Scientia Iranica E (2018) 25(3), 1656–1670

Sharif University of Technology
Scientia Iranica
Transactions E: Industrial Engineering
http://scientiairanica.sharif.edu

Pricing and lot sizing of a decaying item under group dispatching with time-dependent demand and decay rates

A.A. Taleizadeh[a,*] and A. Rasuli-Baghban[b]

a. *School of Industrial Engineering, College of Engineering, University of Tehran, Tehran, Iran.*
b. *Department of Industrial Engineering, Islamic Azad University, South Tehran Branch, Tehran, Iran.*

Received 7 July 2015; received in revised form 13 May 2016; accepted 6 March 2017

KEYWORDS
Economic order quantity;
Economic lot sizing;
Pricing;
Inventory;
Deterioration;
Replenishment;
Group dispatching.

Abstract. Determining appropriate inventory and pricing policies is an important issue in scientific and industrial research. Here, an inventory control model of a decaying item with zero lead time is studied. Two mathematical models under different assumptions are developed. In the first model, deterioration rate is time-dependent and demand rate is price-sensitive while in the second model, deterioration rate is constant and demand rate is time- and price-dependent. The aim of this research is optimizing total cost by deriving decision variables such as dispatch cycle length, order quantity, and wholesale price. To optimize the total cost, a shipment group dispatching policy is used.

1. Introduction

Determining optimal inventory control policy and selling price for different products is one of the main issues in industrial and scientific research, especially when the product is perishable. Recently, due to globalization flow, increasing costs, time-sensitivity occurrence of an action, and running out of resources, researchers have focused on supply chains coordination [1]. Here, we investigate shipment consolidation, pricing, and inventory strategies of a seller selling a decaying item. Thus, some research related to pricing, inventory, and shipment consolidation decisions for deteriorating products is reviewed from the literature.

Since price is one of the main factors for customers to decide about buying a product, jointly determination of inventory and pricing decisions is much important and, first, it was studied by Whithin [2]. Chen et al. [3] modeled a joint inventory-pricing problem for a periodic-review system. Ray et al. [4] analyzed the joint operation-marketing decision-making in a periodic review inventory system for a firm with stochastic and price-sensitive demand. Huang et al. [5] modeled the coordination and selection of suppliers such that pricing and replenishment decisions in a three-level chain as a dynamic non-cooperative game model were optimized. Polatoglu [6] developed a joint inventory-pricing model using a single-period problem for which demand rate was assumed a linear function. Zhu [7] formulated the integrated pricing and inventory control problem in a random demand condition and finite planning horizon with return and expediting. You et al. [8] developed a seasonal inventory model with trial periods during which customers could return the products without any penalty. Su and Geunes [9] studied the price promotions effects on the total profit in a two-stage chain under deterministic demand.

*. *Corresponding author. Tel.: +98 21 8208-4486; Fax: +98 21 8801-3102*
E-mail addresses: Taleizadeh@ut.ac.ir (A.A. Taleizadeh) st_a_rasouli@azad.ac.ir (A. Rasuli-Baghban)

doi: 10.24200/sci.2017.4449

Mutlu and Cetinkaya [10] concentrated on a channel and compared the profits of both decentralized and centralized channels, where demand rate depended on selling price. Maddah and Bish [11] investigated a joint pricing-inventory problem in a newsboy system.

Recently, many researchers have focused on inventory control models of deteriorating products. Maity and Maiti [12] presented optimal production quantity and advertising expenditure of a multi-product inventory control model with inflation and time discounting under different constraints. Yu et al. [13] studied an inventory problem for a VMI system where both raw material and finished products were perishable. Hongjie et al. [14] extended an inventory control model for a decaying item in which vendor-managed inventory system was used. Mahata [15] formulated an inventory-production model for deteriorating products with delayed payment. Taleizadeh and Nematollahi [16] extended an inventory control model in which the influences of inflation and time value of money on best strategies of deteriorating products were examined. Lee and Chung [17] used system dynamics to propose a new order system for deteriorating products and prepared a systematical simulation.

In shipment consolidation policy, the orders of customer are combined to make a larger batch to deliver to the customers. This policy is used to decrease the dispatching cost. Indeed, since several shipments are combined during a cycle, consolidation makes increase in carrying costs. Thus, replenishment and consolidation decisions must be made simultaneously. Time-Based Consolidation (TBC) and Quantity-Based Consolidation (QBC) are two types of this policy. In the first one, accumulated orders of customers are dispatched within each period. But in the second type, orders are distributed when the cumulative orders become larger than economic values.

Cetinkaya and Bookbinder [18] determined optimal (QBC) policy and related optimal cycle length. Cetinkaya et al. [19] analyzed both quantity- and time-based consolidation policies comparatively. Wong et al. [20] extended a shipment consolidation policy and the effects of consolidation were studied in their research. Marklund [21] extended a model to examine the effects of consolidation and replenishment. Howard and Marklund [22] evaluated the effects of time-based consolidation and stock allocation in a chain. Taleizadeh et al. [23] extended a joint replenishment problem under prepayment strategy for imported raw material with several operating limitations. Taleizadeh et al. [24,25] extended a multi-product single-machine imperfect production system without and with shortage. Ulku and Bookbinder [26] optimized the vendor's profit when the selling price depended on arrival times of orders. Sajadieh and Jokar [27] focused on a two-echelon chain and extended a joint production-marketing-inventory problem to optimize total profit. Olsson [28] developed a based-stock model for perishable items. Also, demands were considered as Poisson random variable. On the contrary, lifetime and lead-time were assumed to be fixed. Herbon et al. [29] extended an inventory management problem with perishable products. Maximization of retailer's profit was the goal by considering customer's satisfaction. Taleizadeh [30,31] developed a lot-sizing model for evaporating and deteriorating products with partial backordering. Diabat et al. [32] considered integrated inventory and routing problems for perishable products. Lu et al. [33] considered an inventory system with limited replenishment capacity for perishable goods. Also, the demand rate depended on the stock quantity. Gallego and Hu [34] studied dynamic pricing of complementary and substitutable perishable assets in an oligopolistic market. An integrated production-distribution model was developed by Tayal et al. [35] in a two-echelon supply chain for perishable goods. Taleizadeh et al. [36] studied optimal quantity and multi-discount price for perishable items. They assumed a time-dependent demand function under two scenarios. A new multi-product economic order quantity problem was considered by Maleki Vishkaei et al. [37]. They assumed that defective items were screened out 100% throughout screen process and were sold after screening period. Also, other related research was performed by Taleizadeh et al. [38-48], and Teimouri and Kazemi [49].

Generally, up to now, many outstanding studies about pricing, inventory control, and shipment consolidation for decaying items have been separately developed, but none of them has considered optimal shipment consolidation, inventory, and pricing policies together. The above-mentioned hint is an important gap in this context, and a motivation for this research. Here, a joint inventory-pricing model of a decaying product under shipment consolidation policy is extended.

In the next section, the problem description is provided.

2. Problem description

Consider an inventory system selling deteriorating product for which deterioration rate is linearly time-proportional, $\theta(t) = bt$. The seller wants to apply a TBC policy using which orders are consolidated and distributed in every period of time T. The dispatch cycle length is a period within two deliveries and ordering cycle includes at least one distribution cycle and vendor goes to replenish whenever the on-hand inventory reaches zero. The lead time is zero and shortage is not allowed. Two scenarios under different assumptions are studied. In the first scenario, deterioration rate is time-dependent and demand rate

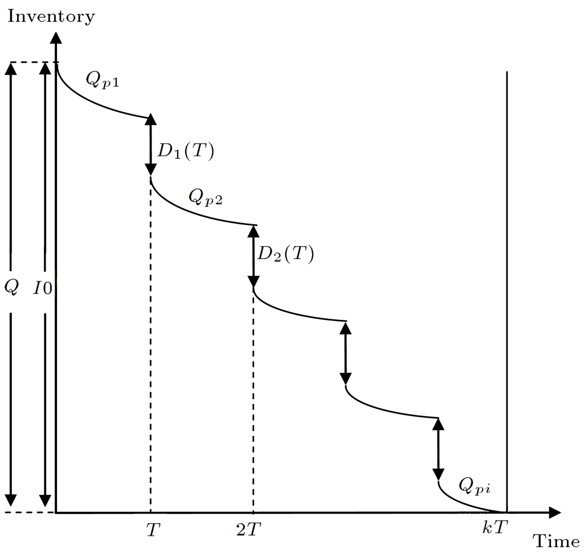

Figure 1. Inventory curve of vendor for deteriorating products.

is price-sensitive while in the second one, deterioration rate is constant and demand rate is time- and price-dependent. The main aim is to extend two models to optimize the dispatch cycle length, selling price, and replenishment quantity for the explained two scenarios such that the total cost is minimized or total profit is maximized. The proposed models in this paper are applicable for every deteriorating product, such as dairy products, vegetables, and whatever is being perished as time passes.

