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Two-dimensional uncertain linguistic generalized normalized weighted geometric Bonferroni mean and its application to multiple-attribute decision making

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Abstract. 2-Dimensional Uncertain Linguistic Variables (2DULVs) are powerful tools to express the fuzzy or uncertain information, and the weighted Bonferroni mean can not only take the attribute importance into account, but also capture the interrelationship between the attributes. However, the traditional Bonferroni mean can only deal with the crisp numbers. In this paper, Bonferroni mean was extended to process the 2DULVs. Firstly, we proposed the Normalized Weighted Geometric Bonferroni Mean (NWGBM) operator and the Generalized Normalized Weighted Geometric Bonferroni Mean (GNWGBM) operator, which had the characteristic of reducibility and considered the interrelationships between two attributes. Then, we introduced the computation rules, characteristics, the expected value, and comparison method of the 2DULVs. Further, we developed the 2-Dimensional Uncertain Linguistic Normalized Weighted Geometric Bonferroni Mean (2DULNWGBM) and the 2-Dimensional Uncertain Linguistic Generalized Normalized Weighted Geometric Bonferroni Mean (2DULGNWGBM), and explored some properties and discussed some special cases of them. Finally, we developed a new decision making method based on these operators. An example is given to compare the method with the existing methods.

1. Introduction

Multiple-Attribute Decision Making (MADM) has wide applications in many fields. Since the time Churchman et al. [1] proposed the MADM methods and applied them to decision problem of "choice of enterprise investment policy", the classical MADM theory has been well established. However, because of the decision making complexity, most of these problems are uncertain and we can call them the Uncertain MADM (UMADM). In the UMADM problems, sometimes, it is difficult to express the attribute values by real numbers; however, it may be easy to do so by the linguistic terms. Up to now, the research on the linguistic MADM problems has led to many results [2-7]. Zadeh [8-10] firstly proposed the concept of the Linguistic Variable (LV). Yager [11] extended Ordered Weighted Averaging (OWA) operator to LVs. Herrera and Martínez [12] developed a method of 2-tuple to overcome the linguistic information loss caused by the linguistic operations. Xu [13] extended the hybrid average operator and the hybrid weighted mean to the LVs. Xu [14] proposed some triangular fuzzy linguistic operators. Liu and Su [3] proposed some trapezoidal fuzzy linguistic aggregation operators. Zhou and Chen [4] proposed some linguistic generalized power aggregation operators. Merigó and Gil-Lafuente [5] extended induced generalized OWA operator and the

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induced quasi-arithmetic Choquet integral operator to LVs.

In order to process more complex decision problems, Zhu et al. [15] defined the 2-dimensional linguistic information, where the I-dimension of linguistic variables was used to express the evaluation results and the II-dimension of linguistic variables was used to express the subjective evaluation of the reliability of I-dimension information. Obviously, the 2-dimensional linguistic variables can conveniently express the evaluation results for decision objective. Liu and Zhang [16] further proposed 2-Dimensional Uncertain Linguistic Variables (2DULVs). Zhang et al. [17] proposed 2-dimensional semantic evidence reasoning method. Yu et al. [18] converted 2-dimensional linguistic information into the generalized triangular fuzzy numbers, and proposed some weighted operator and OWA operator for 2DULVs. Further, Liu and Yu [19] extended the power aggregation operators to process 2DULVs. Chu and Liu [20] extended Heronian mean operators to process the 2DULVs. Liu and Wang [21] defined new operational rules for 2DULVs and developed some new operators. Liu et al. [22] proposed some generalized hybrid operators for the 2DULVs.

