theta t}} - \frac{v(\bar{Q} + 1) F R}{4Q}, \tag{60} \]

\[ C = \frac{a v(\bar{Q} + 1)}{be^{\theta t}} + \frac{v C_D (\bar{Q} + 1)}{4e^{\theta t}} + \frac{a^2}{4b} + \frac{b C_D^2}{4} - \frac{a C_D}{2}, \tag{61} \]

Eq. (58) is a quadratic polynomial of which the discriminant is:

\[ \Delta = B^2 - 4AC. \tag{62} \]

Based on the sign of \( \Delta \), the following cases can occur:

1. When \( \Delta > 0 \), two real roots exist;
2. When \( \Delta = 0 \), a single root exists;
3. When \( \Delta < 0 \), there is no real root.

Now, we can use the following solution procedure to solve the problem:

**Step 1.** Determine \( \bar{Q}_{\text{max}} \) using Eq. (50);

**Step 2.** For \( \bar{Q} = 1 \ldots \bar{Q}_{\text{max}} \), determine the coefficients of polynomial (58);

**Step 3.** Determine all acceptable roots of period length using Eq. (58) and MATLAB;

**Step 4.** Determine \( p \) for all acceptable roots of period length from the previous step;

**Step 5.** For all combinations of order quantity, selling price, and period length, calculate the related total profit and check the concavity using Theorem 2;

**Step 6.** By comparison of the obtained total profits, the related optimal decision variables can be applied.

5. Practical and computational results

Consider a milk producer company for which deteriorating rate is linearly time-proportional, \( \theta(t) = bt \), \( b \in [0, 1] \), and demand rate is price-sensitive and is a linear function of wholesale price, \( D(p) = (a - Bp)T \). Moreover, deterioration rate may be constant and demand rate can be time- and price-sensitive and a function of time and selling price \( D(p, t) = (a - bt)T e^{\theta t} - \lambda T e^{\theta t} \). These two conditions can be analyzed using the first and the second models developed in this paper. Decision variables, namely, cycle length, selling price, and order quantity, should be determined such that total cost is minimized using the proposed solution method.

5.1. Example 1

For the first developed model, consider \( F_D = 5 \), \( F_R = 40 \), \( a = 20 \), \( C_D = 3 \), \( h = 1 \), \( w = 50 \), \( B = 0.02 \), \( \theta = 0.04 \), \( b = 0.05 \), and \( C_R = 40 \).

**Step 1.** Using Eq. (22), \( \bar{Q} = 25 \);

**Step 2.** From \( \bar{Q} = 1 \) to 25, the values of Eqs. (31) to (34) are determined;

**Step 3.** All acceptable real roots of cycle length are determined and reported in Table 1;

**Step 4.** For all acceptable values of \( T \), the wholesale prices are determined (see Table 1);

**Step 5.** For all groups of decision variables, the concavity of the objective function is checked and the related values are shown in Table 1;

**Step 6.** The lowest cost, i.e. \( TC^* = 149.2175 \), corresponds to \( Q^* = 10 \), \( T^* = 0.3179 \), and \( P^* = 90.1049 \).

5.2. Example 2

Now, consider \( F_D = 5 \), \( C_D = 3 \), \( b = 1 \), \( b = 0.04 \), \( F_R = C_R = 40 \), \( w = 50 \), \( v = -0.98 \), \( \theta = 0.04 \), and \( a = 20 \).

**Step 1.** From Eq. (50), \( \bar{Q} = 40 \);

**Step 2.** From \( \bar{Q} = 1 \) to 40, the values of Eqs. (59) to (61) are calculated;

**Step 3.** All acceptable roots of cycle length are calculated and reported in Table 2;

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<td>0.0417</td>
<td>295.829</td>
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</tbody>
</table>
Step 4. $p$ is determined for all acceptable roots of $T$ (see Table 2);

Step 5. For all groups of decision variables, the concavity of the objective function is checked and the related values are shown in Table 2;

Step 6. The highest profit, i.e. $TP^* = 221.724$, corresponds to $Q^* = 27, T^* = 0.0332$, and $P^* = 306.794$.

6. Sensitivity analysis

To examine the sensitivity of the variables with respect to input of the model, sensitivity analysis is performed and results are shown in Tables 3 and 4. In both tables, zero represents that changes of the parameters have no effect on optimal solution.

In the first model, sensitivity analysis has been conducted on the parameters $w$, $F_D$, $F_R$, $h$, and $\theta$.

<table>
<thead>
<tr>
<th>Percentage of parameter changes</th>
<th>Percentage of decision variable change</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>$T$</td>
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<td>+0.25</td>
<td>-0.1827</td>
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<td>-40.5159</td>
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</tr>
<tr>
<td>$h$</td>
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<td>-6.134</td>
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<td>$\theta$</td>
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<tr>
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<td>$F_D$</td>
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<td>-74.551</td>
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Table 4. Effects of parameter changes on optimal values of the first model.

<table>
<thead>
<tr>
<th>Percentage of parameter changes</th>
<th>Percentage of decision variable change</th>
<th>$T$</th>
<th>$p$</th>
<th>$Q$</th>
<th>$TC$</th>
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since they have more important effects on the profit, decision making, and managerial insights. According to Tables 3 and 4, when holding cost increases, order quantity and dispatch cycle length decrease, and total cost increases. However, when $w$ increases, selling price increases, and period length and total cost decrease. Moreover, when $F_R$ becomes larger, order quantity, total cost, and dispatch cycle length increase. When $F_D$ becomes larger, total cost and dispatch cycle length increase and selling price decreases and when decaying rate increases, total cost increases and both order quantity and dispatch cycle length decrease. It is noticeable that QBC causes a considerable decrease in cost. Table 3 represents the sensitivity analysis for the first model. For the second model, sensitivity analysis has been conducted on the parameters $v$, $F_D$, $F_R$, $h$, and $\theta$. Table 4 represents the sensitivity analysis for the second model. According to the results, when the waiting cost is high, the vendor dispatches smaller orders in order to decrease the waiting costs, because
the waiting time of customers increases as dispatch cycle length becomes larger.

7. Conclusion

In this paper, two mathematical models were presented for an integrated pricing-inventory problem of a single decaying product (since the number of deteriorating products increased every day) and optimum quantities of period length, order quantity, and selling price were obtained under two different scenarios. Then, by using several lemmas and theorems, the convexity and concavity of the functions of the two extended models were proved and different solution methods (algorithms) were developed to solve the models. In order to decrease the cost of transportation, time-based consolidation policy was applied. The developed models in this paper were comprehensive and considered as different forms of demand functions and deteriorating rates. Also, all costs of an inventory system were taken into consideration in the extended models. Finally, two examples were provided to show the applicability of the proposed policies. This model was developed under certain environment, and shortage was not permitted. Also, a single-stage problem was considered and developed and no contract was used between the stockholders. Therefore, for the future studies, permissible delay in payments contract and considering promotions, stochastic or fuzzy demand, multi-level supply chain, and permitted shortage could be of interest.

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References


Biographies

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