Figure 1 indicates vendor's inventory level in which Q is the replenishment quantity, Q_{pi} is the deteriorated quantity in each dispatch cycle ($i = 1...k$), and $D_i(T)$ shows the demand rate in each dispatch cycle. The following notations are used:

Variables:

T	Dispatch cycle length
P	Wholesale price
Q	Order quantity

Parameters:

$I_i(t)$	The level of inventory in the ith dispatch cycle at time t
θ	The decaying rate
$\theta(t)$	Time-dependent decaying rate
$D(p)$	The price-sensitive demand rate
$D_i(T)$	The demand during the ith dispatch cycle
F_D	The fixed dispatch cost
F_R	The fixed order cost
C_D	Dispatch cost per unit
C_R	Purchasing cost per unit
w	The waiting cost of product (unit/ time)
h	The carrying cost of product (unit/ time)
k	The number of dispatching periods during a replenishment cycle
v	Time-dependent deterioration rate
S_j	The jth demand arrival time in a dispatch cycle
$Z(t)$	Deteriorated quantity in the first model
Q_p	Deteriorated quantity in the second model

3. The first scenario

In this scenario, we assume that decaying rate depends on time and is a continuous function of time, $\theta(t) = bt$, $b \in [0, 1]$, and demand rate is price-sensitive and a linear function of wholesale price, $D(p) = (a - Bp)T$. Decision variables that should be determined are dispatch cycle length, wholesale price, and order quantity.

3.1. Mathematical model

Here, a mathematical model of a joint pricing-inventory problem for a time-dependent decaying product using a time-based group dispatching policy is developed. From Figure 1, the changes of inventory level are presented as follows:

$$\begin{cases} \frac{dI_i(t)}{dt} = -\theta(t)I_i(t) - (i-1)D(p); \quad t \in [(i-1)T, iT]. \\ I(kT) = 0 \end{cases} \tag{1}$$

By solving this equation, we have:

$$I_i(t) = -(i-1)D(p)\left(t + \frac{bt^3}{6}\right)e^{-\frac{bt^2}{2}} + I(0)e^{-\frac{bt^2}{2}}. \tag{2}$$

To obtain the optimum values of variables, first, the cost function should be modeled. Since each replenishment period includes k (random variable) distributing cycles with length T, the expected value of cycle length is $E(k)T$.

Ordering cost:

The order quantity is demand plus deteriorated product as shown in Eq. (3):

$$Q = Z(t) + \sum_{i=1}^{k-1} D_i(T), \tag{3}$$

where the quantity deteriorated during each cycle is:

$$Z(t) = I(t)_{\text{without deterioration}} - I(t)_{\text{with deterioration}}. \tag{4}$$

Inventory level without deterioration is obtained using Eq. (5):

$$I(t)_{\text{without deterioration}} = \lim_{b \to 0} I(t)_{\text{with deterioration}}$$

$$= 1 - t(i-1)D(p) + I(0). \quad (5)$$

Thus, using Eqs. (2) and (5) and using $I(0) = D(P)(t + \frac{bt^3}{6}) + I(t)e^{\frac{bt^2}{2}}$, Eq. (4) changes to:

$$Z(t) = I(t)(e^{\frac{bt^2}{2}} - 1) + (i-1)D(p)\frac{bt^3}{6}. \quad (6)$$

Since $I(kT) = 0$, deteriorated quantity at the end of each replenishment cycle is:

$$Z(KT) = (i-1)D(p)\frac{bK^3T^3}{6}. \quad (7)$$

Therefore, replenishment quantity is:

$$Q = Z(KT) + KD(p)T$$

$$= (i-1)D(p)\frac{bK^3T^3}{6} + KD(p)T. \quad (8)$$

Finally, the expected replenishment quantity according to Eq. (8) is equal to:

$$E(Q) = (i-1)D(p)\frac{bE(K^3)T^3}{6} + E(k)D(p)T. \quad (9)$$

Thus, the expected related cost is:

$$E(R_c) = F_R + C_R E(Q)$$

$$= F_R + C_R \frac{(a-Bp)bE(K^3)T^3}{6}$$

$$+ C_R E(k)(a-Bp)T. \quad (10)$$

Dispatch cost:
Based on existing k dispatch cycles and dispatch quantity, which is $E(D_i(T))$, the dispatch cost can be determined as follows:

$$\text{dispatch cost} = F_D E(k) + C_D E(k) E(D_i(T))$$

$$= F_D E(k) + C_D E(k)(a-bp)T. \quad (11)$$

Holding cost:
The level of inventory at time t, after substitution of Eq. (8) in Eq. (2), is:

$$I(t) = D(p)\left[(kT - t) + \frac{b}{6}(k^3T^3 - t^3)\right]e^{-\frac{bt^2}{2}}.$$

Utilizing the Taylor series expansion, $e^{-\frac{bt^2}{2}} = 1 - \frac{bt^2}{2} + \frac{b^2t^4}{4}$, the cyclic inventory carrying cost is given by:

$$hE\left(\sum_{i=1}^{k}\left(\int_{(i-1)T}^{iT} I(t)dt\right)\right)$$

$$= hE\Bigg(\sum_{i=1}^{k}\Bigg(\int_{(i-1)T}^{iT} D(p)\Bigg[(kT - t)$$

$$+ \frac{b}{6}(k^3T^3 - t^3)\Bigg]e^{-\frac{bt^2}{2}}\Bigg)\Bigg)$$

$$= hE\left(\frac{k^2D(p)T^2}{2} + \frac{D(p)bk^4T^4}{12}\right)$$

$$= \frac{h\lambda T^2 E(k^2)}{2} + \frac{h\lambda bT^4 E(k^4)}{12}. \quad (12)$$

The cost of waiting:
Using the definition of S_j, the waiting time of customer is $T - S_j$ (see Figure 2). Therefore, the cost of waiting for customer is:

$$wE\left[(T - S_1) + (T - S_2) + ... + (T - S_{N(T)})\right]$$

$$= wE\left[D_i(T)T - \sum_{n=1}^{D_i(T)} S_n\right] = wE(k)\frac{\lambda T^2}{2}, \quad (13)$$

Using Eqs. (10) to (13), the total cost function is:

$$\text{Total cost function } = F_R + C_R\lambda T E(k)$$

$$+ \frac{C_R\lambda bT^3E(k^3)}{6} + F_D E(k) + C_D E(k)\lambda T$$

$$+ \frac{h\lambda T^2E(k^2)}{2} + \frac{h\lambda bT^4E(k^4)}{12} + wE(k)\frac{\lambda T^2}{2}. \quad (14)$$

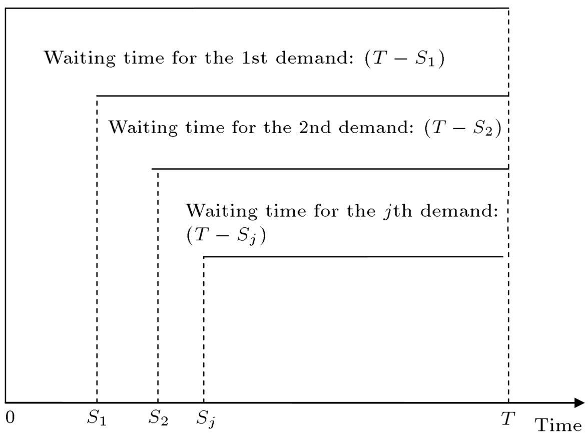

Figure 2. Customer waiting time.