The information aggregation operators are receiving more and more attention [23-34]. In addition, Bonferroni [35] originally proposed a Bonferroni Mean (BM), which could capture the interrelationships of the attributes. Further, Yager [36] proposed a Bonferroni OWA (BOWA) operator, Weighted Bonferroni Mean (WBM), and Bonferroni choquet integral. Xu [37] proposed some uncertain Bonferroni mean operators based on interval numbers. Zhu et al. [38] extended BM to process triangular fuzzy information, and developed some new BM operators. Xu and Yager [39] developed some BM operators for the intuitionistic fuzzy information. Beliakov et al. [40] proposed the Generalized Bonferroni Mean (G-BM). Xu and Chen [41] extended the BM to Interval-valued Intuitionistic Fuzzy (IIF) sets. Xia et al. [42] extended the Geometric BM (GBM) to Intuitionistic Fuzzy Set (IFS). Wei et al. [43] extended the BM operators to the uncertain linguistic information. Liu et al. [32] proposed some intuitionistic uncertain linguistic weighted Bonferroni OWA operators. Liu and Wang [44] proposed some single-valued neutrosophic normalized weighted Bonferroni Mean. Wei et al. [45] proposed some generalized BM operators for 2-tuple linguistic information and applied them to MADM problems. We can note that the BM operators extended above do not have the characteristic of reducibility. To overcome this shortcoming, Xia et al. [46] developed the extended BM, GBM, GWBM, and GWGBM operators, which had reducibility, and further extended them to IFS. However, these operators only considered the relationships between individual attributes and the other ones. Further, Zhou and He [47] proposed the Intuitionistic Fuzzy Generalized Weighted GBM (IFGWGBM) operators, which had reducibility and reflected the interrelationships of the attributes. Zhou and He [48] proposed the Normalized WBM (NWBM) and the Generalized NWBM (GNWBM), and the Intuitionistic Fuzzy NWBM (IFNWBM) and the Generalized Intuitionistic Fuzzy NWBM (GIFNWBM), which had reducibility and reflected the interrelationship between two attributes. Tian et al. [49] proposed some simplified neutrosophic linguistic normalized weighted BM operators and applied them to the multi-criteria decision making problems. Liu et al. [33] proposed the multi-valued neutrosophic number Bonferroni mean operators.

Motivated by the ideas of NWBM, GNWBM, IFNWBM, and GIFNWBM operators proposed by Zhou and He [48], this paper is to propose the Normalized Weighted GBM (NWGBM) and the Generalized NWGBM (GNWGBM), the 2-Dimensional Uncertain Linguistic NWGBM (2DULNWGBM) and the 2-Dimensional Uncertain Linguistic Generalized NWGBM (2DULGNWGBM). Based on these operators, the MADM approach with 2DULVs is developed.

The rest of this study is organized as follows. Section 2 briefly reviews some basic concepts. Section 3 develops the Normalized Weighted Geometric Bonferroni Mean (NWGBM) and the Generalized Normalized Weighted Geometric Bonferroni Mean (GNWGBM). Section 4 introduces 2DULVs. In Section 5, we propose 2-Dimensional Uncertain Linguistic NWGBM (2DULNWGBM) and 2-Dimensional Uncertain Linguistic GNWGBM (2DULGNWGBM). In Section 6, we use the 2DULGNWGBM operator for the MADM problems and give the decision making steps. Section 7 gives an example to show the effectiveness of the proposed method. Finally, Section 8 gives some conclusions.

2. Preliminaries

2.1. Uncertain linguistic variables

In order to express the qualitative information, decision makers generally need to pre-set a linguistic evaluation term set. Let $S = \{s_\alpha | \alpha = 0, 1, \cdots, L-1\}$ be a linguistic set, where s_α denotes a linguistic variable and L is an odd number. For example, when $L = 5$, we can define $S = \{s_0, s_1, s_2, s_3, s_4\}$ = (very bad,bad, fair, good, very good). The characteristics of the term set, S, can be found in [8-10,18,36].

In order to minimize the linguistic information loss in the linguistic operational process, it is very necessary to extend the original discrete linguistic set $S = (s_0, s_1, \cdots, s_{L-1})$ to the continuous linguistic set $\bar{S} = \{s_\alpha | \alpha \in [0, q]\}$, where q is a sufficiently large positive integer [50].