Dividing Eq. (16) over $E(k)T$ yields:

$$TC(p,T,Q) = \frac{F_R}{E(k)T} + C_R\lambda + \frac{C_R\lambda b T^2 E(k^3)}{6E(k)} + \frac{F_D}{T} + C_D\lambda + \frac{h\lambda T E(k^2)}{2E(k)} + \frac{h\lambda b T^3 E(k^4)}{12E(k)} + \frac{w\lambda T}{2}. \quad (15)$$

3.2. Solution method

The following Lemma is presented to derive a solution method.

Lemma 1. The following equations can be used to derive the optimal solutions:

$$E(k) = \frac{\bar{Q}}{\lambda T}, \quad (16)$$

$$E(k^2) = \frac{\bar{Q}(\bar{Q}+1)}{\lambda^2 T^2}, \quad (17)$$

$$E(k^3) = \frac{\bar{Q}(\bar{Q}+1)(\bar{Q}+2)}{\lambda^3 T^3}, \quad (18)$$

$$E(k^4) = \frac{\bar{Q}(\bar{Q}+1)(\bar{Q}+2)(\bar{Q}+3)}{\lambda^4 T^4}. \quad (19)$$

Proof. Let $f(0)$ denote the distribution function of $D_i(T)$, and $f^{(k)}(0)$ denote the k-fold convolution of $f(0)$. From $k = \inf\left\{k : \sum_{i=1}^{k} D_i(T) \geq Q\right\}$, we have $P[k \geq k+1] = f^{(k)}(Q)$ and, thus, $P[k \leq k+1] = 1 - f^{(k)}(Q)$. Since $f(0)$ is a Poisson distribution with parameter λT, k-fold convolution of $f(0)$ is a Poisson distribution with parameter $k\lambda T$, where:

$$f^{(k)}(Q) = \sum_{i=0}^{Q} \frac{(k\lambda T)^i e^{-k\lambda T}}{i!}.$$

Then, for $k = 1, 2, \ldots$, we have:

$$p[K \leq k] = 1 - \sum_{i=0}^{Q} \frac{(k\lambda T)^i e^{-k\lambda T}}{i!},$$

of which the right side is a Q-stage P.D.F. with parameter λT and expected value of $\frac{(Q+1)}{\lambda T}$.

For simplicity, by using Lemma 1 and substituting $\bar{Q} = Q+1$ and $\lambda = a-bp$, the expected long-run average cost changes to:

$$TC(P,T,Q) = \frac{F_R(a-Bp)}{\bar{Q}} + C_R(a-Bp) + \frac{bC_R(\bar{Q}+1)(\bar{Q}+2)}{6(a-Bp)} + \frac{F_D}{T} + C_D(a-Bp) + \frac{h(\bar{Q}+1)}{2} + \frac{hb(\bar{Q}+1)(\bar{Q}+2)(\bar{Q}+3)}{12(a-Bp)^2} + \frac{w(a-bp)T}{2}). \quad (20)$$

Lemma 2. For each couple of T and p, the optimum quantity of $\bar{Q}$ should satisfy Inequality (21):

$$\bar{Q}^*(\bar{Q}^*-1) + \frac{b\bar{Q}^*(\bar{Q}^*+1)(\bar{Q}^*-1)}{(a-Bp)} \left(\frac{2C_R}{3h} + \frac{(\bar{Q}^*+2)}{2(a-Bp)}\right) \leq \frac{2F_R(a-Bp)}{h} \leq \bar{Q}^*(\bar{Q}^*+1) + \frac{b\bar{Q}^*(\bar{Q}^*+1)(\bar{Q}^*+2)}{(a-Bp)} \left(\frac{2C_R}{3h} + \frac{(\bar{Q}^*+3)}{2(a-Bp)}\right). \quad (21)$$

Proof. For each couple of p and T, the optimum quantity of $\bar{Q}$ should satisfy $TC(\bar{Q}^*-1) \geq TC(\bar{Q}^*)$ and $TC(\bar{Q}^*+1) \geq TC(\bar{Q}^*)$. Using Eq. (20), the optimality condition for $\bar{Q}$ is:

$$\bar{Q}^*(\bar{Q}^*-1) + \frac{b\bar{Q}^*(\bar{Q}^*+1)(\bar{Q}^*-1)}{(a-Bp)} \left(\frac{2C_R}{3h} + \frac{(\bar{Q}^*+2)}{2(a-Bp)}\right) \leq \frac{2F_R(a-Bp)}{h} \leq \bar{Q}^*(\bar{Q}^*+1) + \frac{b\bar{Q}^*(\bar{Q}^*+1)(\bar{Q}^*+2)}{(a-Bp)} \left(\frac{2C_R}{3h} + \frac{(\bar{Q}^*+3)}{2(a-Bp)}\right).$$

Lemma 3. The following condition should be satisfied by the upper bound of $\bar{Q}$:

$$\bar{Q}^*(\bar{Q}^*-1) + \frac{b\bar{Q}^*(\bar{Q}^*+1)(\bar{Q}^*-1)}{a} \left(\frac{2C_R}{3h} + \frac{(\bar{Q}^*+2)}{2a}\right) \leq \frac{2F_R a}{h} \leq \bar{Q}^*(\bar{Q}^*+1) + \frac{b\bar{Q}^*(\bar{Q}^*+1)(\bar{Q}^*+2)}{a} \left(\frac{2C_R}{3h} + \frac{(\bar{Q}^*+3)}{2a}\right). \quad (22)$$

Proof. $\frac{2F_R(a-bp)}{h}$ in Eq. (21) is decreasing with respect to p. Therefore, the maximum $\bar{Q}$ is derived at $p = 0$.

Theorem 1. The cost function becomes convex is $w \leq \frac{2F_D}{BpT^2} + \frac{Bbp(\bar{Q}+1)(\bar{Q}+2)}{T\lambda^3}\left(\frac{C_R}{3} + \frac{h(\bar{Q}+3)}{2\lambda}\right)$.

Proof. The total cost function is convex if $X.H.X^T = \begin{bmatrix} P & T \end{bmatrix} \times H \times \begin{bmatrix} P & T \end{bmatrix}^T \geq 0$, where:

$$H = \begin{bmatrix} \frac{\partial^2 TP}{\partial p^2} & \frac{\partial^2 TP}{\partial p \partial T} \\ \frac{\partial^2 TP}{\partial T \partial p} & \frac{\partial^2 TP}{\partial T^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{B^2 b(\bar{Q}+1)(\bar{Q}+2)}{\lambda^3}\left(\frac{C_R}{3} + \frac{h(\bar{Q}+3)}{2\lambda}\right) & -\frac{Bw}{2} \\ -\frac{Bw}{2} & \frac{2F_D}{T^3} \end{bmatrix}. \tag{23}$$

Therefore, we have:

$$X \times H \times X^T = \frac{B^2 p^2 b(\bar{Q}+1)(\bar{Q}+2)}{\lambda^3} \left(\frac{C_R}{3} + \frac{h(\bar{Q}+3)}{2\lambda}\right) + \frac{2F_D}{T} - BpTw, \tag{24}$$

and we should show that:

$$X \times H \times X^T = \frac{B^2 p^2 b(\bar{Q}+1)(\bar{Q}+2)}{\lambda^3} \left(\frac{C_R}{3} + \frac{h(\bar{Q}+3)}{2\lambda}\right) + \frac{2F_D}{T} - BpTw \geq 0. \tag{25}$$

Thus, the cost function is convex if and only if:

$$w \leq \frac{2F_D}{BpT^2} + \frac{Bbp(\bar{Q}+1)(\bar{Q}+2)}{T\lambda^3}\left(\frac{C_R}{3} + \frac{h(\bar{Q}+3)}{2\lambda}\right). \tag{26}$$

Using the first derivative of $TC(p,T,Q)$ with respect to p yields:

$$\frac{\partial TC(p,T,Q)}{\partial P} = -B\Bigg(\frac{F_R}{\bar{Q}} + \frac{wT}{2} + C_R + C_D - \frac{bC_R(\bar{Q}+1)(\bar{Q}+2)}{6\lambda^2} - \frac{hb(\bar{Q}+1)(\bar{Q}+2)(\bar{Q}+3)}{6\lambda^3}\Bigg). \tag{27}$$

Moreover, the first derivative of $TC(p,T,Q)$ with respect to T yields:

$$\frac{\partial TC(p,T,Q)}{\partial T} = \frac{w\lambda}{2} - \frac{F_D}{T^2}. \tag{28}$$

Setting Eq. (28) equal to zero gives:

$$p^* = \frac{a}{B} - \frac{2F_D}{BwT^2}. \tag{29}$$

After some algebraic calculations, the following equation is derived:

$$AT^4 + ET^3 + CT + D = 0, \tag{30}$$

where:

$$A = \frac{-bC_R(Q+1)(Q+2)w^2}{24F_D^2}, \tag{31}$$

$$E = \frac{-hb_R(Q+1)(Q+2)(Q+3)w^3}{48F_D^3}, \tag{32}$$

$$C = \frac{w}{2}, \tag{33}$$

$$D = \frac{F_R}{Q} + C_D + C_R. \tag{34}$$

Now, we can use the following algorithm to solve the problem.

Step 1. Compute $\bar{Q}_{\max}$ suing Eq. (22);

Step 2. For ($\bar{Q} = 1 ... \bar{Q}_{\max}$), determine the coefficients of the polynomial shown in Eq. (30);

Step 3. Determine all acceptable roots for T using Eq. (30) and MATLAB software;

Step 4. Calculate p for all acceptable roots of T from the previous step;

Step 5. Now, total cost should be calculated using Eq. (20) and convexity should be examined using Theorem 1;

Step 6. In the comparison of the derived total costs, the lowest cost shows the related optimal solutions.

4. The second scenario

In this scenario, we assume that decaying rate is constant and demand rate is time- and price-sensitive and a function of time and selling price $D(p,t) = (a - bp)Te^{vt} = \lambda Te^{vt}$. Decision variables, namely, dispatch cycle length, order quantity, and selling price should be determined to optimize the total profit.

4.1. Mathematical modeling

In this scenario, Eqs. (1) and (2) will change to Eqs. (35) and (36) as follows:

$$\frac{dI_i(t)}{dt} = -\theta I_i(t) - (i-1)D(p,t) \quad t \in [(i-1)T, iT], \tag{35}$$

$$I_i(t) = \frac{(i-1)(a-bp)e^{vt}}{(v+\theta)}\left[e^{(v+\theta)(kT-t)} - 1\right]. \tag{36}$$

Therefore, the order quantity is:

$$Q = I(0) = \frac{(a-bp)}{(v+\theta)}\left[e^{(v+\theta)kT} - 1\right]. \tag{37}$$

According to the description provided for the previous case, $E(D_i(T)) = \lambda e^{vt}T = (a-bp)e^{vt}T$ and the expected income is:

$$PE\left(\sum_{i=1}^{k-1}\left(\int_{(i-1)T}^{iT} D_i(T)dt\right)\right)$$

$$= PE\left(\sum_{i=1}^{k-1}\left(\int_{(i-1)T}^{iT} (a-bp)e^{vt}Tdt\right)\right)$$

$$= P(a-bp)T^2E(k) + \frac{P(a-bp)T^3vE(k^2)}{2}. \tag{38}$$

To derive the cost function, we act as follows.

Ordering cost:
The ordering cost of this case, using the logic of the previous case, is:

$$Q = Q_p + \sum_{i=1}^{k-1} D_i(T). \tag{39}$$

Moreover, quantity deteriorated during each cycle is:

$$Q_p = I(0) - I(KT) - \sum_{i=1}^{k-1} D_i(t), \tag{40}$$

where:

$$Q = I(0) = \frac{(a-bp)}{(v+\theta)}\left[e^{(v+\theta)kT} - 1\right],$$

$$I(KT) = 0, \quad \sum_{i=1}^{k}\left(\int_{(i-1)T}^{iT} (a-bp)e^{vt}Tdt\right),$$

and the expected deteriorated quantity is:

$$E(Q_p) = E(I(0)) - E(I(KT))$$

$$- E\left(\sum_{i=1}^{k}\left(\int_{(i-1)T}^{iT} (a-bp)e^{vt}Tdt\right)\right)$$

$$= \lambda E(k)T + \frac{\lambda(v+\theta)T^2E(k^2)}{2}$$

$$+ \lambda E(k)T^2 + \frac{\lambda vT^3E(k^2)}{2}. \tag{41}$$

Finally, using the Taylor series expansion, $e^{\theta Tk} = 1 + \theta Tk + \frac{(\theta Tk)^2}{2}$, Eq. (41) changes to:

$$E(Q) = E(I(0)) = (a-bp)E(k)T$$

$$+ \frac{(a-bp)(v+\theta)T^2E(k^2)}{2}. \tag{42}$$

And the expected ordering cost is:

$$E(R_c) = F_R + C_R E(Q) = F_R + C_R(a-bp)TE(k)$$

$$+ \frac{C_R(a-bp)T^2(v+\theta)E(k^2)}{2}. \tag{43}$$

Dispatch cost:
For the distribution cost, similar to the previous case, we have:

$$E(D_C) = F_D E(k) + C_D E(k)E(D_i(T)) = F_D E(k)$$

$$+ C_D E\left(\sum_{i=1}^{k}\left(\int_{(i-1)T}^{iT} (a-bp)e^{vt}Tdt\right)\right)$$

$$= F_D E(k) + C_D(a-bp)T^2E(k)$$

$$+ \frac{C_D(a-bp)T^3vE(k^2)}{2}. \tag{44}$$

Carrying cost:
From Eq. (36), the level of inventory at time t is $I(t) = \frac{(i-1)(a-bp)e^{vt}}{(v+\theta)}\left[e^{(v+\theta)(kT-t)} - 1\right]$. Utilizing the Taylor series expansion, $e^{\theta Tk} = 1 + \theta Tk + \frac{(\theta Tk)^2}{2}$, the cyclic inventory carrying cost is:

$$hE\left(\sum_{i=1}^{k}\left(\int_{(i-1)T}^{iT} I(t)dt\right)\right)$$

$$= hE\left(\sum_{i=1}^{k}\left[\int_{(i-1)T}^{iT}\left(\frac{D(p)e^{vt}}{(v+\theta)}(e^{(v+\theta)(kT-t)} - 1)\right)dt\right]\right)$$

$$= \frac{-h(a-bp)v^2T^2E(k^2)}{2\theta(v+\theta)}$$

$$+ \frac{h(a-bp)(v+\theta)T^2E(k^2)}{2\theta}$$

$$- \frac{h(a-bp)vT^2E(k^2)}{2(v+\theta)}. \tag{45}$$

Using Eqs. (38), (43), (44), and (45), the total profit function is:

$$\text{Total Profit Function} = E(k)\lambda T^2 p + \frac{p\lambda vT^3E(k^2)}{2}$$

$$- \left\{\begin{array}{l} F_R + C_R\lambda TE(k) + \frac{C_R\lambda T^2(v+\theta)E(k^2)}{2} \\ + F_D E(k) + C_D E(k)\lambda T^2 + \frac{C_D E(k^2)\lambda vT^3}{2} \\ - \frac{h\lambda v^2T^2E(k^2)}{2\theta(v+\theta)} + \frac{h\lambda(v+\theta)T^2E(k^2)}{2\theta} \\ - \frac{h\lambda vT^2E(k^2)}{2(v+\theta)} \end{array}\right\}. \tag{46}$$

Dividing the above total profit function by $E(k)T$, we have:

$$\text{Total Profit Function} = \lambda Tp + \frac{p\lambda vT^2E(k^2)}{2E(k)} - \left\{ \begin{array}{l} \frac{F_R}{E(k)T} + C_R\lambda + \frac{C_R\lambda T(v+\theta)E(k^2)}{2E(k)} + \frac{F_D}{T} \\ + \frac{C_DE(k^2)\lambda vT^2}{2E(k)} - \frac{h\lambda v^2TE(k^2)}{2\theta(v+\theta)E(k)} + C_D\lambda T \\ + \frac{h\lambda(v+\theta)TE(k^2)}{2\theta E(k)} - \frac{h\lambda vTE(k^2)}{2(v+\theta)E(k)} \end{array} \right\}. \quad (47)$$