Definition 1 [50]. Suppose $\tilde{s} = [s_a, s_b], s_a, s_b \in \bar{S}$ and $a \leq b$; s_a and s_b are the lower and upper limits of $\tilde{s}$; then, $\tilde{s}$ is called the Uncertain Linguistic Variable (ULV).

In general, we define $\tilde{S}$ as the set of all ULVs. For any two ULVs $\tilde{s}_1 = [s_{x1}, s_{y1}]$ and $\tilde{s}_2 = [s_{x2}, s_{y2}]$, we have [50]:

$$\tilde{s}_1 \oplus \tilde{s}_2 = [s_{x1}, s_{y1}] \oplus [s_{x2}, s_{y2}] = [s_{x1+x2}, s_{y1+y2}], \quad (1)$$

$$\tilde{s}_1 \otimes \tilde{s}_2 = [s_{x1}, s_{y1}] \otimes [s_{x2}, s_{y2}] = [s_{x1\times x2}, s_{y1\times y2}], \quad (2)$$

$$\tilde{s}_1 / \tilde{s}_2 = [s_{x1}, s_{y1}] / [s_{x2}, s_{y2}] = [s_{x1/y2}, s_{y1/x2}], \quad (3)$$

$$\lambda \tilde{s}_1 = \lambda [s_{x1}, s_{y1}] = [s_{\lambda * x1}, s_{\lambda * y1}] \qquad (\lambda \geq 0), \quad (4)$$

$$\lambda(\tilde{s}_1 \oplus \tilde{s}_2) = \lambda \tilde{s}_1 \oplus \lambda \tilde{s}_2 \qquad (\lambda \geq 0), \quad (5)$$

$$(\lambda_1 + \lambda_2)\tilde{s}_1 = \lambda_1 \tilde{s}_1 \oplus \lambda_2 \tilde{s}_1 \qquad (\lambda_1 \geq 0, \lambda_2 \geq 0). \quad (6)$$

2.2. Bonferroni Mean (BM)

BM operator has a notable feature, i.e. it can capture the interrelationship between the attributes, which is defined as follows.

Definition 2 [35]. Let a_i $(i = 1, 2, \cdots, n)$ be a set of nonnegative numbers and $p, q \geq 0$. If:

$$\mathrm{BM}^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{1/p+q}, \quad (7)$$

then, $\mathrm{BM}^{p,q}$ is called the Bonferroni Mean (BM).

The BM operator has the properties of idempotency, monotonicity, and boundedness.

In addition, we can change this aggregation operator by another interesting way shown in the following:

$$\mathrm{BM}^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^p \left(\frac{1}{n-1} \sum_{j=1, j\neq i}^{n} a_j^q \right) \right)^{1/p+q}. \quad (8)$$

Obviously, the part $\frac{1}{n-1} \sum_{j=1,j\neq i}^{n} a_j^q$ is the power average. If we will further denote $u_i^q = \frac{1}{n-1} \sum_{j=1,j\neq i}^{n} a_j^q$, then have:

$$\mathrm{BM}^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^p u_i^q \right)^{1/p+q}. \quad (9)$$

Definition 3 [42]. Let a_i $(i = 1, 2, \cdots, n)$ be a set of nonnegative numbers and $p, q \geq 0$. If:

$$\mathrm{GBM}^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{1}{n(n-1)}}, \quad (10)$$

then, $\mathrm{GBM}^{p,q}$ is called the Geometric BM (GBM).

As mentioned above, the BM and GBM can consider the interrelationships of any two input arguments. Further, Beliakov et al. [40] extended the BM to the Generalized BM (G-BM) by taking the interrelationships of any three arguments into account.

Definition 4 [40]. Let $p, q, r \geq 0$, and a_i $(i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers. If:

$$\text{G-BM}^{p,q,r}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)(n-1)} \sum_{\substack{i,j,r=1 \\ i \neq j \neq r}}^{n} a_i^p a_j^q a_k^r \right)^{1/p+q+r}, \quad (11)$$

then, $\text{G-BM}^{p,q,r}$ is called the Generalized BM (G-BM).

However, BM, GBM, and G-BM operators can only consider interrelationships between a_i and other data a_j, and they ignore their own weights. In the following, we will introduce the Weighted BM (WBM), Weighted GBM (WGBM), Generalized Weighted BM (GWBM), and Generalized WGBM (GWGBM).

Definition 5 [39]. Let $p, q \geq 0$, and a_i $(i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $a_i (i = 1, 2, \cdots, n)$, where w_i represents the importance degree of a_i, satisfying $w_i > 0$, $\sum_{i=1}^{n} w_i = 1$. If:

$$\mathrm{WBM}^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i a_i^p)(w_j a_j^q) \right)^{1/p+q}, \quad (12)$$

then, $\mathrm{WBM}^{p,q}$ is called the Weighted BM (WBM).

Definition 6 [42]. Let $p, q \geq 0$, and a_i $(i = 1, 2, \cdots, n)$ be a collection of nonnegative nonnegative numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of a_i $(i = 1, 2, \cdots, n)$, where w_i represents the importance degree of a_i, satisfying $w_i > 0$, $\sum_{i=1}^{n} w_i = 1$. If:

$$\mathrm{WGBM}^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(pa_i^{w_i} + qa_j^{w_j}\right)^{\frac{1}{n(n-1)}}, \quad (13)$$

then, $\text{WGBM}^{p,q}$ is called the Weighted GBM (WGBM).

Definition 7 [46]. Let $p,q,r \geq 0$, and a_i $(i = 1,2,\cdots,n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of a_i $(i = 1,2,\cdots,n)$, where w_i represents the importance degree of a_i, satisfying $w_i > 0$, $\sum_{i=1}^n w_i = 1$. If:

$$\text{GWBM}^{p,q,r}(a_1, a_2, \cdots, a_n) = \left(\sum_{i,j,r=1}^{n} \left(w_i w_j w_k a_i^p a_j^q a_k^r \right) \right)^{1/p+q+r}, \tag{14}$$

then, $\text{GWBM}^{p,q,r}$ is called the Generalized WBM (GWBM).

Definition 8 [46]. Let $p,q,r \geq 0$, and a_i $(i = 1,2,\cdots,n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $a_i (i = 1,2,\cdots,n)$, where w_i represents the importance degree of a_i, satisfying $w_i > 0$, $\sum_{i=1}^n w_i = 1$. If:

$$\text{GWGBM}^{p,q,r}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q+r} \prod_{i,j,r=1}^{n} (pa_i + qa_j + ra_k)^{w_i w_j w_k}, \tag{15}$$

then, $\text{GWGBM}^{p,q,r}$ is called the Generalized WGBM (GWGBM).

Further, Zhou and He [48] noted that the BM could not be obtained from the WBM when the weights of the aggregated parameters were equal; that is to say, the WBM did not have reducibility. It seemed to be counterintuitive. Similarly, the GBM could not be obtained from the WGBM when the weights were equal. At the same time, the GWBM and GWGBM only considered the complete correlation among any three attribute values and could not reflect the relationships between individual attributes and the other ones. In order to further overcome these shortcomings, Zhou and He [48] proposed the following Normalized WBM (NWBM) and the Generalized NWBM (GNWBM).

Definition 9 [48]. Let $p,q \geq 0$, and a_i $(i = 1,2,\cdots,n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of a_i $(i = 1,2,\cdots,n)$ and meets $w_i \geq 0$, $\sum_{i=1}^n w_i = 1$. If:

$$\text{NWBM}^{p,q}(a_1, a_2, \cdots, a_n) = \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{w_i w_j}{1 - w_i} a_i^p a_j^q \right)^{1/p+q}, \tag{16}$$

then, $\text{NWBM}^{p,q}$ is called the Normalized Weighted BM (NWBM).