By substituting $E(k) = \frac{Q+1}{(a-bp)e^{vt}T}$ and $\lambda = a - bp$, and assuming $\bar{Q} = Q + 1$, the expected long-run average profit changes to:

$$\text{Total Profit Function} = \lambda Tp + \frac{pvT(\bar{Q}+1)}{2e^{vt}} - \left\{ \begin{array}{l} \frac{F_R\lambda e^{vt}}{\bar{Q}} + C_R\lambda + \frac{C_R(v+\theta)(\bar{Q}+1)}{2e^{vt}} + \frac{F_D}{T} \\ + C_D\lambda T + \frac{C_DvT(\bar{Q}+1)}{2e^{vt}} - \frac{hv^2(\bar{Q}+1)}{2\theta(v+\theta)e^{vt}} \\ + \frac{h(v+\theta)(\bar{Q}+1)}{2\theta e^{vt}} - \frac{hv(\bar{Q}+1)}{2(v+\theta)e^{vt}} \end{array} \right\}. \quad (48)$$

4.2. Solution method

Lemma 4. For each couple of p and T, the optimum quantity of $\bar{Q}$ should satisfy Inequality (49):

$$\left\{ \begin{array}{l} \frac{(v+\theta)\bar{Q}^*(\bar{Q}^*-1)}{\theta e^{vt}} + \frac{(v+\theta)C_R\bar{Q}^*(\bar{Q}^*+1)}{e^{vt}h} - \frac{v^2\bar{Q}^*(\bar{Q}^*-1)}{\theta(v+\theta)e^{vt}} \\ - \frac{pvT\bar{Q}^*(\bar{Q}^*-1)}{e^{vt}h} - \frac{v\bar{Q}^*(\bar{Q}^*-1)}{(v+\theta)e^{vt}} + \frac{C_DvT\bar{Q}^*(\bar{Q}^*-1)}{e^{vt}h} \end{array} \right\} \leq \frac{2F_R(a-bp)e^{vt}}{h} \leq \left\{ \begin{array}{l} \frac{(v+\theta)\bar{Q}^*(\bar{Q}^*+1)}{\theta e^{vt}} + \frac{(v+\theta)C_R\bar{Q}^*(\bar{Q}^*+1)}{e^{vt}h} - \frac{v^2\bar{Q}^*(\bar{Q}^*+1)}{\theta(v+\theta)e^{vt}} \\ - \frac{pvT\bar{Q}^*(\bar{Q}^*+1)}{e^{vt}h} - \frac{v\bar{Q}^*(\bar{Q}^*+1)}{(v+\theta)e^{vt}} + \frac{C_DvT\bar{Q}^*(\bar{Q}^*+1)}{e^{vt}h} \end{array} \right\}. \quad (49)$$

Proof. For each couple of p and T, the optimum quantity of $\bar{Q}$ should satisfy $TP(\bar{Q}^*-1) \leq TP(\bar{Q}^*)$ and $TP(\bar{Q}^*+1) \leq TP(\bar{Q}^*)$. Using Eq. (48), the optimality condition for $\bar{Q}$ is obtained as Eq. (49).

Lemma 5. The upper bound of $\bar{Q}$ satisfies the following condition:

$$\bar{Q}_u^*(\bar{Q}_u^*-1) + \frac{\theta C_R\bar{Q}_u^*(\bar{Q}_u^*-1)}{h} \leq \frac{2F_Ra}{h} \leq \bar{Q}_u^*(\bar{Q}_u^*+1) + \frac{\theta C_R\bar{Q}_u^*(\bar{Q}_u^*+1)}{h}. \quad (50)$$

Proof. The maximum amount of $\frac{2F_R(a-bp)e^{vt}}{h}$ or the maximum amount of replenishment is obtained when the demand is maximum. Thus, the maximum quantity of $\bar{Q}$ is derived at $p = v = 0$.

Theorem 2. The profit function is concave if and only if $F_D \geq \left[a + bC_D - 3bp + \frac{v(\bar{Q}+1)}{2e^{vt}}\right]pT^2$.

Proof. The total profit function is concave if $X.H.X^T = \begin{bmatrix} P & T \end{bmatrix} \times H \times \begin{bmatrix} P & T \end{bmatrix}^T \leq 0$, where:

$$H = \begin{bmatrix} \frac{\partial^2TP}{\partial p^2} & \frac{\partial^2TP}{\partial p\partial T} \\ \frac{\partial^2TP}{\partial T\partial p} & \frac{\partial^2TP}{\partial T^2} \end{bmatrix} = \begin{bmatrix} -2bT & a-2bp+bC_D+\frac{v(\bar{Q}+1)}{2e^{vt}} \\ a-2bp+bC_D+\frac{v(\bar{Q}+1)}{2e^{vt}} & -\frac{2F_D}{T^3}, \end{bmatrix} \quad (51)$$

therefore, we have:

$$X \times H \times X^T = -3bp + a + bC_D + \frac{v(\bar{Q}+1)}{2e^{vt}} - \frac{F_D}{pT^2}. \quad (52)$$

And we should show that:

$$X \times H \times X^T = -3bp + a + bC_D + \frac{v(\bar{Q}+1)}{2e^{vt}} - \frac{F_D}{pT^2} \leq 0. \quad (53)$$

In order to be sure that the profit function is concave, the following inequality should be held:

$$F_D \geq \left[a + bC_D - 3bp + \frac{v(\bar{Q}+1)}{2e^{vt}}\right]pT^2. \quad (54)$$

Now, the first derivative of total profit function with respect to p is:

$$\frac{\partial TP(p,T,Q)}{\partial P} = aT - 2bpT + \frac{bF_Re^{vt}}{\bar{Q}} + bC_R + bTC_D + \frac{vT(\bar{Q}+1)}{2e^{vt}}. \quad (55)$$

Moreover, the first derivative with respect to T becomes:

$$\frac{\partial TP(p,T,Q)}{\partial T} = \frac{F_D}{T^2} - C_D\lambda + p\lambda - \frac{C_Dv(\bar{Q}+1)}{2e^{vt}} + \frac{pv(\bar{Q}+1)}{2e^{vt}}. \quad (56)$$

By setting Eq. (55) equal to zero, we have:

$$p^* = \frac{a}{2b} + \frac{F_Re^{vt}}{2T\bar{Q}} + \frac{C_R}{2T} + \frac{C_D}{2} + \frac{v(\bar{Q}+1)}{2be^{vt}}. \quad (57)$$

Substitution of Eq. (57) in Eq. (56) yields the following equation:

$$AT^2 + BT + C = 0, \tag{58}$$

where:

$$A = -\frac{b{F_R}^2 e^{2vt}}{4\bar{Q}^2} - \frac{bF_R C_R e^{vt}}{2\bar{Q}} - \frac{b{C_R}^2}{4} + F_D, \tag{59}$$

$$B = -\frac{v(\bar{Q}+1)C_R}{4e^{vt}} - \frac{v(\bar{Q}+1)F_R}{4\bar{Q}}, \tag{60}$$

$$C = \frac{av(\bar{Q}+1)}{be^{vt}} + \frac{vC_D(\bar{Q}+1)}{4e^{vt}} + \frac{a^2}{4b} + \frac{bC_D^2}{4} - \frac{aC_D}{2}. \tag{61}$$

Eq. (58) is a quadratic polynomial of which the discriminant is:

$$\Delta = B^2 - 4AC. \tag{62}$$

Based on the sign of Δ, the following cases can occur:

1. When $\Delta > 0$, two real roots exist;
2. When $\Delta = 0$, a single root exists;
3. When $\Delta < 0$, there is no real root.

Now, we can use the following solution procedure to solve the problem:

Step 1. Determine $\bar{Q}_{\max}$ using Eq. (50);

Step 2. For ($\bar{Q} = 1...\bar{Q}_{\max}$), determine the coefficients of polynomial (58);

Step 3. Determine all acceptable roots of period length using Eq. (58) and MATLAB;

Step 4. Determine p for all acceptable roots of period length from the previous step;

Step 5. For all combinations of order quantity, selling price, and period length, calculate the related total profit and check the concavity using Theorem 2;

Step 6. By comparison of the obtained total profits, the related optimal decision variables can be applied.