Definition 10 [48]. Let $p,q,r \geq 0$, and a_i $(i = 1,2,\cdots,n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of a_i $(i = 1,2,\cdots,n)$ with $w_i \geq 0$, $\sum_{i=1}^n w_i = 1$. If:

$$\text{GNWBM}^{p,q,r}(a_1, a_2, \cdots, a_n) = \left(\sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)} a_i^p a_j^q a_j^r \right)^{1/p+q+r}, \tag{17}$$

then, $\text{GNWBM}^{p,q,r}$ is called the Generalized NWBM (GNWBM).

Zhou and He [48] have proved that NWBM and GNWBM have the desirable properties, including reducibility, commutativity, idempotency, monotonicity, and boundedness, and can reflect the relationships between individual attributes and the other ones.

3. The normalized weighted geometric BM and generalized normalized weighted geometric BM

As mentioned above, the WGBM and GWGBM have some drawbacks. According to definitions of the NWBM and GNWBM, we can define the Normalized WGBM (NWGBM) and the Generalized Normalized WGBM (GNWGBM) to overcome these disadvantages, which are shown as follows.

3.1. NWGBM

Definition 11. Let $p,q \geq 0$, and a_i $(i = 1,2,\cdots,n)$ be a set of nonnegative numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of a_i $(i = 1,2,\cdots,n)$, satisfying $w_i > 0$, $\sum_{i=1}^n w_i = 1$. If:

$$\text{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{w_i w_j}{1-w_i}}, \tag{18}$$

then, $\text{NWGBM}^{p,q}$ is called the Normalized WGBM (NWGBM).

Further, we can prove that NWGBM has the following properties.

Theorem 1 (Reducibility).
Let $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$; then:

$$\text{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n) = \text{GBM}^{p,q}(a_1, a_2, \cdots, a_n). \tag{19}$$

Proof. Since $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, according to Eq. (18), we have:

$$\mathrm{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n)$$

$$= \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{w_i w_j}{1-w_i}}$$

$$= \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{1}{n(n-1)}}$$

$$= \mathrm{GBM}^{p,q}(a_1, a_2, \cdots, a_n). \quad (20)$$

Theorem 2 (Idempotency).
Let $a_j = a, j = 1, 2, \cdots, n$; then:

$$\mathrm{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n) = a. \quad (21)$$

Proof. Since $a_i = a, i = 1, 2, \cdots, n$, according to Eq. (18), we have:

$$\mathrm{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n)$$

$$= \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{w_i w_j}{1-w_i}}$$

$$= \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa + qa)^{\frac{w_i w_j}{1-w_i}}$$

$$= \frac{1}{p+q} \left((p+q)^{\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{w_i w_j}{1-w_i}} a^{\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{w_i w_j}{1-w_i}} \right)$$

$$= \frac{1}{p+q}((p+q)a) = a. \quad (22)$$

Theorem 3 (Commutativity).
Let b_i $(i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers and $(b_1, b_2, \cdots, b_n)$ be any permutation of $(a_1, a_2, \cdots, a_n)$; then:

$$\mathrm{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n)$$

$$= \mathrm{NWGBM}^{p,q}(b_1, b_2, \cdots, b_n). \quad (23)$$

Proof. Since $(b_1, b_2, \cdots, b_n)$ is any permutation of $(a_1, a_2, \cdots, a_n)$, then:

$$\frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{w_i w_j}{1-w_i}}$$

$$= \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pb_i + qb_j)^{\frac{w_i w_j}{1-w_i}}.$$

Therefore:

$$\mathrm{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n)$$

$$= \mathrm{NWGBM}^{p,q}(b_1, b_2, \cdots, b_n).$$

Theorem 4 (Monotonicity).
Let a_i $(i = 1, 2, \cdots, n)$ and b_i $(i = 1, 2, \cdots, n)$ be two collections of nonnegative numbers. If $a_i \geq b_i$ for all i, then:

$$\mathrm{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n)$$

$$\geq \mathrm{NWGBM}^{p,q}(b_1, b_2, \cdots, b_n). \quad (24)$$

Proof. Since $a_i \geq b_i$ for all i, and $p, q > 0$, then:

$$pa_i + qa_j \geq pb_i + qb_j,$$

$$(pa_i + qa_j)^{\frac{w_i w_j}{1-w_i}} \geq (pb_i + qb_j)^{\frac{w_i w_j}{1-w_i}}.$$

Further, we have:

$$\frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{w_i w_j}{1-w_i}}$$

$$\geq \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pb_i + qb_j)^{\frac{w_i w_j}{1-w_i}},$$

i.e.:

$$\mathrm{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n)$$

$$\geq \mathrm{NWGBM}^{p,q}(b_1, b_2, \cdots, b_n).$$

Theorem 5 (Boundedness).

$$\min(a_1, a_2, \cdots, a_n) \leq \mathrm{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n)$$

$$\leq \max(a_1, a_2, \cdots, a_n). \quad (25)$$

Proof. Let $a = \min(a_1, a_2, \cdots, a_n)$, $b = \max(a_1, a_2, \cdots, a_n)$; according to Theorem 2, we have:

$$\mathrm{NWGBM}^{p,q}(a, a, \cdots, a) = a,$$

$$\mathrm{NWGBM}^{p,q}(b, b, \cdots, b) = b.$$

Since $a \leq a_i \leq b$ for all i, based on Theorem 4, we have:

$$\text{NWGBM}^{p,q}(a, a, \cdots, a) \leq \text{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n) \leq \text{NWGBM}^{p,q}(b, b, \cdots, b).$$

Further, $a \leq \text{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n) \leq b$, i.e.:

$$\min(a_1, a_2, \cdots, a_n) \leq \text{NWGBM}^{p,q}(a_1, a_2, \cdots, a_n) \leq \max(a_1, a_2, \cdots, a_n).$$

Further, we will explore some specials of the NWGBM for the parameters p and q.

1. If $q = 0$, then Eq. (18) reduces to the weighted geometric mean operator as follows:

$$\text{NWGBM}^{p,0}(a_1, a_2, \cdots, a_n) = \frac{1}{p}\prod_{i=1}^{n}(pa_i)^{w_i} = \prod_{i=1}^{n} a_i^{w_i}. \quad (26)$$

Thus, when $q = 0$, $\text{NWGBM}^{p,0}$ does not have any relationship with p;

2. If $p = q$, then Eq. (18) reduces to the following form:

$$\text{NWGBM}^{p,p}(a_1, a_2, \cdots, a_n) = \frac{1}{2p}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(p(a_i + a_j))^{\frac{w_i w_j}{1-w_i}} = \frac{1}{2}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(a_i + a_j)^{\frac{w_i w_j}{1-w_i}}. \quad (27)$$

Likewise, when $p = q$, $\text{NWGBM}^{p,p}$ does not have any relationship with p.

3.2. GNWGBM

In this section, we will propose the Generalized Normalized Weighted Geometric Bonferroni Mean (GNWGBM), which considers the correlation of any three aggregated arguments.

Definition 12: Let $p, q, r \geq 0$, and a_i $(i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of a_i $(i = 1, 2, \cdots, n)$, where w_i represents the importance degree of a_i, satisfying $w_i > 0$, $\sum_{i=1}^{n} w_i = 1$. If:

$$\text{GNWGBM}^{p,q,r}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q+r}\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}(pa_i + qa_j + ra_k)^{\frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)}}, \quad (28)$$

then, $\text{GNWGBM}^{p,q,r}$ is called the Generalized Normalized Weighted Geometric BM (GNWGBM).