5. Practical and computational results

Consider a milk producer company for which deteriorating rate is linearly time-proportional, $\theta(t) = bt$, $b \in [0, 1]$, and demand rate is price-sensitive and is a linear function of wholesale price, $D(p) = (a - Bp)T$. Moreover, deterioration rate may be constant and demand rate can be time- and price-sensitive and a function of time and selling price $D(p, t) = (a - bp)Te^{vt} = \lambda Te^{vt}$. These two conditions can be analyzed using the first and the second models developed in this paper. Decision variables, namely, cycle length, selling price, and order quantity, should be determined such that total cost is minimized using the proposed solution method.

5.1. Example 1

For the first developed model, consider $F_D = 5$, $F_R = 40$, $a = 20$, $C_D = 3$, $h = 1$, $w = 50$, $B = 0.02$, $\theta = 0.04$, $b = 0.05$, and $C_R = 40$.

Step 1. Using Eq. (22), $\bar{Q} = 25$;

Step 2. From $\bar{Q} = 1$ to 25, the values of Eqs. (31) to (34) are determined;

Step 3. All acceptable real roots of cycle length are determined and reported in Table 1;

Step 4. For all acceptable values of T, the wholesale prices are determined (see Table 1);

Step 5. For all groups of decision variables, the concavity of the objective function is checked and the related values are shown in Table 1;

Step 6. The lowest cost, i.e. $TC^* = 149.2175$, corresponds to $Q^* = 10$, $T^* = 0.3179$, and $P^* = 90.1049$.

5.2. Example 2

Now, consider $F_D = 5$, $C_D = 3$, $h = 1$, $b = 0.04$, $F_R = C_R = 40$, $w = 50$, $v = -0.98$, $\theta = 0.04$, and $a = 20$.

Step 1. From Eq. (50), $\bar{Q} = 40$;

Step 2. From $\bar{Q} = 1$ to 40, the values of Eqs. (59) to (61) are calculated;

Step 3. All acceptable roots of cycle length are calculated and reported in Table 2;

Table 1. Results for the first example.

$\bar{Q}$	T	p	TC	Convexity
1	1.1846	99.2874	36.4358	No
2	0.8894	98.7358	44.4573	No
3	0.7235	98.0896	54.0442	No
4	0.6124	97.3336	64.5559	No
5	0.5312	96.4561	76.0043	No
6	0.4690	95.4537	88.4376	No
7	0.4196	94.3203	101.9309	No
8	0.3794	93.0529	116.5323	No
9	0.3460	91.6469	132.2982	No
10	**0.3179**	**90.1049**	**149.2175**	**Yes**
11	0.2939	88.4229	167.3233	Yes
12	0.2732	86.6020	186.6376	Yes
13	0.2552	84.6454	207.1067	Yes
14	0.2393	82.5372	228.8869	Yes
15	0.2253	80.2995	251.7767	Yes
16	0.2127	77.8963	276.1084	Yes
17	0.2015	75.3708	301.4834	Yes
18	0.1914	72.7029	328.0874	Yes
19	0.1822	69.8767	356.0674	Yes
20	0.1739	66.9325	385.0502	Yes
21	0.1662	63.7976	415.7086	Yes
22	0.1592	60.5439	447.3766	Yes
23	0.1528	57.1695	480.0718	Yes
24	0.1469	53.6600	513.9276	Yes
25	0.1414	49.9849	549.2252	Yes

Table 2. Results for the second example.

$\bar{Q}$	T	p	TP	Concavity	$\bar{Q}$	T	p	TP	Concavity
1	0.1211	477.692	–79.76	No	19	0.0467	293.458	43.9788	Yes
						0.1658	–23.9551	Infeasible	No
2	0.0937	470.047	–78.92	No	20	0.0448	291.399	62.4998	Yes
						0.1525	–33.2788	Infeasible	No
3	0.0839	458.444	–78.91	No	21	0.0429	290.946	82.3092	Yes
						0.1422	–43.7487	Infeasible	No
4	0.0785	445.172	–78.18	No	22	0.0411	291.545	103.293	Yes
						0.1342	–55.2696	Infeasible	No
5	0.0749	431.197	–76.74	No	23	0.0393	293.094	125.331	Yes
						0.1278	–67.7387	Infeasible	No
6	0.0721	417.005	–74.62	No	24	0.0377	295.488	148.302	Yes
						0.1225	–81.0526	Infeasible	No
7	0.0698	402.887	–71.81	No	25	0.0361	298.63	172.093	Yes
						0.1182	–95.1123	Infeasible	No
8	0.0678	389.049	–68.27	No	26	0.0346	302.426	196.598	Yes
						0.1146	–109.826	Infeasible	No
9	0.0659	375.652	–63.93	No	**27**	**0.0332**	**306.794**	**221.724**	**Yes**
						0.1116	-125.111	Infeasible	No
10	0.0641	362.842	–58.70	No	28	0.0319	311.661	207.200	Yes
						0.1090	–140.895	Infeasible	No
11	0.0622	350.753	–52.49	No	29	0.0306	316.963	198.508	Yes
						0.1067	–157.112	Infeasible	No
12	0.0604	339.516	–45.20	No	30	0.0295	322.644	185.415	Yes
	1.4473	6.077	–404.05	No		0.1048	–173.708	Infeasible	No
13	0.0585	329.254	–36.70	No	31	0.0284	328.655	173.87	Yes
	0.6189	5.392	–405.82	No		0.1031	–190.633	Infeasible	No
14	0.0566	320.081	–26.892	No	32	0.0274	334.956	169.102	Yes
	0.4009	3.622	–391.39	No	33	0.1016	–207.848	Infeasible	No
15	0.0547	312.095	–15.68	No	34	0.0264	341.51	157.005	Yes
	0.3012	0.671	–375.24	No	35	0.1003	–225.316	Infeasible	No
16	0.0527	305.368	–3.00	No	36	0.0255	348.288	141.67	Yes
	0.2446	–3.536	Infeasible	No	37	0.0991	–243.007	Infeasible	No
17	0.0507	299.943	11.1683	Yes	38	0.0247	355.263	136.140	Yes
	0.2084	–9.042	Infeasible	Yes	39	0.0980	–260.895	Infeasible	No
18	0.0487	295.829	26.844	Yes	40	0.0239	362.412	127.115	Yes
	0.1837	–15.8577	Infeasible	Yes					

Step 4. p is determined for all acceptable roots of T (see Table 2);

Step 5. For all groups of decision variables, the concavity of the objective function is checked and the related values are shown in Table 2;

Step 6. The highest profit, i.e. $TP^* = 221.724$, corresponds to $Q^* = 27$, $T^* = 0.0332$, and $P^* = 306.794$.

6. Sensitivity analysis

To examine the sensitivity of the variables with respect to input of the model, sensitivity analysis is performed and results are shown in Tables 3 and 4. In both tables, zero represents that changes of the parameters have no effect on optimal solution.

In the first model, sensitivity analysis has been conducted on the parameters w, F_D, F_R, h, and θ,

Table 3. Effects of parameter changes on optimal values of the first model.

		Percentage of decision variable change			
Percentage of parameter changes		T	p	Q	TC
w	+0.25	–0.1827	+2.1963	0	–8.5021
	–0.25	+28.3108	–3.6605	0	+18.0279
	+0.5	–31.1104	+3.6606	0	–12.1140
	–0.5	+77.3513	–1.0982	0	+59.1799
	+0.75	–40.5159	+4.7065	0	–12.8475
	–0.75	+18.5431	–3.2945	0	+18.9663
h	+0.25	–6.134	0	0	+1.2084
	–0.25	+7.958	0	+10	–1.1043
	+0.5	–1.1073	0	–10	+2.4776
	–0.5	+1.8811	0	+20	–2.0183
	+0.75	–1.5162	0	–10	+3.7840
	–0.75	+3.4727	0	+40	–2.5549
θ	+0.25	–7.1091	0	–10	+3.2844
	–0.25	+1.0003	0	+20	–3.1287
	+0.5	–1.2151	0	–20	+6.6391
	–0.5	+2.5794	0	+40	–5.8918
	+0.75	–1.6923	0	–20	+1.0037
	–0.75	+5.8226	0	+60	–7.4153
F_R	+0.25	+0.597	0	+20	+1.3266
	–0.25	–0.597	0	–20	–1.3259
	+0.5	+0.0038	0	+40	+2.654
	–0.5	–0.0039	0	–40	–2.650
	+0.75	+0.0057	0	+60	+3.982
	–0.75	–0.0058	0	–80	–3.9752
F_D	+0.25	+22.900	–3.0465	0	+18.679
	–0.25	–24.032	+3.0464	0	–15.518
	+0.5	+44.510	–5.490	0	+39.357
	–0.5	–49.040	+5.490	0	–24.364
	+0.75	+64.863	–8.236	0	+61.500
	–0.75	–74.551	+8.236	0	–6.365

Table 4. Effects of parameter changes on optimal values of the first model.