Similar to NWGBM, the GNWGBM has reducibility, commutativity, idempotency, monotonicity, and boundedness.

We will discuss some specials of the GNWGBM as follows:

1. If $r = 0$, then Eq. (28) reduces to the Normalized Weighted Geometric Bonferroni Mean (NWGBM) operator as follows:

$$\text{GNWGBM}^{p,q,0}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(pa_i + qa_j)^{\frac{w_i w_j}{1-w_i}}. \quad (29)$$

2. If $p = q = r$, then Eq. (28) reduces to the following form:

$$\text{NWGBM}^{p,p,p}(a_1, a_2, \cdots, a_n) = \frac{1}{3p}\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}(p(a_i + a_j + a_k))^{\frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)}} = \frac{1}{3}\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}(a_i + a_j + a_k)^{\frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)}}. \quad (30)$$

Likewise, when $p = q = r$, $\text{NWGBM}^{p,p,p}$ does not have any relationship with p.

3.3. The DULV

In this section, we will give the definition of 2DULVs; moreover, the operational laws and comparison method for 2DULVs are given.

Definition 13 [19]. Let $\hat{s} = ([\dot{s}_a, \dot{s}_b][\ddot{s}_c, \ddot{s}_d])$, where $[\dot{s}_a, \dot{s}_b]$ is I class of ULVs, which is the judgment result of assessing the object given by the decision makers, $\dot{s}_a, \dot{s}_b \in S_{\text{I}} = (\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_{l-1})$; $[\ddot{s}_c, \ddot{s}_d]$ is II class of ULVs, which denotes the subjective evaluation of the reliability of the judgment result in I class information, $\ddot{s}_c, \ddot{s}_d \in S_{\text{II}} = (\ddot{s}_0, \ddot{s}_1, \cdots, \ddot{s}_{t-1})$. Then, we can call $\hat{s}$ a 2-dimensional uncertain linguistic variable.

Similarly, in order to reduce the distortion of linguistic information, we will extend the discrete linguistic set to the continuous linguistic set, i.e. $\dot{s}_a, \dot{s}_b \in \bar{S}_I = \{\dot{s}_\alpha | \alpha \in [0, q]\}$, and $\ddot{s}_c, \ddot{s}_d \in \bar{S}_{\text{II}} = \{\ddot{s}_\alpha | \alpha \in [0, q']\}$. Furthermore, we can define $\hat{S}$ as the set of all 2DULVs.

Liu and Yu [19] think that I class information is the judgment result of assessing objects and it can keep the operational rules of the traditional uncertain linguistic variables. Moreover, II class information denotes the credibility of I class information; in the operational process, it can get the lowest credibility. Thus, the operational rules of 2DULVs can be defined as follows [19]:

Let $\hat{s}_1 = ([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}])$ and $\hat{s}_2 = ([\dot{s}_{a2}, \dot{s}_{b2}][\ddot{s}_{c2}, \ddot{s}_{d2}])$ be two 2DULVs; then, we have:

$$\hat{s}_1 \oplus \hat{s}_2 = ([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}]) \oplus ([\dot{s}_{a2}, \dot{s}_{b2}][\ddot{s}_{c2}, \ddot{s}_{d2}]) = ([\dot{s}_{a1+a2}, \dot{s}_{b1+b2}][\ddot{s}_{\min(c1,c2)}, \ddot{s}_{\min(d1,d2)}]), \tag{31}$$

$$\hat{s}_1 \otimes \hat{s}_2 = ([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}]) \otimes ([\dot{s}_{a2}, \dot{s}_{b2}][\ddot{s}_{c2}, \ddot{s}_{d2}]) = ([\dot{s}_{a1\times a2}, \dot{s}_{b1\times b2}][\ddot{s}_{\min(c1,c2)}, \ddot{s}_{\min(d1,d2)}]), \tag{32}$$