		Percentage of decision variable change			
Percentage of parameter changes		T	p	Q	TC
v	+0.25	25.96403	2.537816	0	–1365.93
	–0.25	–19.7392	–18.4472	0	597.1643
	+0.5	47.66556	–12.0663	0	–3578.92
	–0.5	–31.1491	–41.7026	0	808.7256
	+0.75	62.35726	–38.4073	0	–6522.12
	–0.75	–37.9943	–63.2624	0	873.5994
h	+0.25	0	0	0	32.8854
	–0.25	0	0	+6.25	–32.885
	+0.5	0	0	–6.25	65.77053
	–0.5	0	0	+12.5	–65.7701
	+0.75	0	0	–6.25	98.65569
	–0.75	0	0	+25	–98.6551
θ	+0.25	0	0	–6.25	52.61651
	–0.25	0	0	+12.5	–52.6159
	+0.5	0	0	–12.5	105.2327
	–0.5	0	0	+25	–105.232
	+0.75	0	0	–12.5	157.8489
	–0.75	0	0	+37.5	–157.849
F_R	+0.25	–1.81257	1.260573	+12.5	50.66922
	–0.25	1.814735	–1.31092	–12.5	–50.6944
	+0.5	-3.62237	2.474137	+25	101.3129
	–0.5	3.632228	–2.67451	-25	–101.414
	+0.75	–5.43001	3.643026	+37.5	151.9316
	–0.75	5.452087	–4.09444	–50	–152.16
F_D	+0.25	13.02458	–20.3885	0	–133.635
	–0.25	–14.2209	16.95089	0	145.9083
	+0.5	25.11224	–45.6553	0	–257.657
	–0.5	–30.0442	31.45431	0	308.2564
	+0.75	36.44014	–78.0568	0	–373.881
	–0.75	–48.1764	44.26574	0	494.2964

since they have more important effects on the profit, decision making, and managerial insights. According to Tables 3 and 4, when holding cost increases, order quantity and dispatch cycle length decrease, and total cost increases. However, when w increases, selling price increases, and period length and total cost decrease. Moreover, when F_R becomes larger, order quantity, total cost, and dispatch cycle length increase.

When F_D becomes larger, total cost and dispatch cycle length increase and selling price decreases and when decaying rate increases, total cost increases and both order quantity and dispatch cycle length decrease. It is noticeable that QBC causes a considerable decrease in cost. Table 3 represents the sensitivity analysis for the first model. For the second model, sensitivity analysis has been conducted on the parameters v , F_D, F_R, h, and θ. Table 4 represents the sensitivity analysis for the second model. According to the results, when the waiting cost is high, the vendor dispatches smaller orders in order to decrease the waiting costs, because the waiting time of customers increases as dispatch cycle length becomes larger.

7. Conclusion

In this paper, two mathematical models were presented for an integrated pricing-inventory problem of a single decaying product (since the number of deteriorating products increased every day) and optimum quantities of period length, order quantity, and selling price were obtained under two different scenarios. Then, by using several lemmas and theorems, the convexity and concavity of the functions of the two extended models were proved and different solution methods (algorithms) were developed to solve the models. In order to decrease the cost of transportation, time-based consolidation policy was applied. The developed models in this paper were comprehensive and considered as different forms of demand functions and deteriorating rates. Also, all costs of an inventory system were taken into consideration in the extended models. Finally, two examples were provided to show the applicability of the proposed policies. This model was developed under certain environment, and shortage was not permitted. Also, a single-stage problem was considered and developed and no contract was used between the stockholders. Therefore, for the future studies, permissible delay in payments contract and considering promotions, stochastic or fuzzy demand, multi-level supply chain, and permitted shortage could be of interest.

Acknowledgment

The first author would like to thank the financial support of University of Tehran for this research under grant number 30015-1-04.

References

1. Cetinkaya, S. and Lee, C.Y. "Stock replenishment and shipment scheduling for vendor-managed inventory systems", *Management Science*, **46**(2), pp. 217-232 (2000).
2. Whitin, T. "Inventory control and price theory", *Management Science*, **2**, pp. 61-68 (1955).
3. Chen, F.Y., Wang, T., and Xu, T.Z. "Integrated inventory replenishment and temporal shipment consolidation: A comparison of quantity-based and time-based models", *Annals of Operations Research*, **135**, pp. 197-210 (2005).
4. Ray, S., Song, Y., and Verma, M. "Comparison of two periodic review models for stochastic and price-sensitive demand environment", *International Journal of Production Economics,* **128**, pp. 209-222 (2010).
5. Huang, Y., Huang, G.Q., and Newman, S.T. "Coordinating pricing and inventory decisions in a multi-level supply chain: A game theoretic approach", *Transportation Research Part E*, **47**, pp. 115-129 (2011).
6. Polatoglu, L.H. "Optimal order quantity and pricing decisions in single-period inventory systems", *International Journal of Production Economics*, **23**, pp. 175-185 (1991).
7. Zhu, S.X. "Joint pricing and inventory replenishment decisions with returns and expediting", *European Journal of Operation Research*, **216**, pp. 105-112 (2012).
8. You, P.S., Ikuta, S., and Hsieh, Y.C. "Optimal ordering and pricing policy for an inventory system with trial periods", *Applied Mathematical Modeling*, **34**, pp. 3179-3188 (2012).
9. Su, Y. and Geunes, J. "Price promotions, operations cost, and profit in a two-stage supply chain", *Omega*, **40**, pp. 891-905 (2012).
10. Mutlu, F. and Cetinkaya, S. "Pricing decisions in a carrier-retailer channel under price-sensitive demand and contract-carriage with common-carriage option", *Transportation Research Part E*, **51**, pp. 28-40 (2013).
11. Maddah, B. and Bish, E.K. "Joint pricing, assortment, and inventory decisions for a retailer's product-line", *Naval Research Logistics*, **54**, pp. 315-330 (2007).
12. Maity, K. and Maiti, M. "A numerical approach to a multi-objective optimal inventory control problem for deteriorating multi-items under fuzzy inflation and discounting", *Computer and Mathematics with Applications*, **55**, pp. 1794-1807 (2008).
13. Yu, Y., Wang, Z., and Liang, L. "A vendor managed inventory supply chain with deteriorating raw materials and products", *International Journal of Production Economics*, **136**, pp. 266-274 (2012).
14. Hongjie, L., Ruxian, L., Zhigao, L., and Ruijiang, W. "Study on the inventory control of deteriorating items under VMI model based on bi-level programming", *Expert Systems with Application*, **38**, pp. 9287-9295 (2011).
15. Mahata, G.C. "An EPQ-based inventory model for exponentially deteriorating items under retailer partial trade credit policy in supply chain", *Expert Systems with Application*, **39**, pp. 3537-3550 (2012).
16. Taleizadeh, A.A. and Nematollahi, M. "An inventory control problem for deteriorating items with backordering and financial considerations", *Applied Mathematical Modeling*, **38**(1), pp. 93-109 (2013a).
17. Lee, C.F. and Chung, C.P. "An inventory model for deteriorating items in a supply chain with system dynamics analysis", *Procedia-Social and Behavioral Sciences*, **40**, pp. 41-51 (2012).
18. Cetinkaya, S. and Bookbinder, J.H. "Stochastic models for the dispatch of consolidated shipments", *Transportation Research-Part B*, **37**, pp. 747-768 (2003).

19. Cetinkaya, S., Mutlu, F., and Lee, C.Y. "A comparison of outbound dispatch policies for integrated inventory and transportation decisions", *European Journal of Operation Research*, **171**, pp. 1094-1112 (2006).

20. Wong, W.H., Lawrence, C.L., and Hui, Y.V. "Air-freight forwarder shipment planning: A mixed 0-1 model and managerial issues in the integration and consolidation of shipments", *European Journal of Operation Research*, **193**, pp. 86-97 (2009).

21. Marklund, J. "Inventory control in divergent supply chains with time-based dispatching and shipment consolidation", *Naval Research Logistics*, **58**, pp. 59-71 (2010).

22. Howard, C. and Marklund, J. "Evaluation of stock allocation policies in a divergent inventory system with shipment consolidation", *European Journal of Operation Research*, **211**, pp. 298-309 (2011).

23. Taleizadeh, A.A., Moghadasi, H., Niaki, S.T.A., and Eftekhari, A.K. "An EOQ-joint replenishment policy to supply expensive imported raw materials with payment in advance", *Journal of Applied Science*, **8**(23), pp. 4263-4273 (2009).

24. Taleizadeh, A.A., Niaki, S.T.A., and Najafi, A.A. "Multi product single machine production system with stochastic scraped production rate, partial back ordering and service level constraint", *Journal of Computational and Applied Mathematics*, **233**(8), pp. 1834-1849 (2010a).

25. Taleizadeh, A.A., Najafi, A.A., and Niaki, S.T.A. "Multi product EPQ model with scraped items and limited production capacity", *Scientia Iranica, Transaction E*, **17**(1), pp. 58-69 (2010b).

26. Ulku, M.A. and Bookbinder, J.H. "Modelling shipment consolidation and pricing decisions for a manufacturer-distributor", *International Journal of Revenue Management*, **6**(1) (2012).

27. Sajadieh, M.S. and Jokar, M.R.A. "Optimizing shipment, ordering and pricing policies in a two-stage supply chain with price-sensitive demand", *Transportation Research - Part E*, **45**, pp. 564-571 (2009).

28. Olsson, F. "Analysis of inventory policies for perishable items with fixed leadtimes and lifetimes", *Annals of Operations Research*, **217**(1), pp. 399-423 (2014).

29. Herbon, A., Levner, E., and Cheng, T.C.E. "Perishable inventory management with dynamic pricing using time-temperature indicators linked to automatic detecting devices", *International Journal of Production Economics*, **147**, pp. 605-613 (2014).

30. Taleizadeh, A.A. "An economic order quantity model with consecutive payments for deteriorating items", *Applied Mathematical Modeling,* **38**, pp. 5357-5366 (2014a).

31. Taleizadeh, A.A. "An economic order quantity model with partial backordering and advance payments for an evaporating item", *International Journal of Production Economic*, **155**, pp. 185-193 (2014b).

32. Diabat, A., Abdallah, T., and Le, T. "A hybrid tabu search based heuristic for the periodic distribution inventory problem with perishable goods", *Annals of Operations Research*, **242**(2), pp. 1-26 (2014).

33. Lu, L., Zhang, J., and Tang, W. "Optimal dynamic pricing and replenishment policy for perishable items with inventory-level-dependent demand", *International Journal of Systems Science*, **47**(6), pp. 1480-1494 (2016).

34. Gallego, G. and Hu, M., *Dynamic Pricing of Perishable Assets under Competition Management Science*, **60**(5), pp. 1241-1259 (2014) .

35. Tayal, S.H., Singh, S.R., and Sharma, R. "An integrated production inventory model for perishable products with trade credit period and investment in preservation technology", *International Journal of Mathematics in Operational Research*, **8**(2), pp. 137-163 (2016).

36. Taleizadeh, A.A., Satarian, F., and Jamil, A. "Optimal multi-discount selling prices schedule for deteriorating product", *Scientia Iranica*, **22**(6), pp. 2595-2603 (2015).

37. Maleki Vishkaei, B., Pasandideh, S.H.R., and Farhangi, M. "The 100% screening economic order quantity model under shortage and delay in payment", *Scientia Iranica*, **21**(6), pp. 2429-2435 (2015).

38. Taleizadeh, A.A., Niaki, S.T.A., and Aryanezhad, M.B. "Replenish-up-to multi chance-constraint inventory control system with stochastic period lengths and total discount under fuzzy purchasing price and holding costs", *International Journal of System Sciences*, **41**(10), pp. 1187-1200 (2010c).

39. Taleizadeh, A.A., Wee, H.M., and Sadjadi, S.J. "Multi-product production quantity model with repair failure and partial back-ordering", *Computers and Industrial Engineering,* **59**(1), pp. 45-54 (2010d).

40. Taleizadeh, A.A., Niaki, S.T.A., Aryanezhad, M.B., and Fallah-Tafti, A. "A genetic algorithm to optimize multi-product multi-constraint inventory control systems with stochastic replenishments and discount", *International Journal of Advanced Manufacturing Technology,* **51**, (1-4), pp. 311-323 (2010e).

41. Taleizadeh, A.A., Barzinpour, F., and Wee, H.M. "Meta-heuristic algorithms to solve the fuzzy single period problem", *Mathematical and Computer Modeling,* **54**(5-6), pp. 1273-1285 (2011f).

42. Taleizadeh, A.A., Cárdenas-Barrón, L.E., Biabani, J., and Nikousokhan, R. "Multi products single machine EPQ model with immediate rework process", *International Journal of Industrial Engineering Computations*, **3**(2), pp. 93-102 (2012a).

43. Taleizadeh, A.A., Pentico, D.W., Aryanezhad, M.B., and Ghoreyshi, M. "An economic order quantity model with partial backordering and a special sale price", *European Journal of Operational Research*, **221**(3), pp. 571-583 (2012b).

44. Taleizadeh, A.A., Niaki, S.T., Aryanezhad, M.B., and Shafii, N. "A hybrid method of fuzzy simulation and genetic algorithm to optimize multi-product single-constraint inventory control systems with stochastic replenishments and fuzzy-demand", *Information Science*, **220**(20), pp. 425-441 (2013b).

45. Taleizadeh, A.A., Pentico, D.W., Jabalameli, M.S., and Aryanezhad, M.B. "An economic order quantity model with multiple partial prepayments and partial backordering", *Mathematical and Computer Modeling*, **57**(3-4), pp. 311-323 (2013c).

46. Taleizadeh, A.A., Mohammadi, B., Cárdenas-Barron, L.E., and Samimi, H. "An EOQ model for perishable product with special sale and shortage", *International Journal of Production Economics*, **145**(1), pp. 318-338 (2013d).

47. Taleizadeh, A.A., Jalali-Naini, S.G.H., Wee, H.M., and Kuo, T.C. "An imperfect, multi product production system with rework", *Scientia Iranica*, **20**(3), pp. 811-823 (2013e).

48. Taleizadeh, A.A. and Pentico, D.W. "An economic order quantity model with known price increase and partial backordering", *European Journal of Operational Research*, **28**(3), pp. 516-525 (2013f).

49. Teimoury, R. and Kazemi, S.M.M. "An integrated pricing and inventory model for deteriorating products in a two stage supply chain under replacement and shortage", *Scientia Iranica*, **24**(1), pp. 342-354 (2017).

Biographies

Ata Allah Taleizadeh is an Associate Professor in the School of Industrial Engineering, College of Engineering at University of Tehran in Iran. He received his PhD degree in Industrial Engineering from Iran University of Science and Technology. Moreover, he received his BSc and MSc degrees both in Industrial Engineering from Azad University of Qazvin and Iran University of Science and Technology, respectively. His research interests include inventory control and production planning, pricing and revenue optimization, and Game theory. He has published several papers in reputable journals and book chapters, and is now serving as the editor/editorial board member for a number of international journals.

Arezu Rasouli-Baghban received her MSc degree in Industrial Engineering from South Tehran Branch of Islamic Azad University and BSc degree in Mathematics from K. N. Toosi University of Technology. Her research interests include supply chain management, inventory control, and operation research.