$$\hat{s}_1 / \hat{s}_2 = ([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}])/([\dot{s}_{a2}, \dot{s}_{b2}][\ddot{s}_{c2}, \ddot{s}_{d2}]) = ([\dot{s}_{a1/b2}, \dot{s}_{b1/a2}][\ddot{s}_{\min(c1,c2)}, \ddot{s}_{\min(d1,d2)}]), \tag{33}$$

where $a2, b2 \neq 0$:

$$\lambda\hat{s}_1 = \lambda([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}]) = ([\dot{s}_{\lambda\times a1}, \dot{s}_{\lambda\times b1}][\ddot{s}_{c1}, \ddot{s}_{d1}]) \qquad \lambda \geq 0, \tag{34}$$

$$(\hat{s}_1)^\lambda = ([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}])^\lambda = ([\dot{s}_{(a1)^\lambda}, \dot{s}_{(b1)^\lambda}][\ddot{s}_{c1}, \ddot{s}_{d1}]) \qquad \lambda \geq 0. \tag{35}$$

Let $\hat{s}_1 = ([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}])$, $\hat{s}_2 = ([\dot{s}_{a2}, \dot{s}_{b2}][\ddot{s}_{c2}, \ddot{s}_{d2}])$, and $\hat{s}_3 = ([\dot{s}_{a3}, \dot{s}_{b3}][\ddot{s}_{c3}, \ddot{s}_{d3}])$ be any three 2DULVs, and $\lambda, \lambda_1, \lambda_2 \geq 0$; then they have the following characteristics [19]:

$$\hat{s}_1 \oplus \hat{s}_2 = \hat{s}_2 \oplus \hat{s}_1, \tag{36}$$

$$\hat{s}_1 \otimes \hat{s}_2 = \hat{s}_2 \otimes \hat{s}_1, \tag{37}$$

$$\hat{s}_1 \oplus \hat{s}_2 \oplus \hat{s}_3 = \hat{s}_1 \oplus (\hat{s}_2 \oplus \hat{s}_3), \tag{38}$$

$$\hat{s}_1 \otimes \hat{s}_2 \otimes \hat{s}_3 = \hat{s}_1 \otimes (\hat{s}_2 \otimes \hat{s}_3), \tag{39}$$

$$\hat{s}_1 \otimes (\hat{s}_2 \oplus \hat{s}_3) = (\hat{s}_1 \otimes \hat{s}_2) \oplus (\hat{s}_1 \otimes \hat{s}_3), \tag{40}$$

$$\lambda(\hat{s}_1 \oplus \hat{s}_2) = (\lambda\hat{s}_1) \oplus (\lambda\hat{s}_2), \tag{41}$$

$$(\lambda_1 + \lambda_2)\hat{s}_1 = (\lambda_1\hat{s}_1) \oplus (\lambda_2\hat{s}_1). \tag{42}$$

In the following, we will give the expected value and comparison method for 2DULVs:

Definition 14 [19]. Let $\hat{s}_1 = ([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}])$ be a 2DULV; then, the expected value of $\hat{s}_1$, i.e. $E(\hat{s}_1)$, can be defined as follows:

$$E(\hat{s}_1) = \frac{a1 + b1}{2 \times (l-1)} \times \frac{c1 + d1}{2 \times (t-1)}. \tag{43}$$

Example 1. Let $S_{\text{I}} = (\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_8)$ and $S_{\text{II}} = (\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_4)$ be the predefined linguistic sets of I class and II class, for a 2DULV $\hat{s}_1 = ([\dot{s}_4, \dot{s}_7][\ddot{s}_2, \ddot{s}_3])$; its expected value, $E(\hat{s}_1)$, can be calculated as follows:

$$E(\hat{s}_1) = \frac{4+7}{2 \times (9-1)} \times \frac{2+3}{2 \times (5-1)} = 1.3125.$$

Let $\hat{s}_1 = ([\dot{s}_{a1}, \dot{s}_{b1}][\ddot{s}_{c1}, \ddot{s}_{d1}])$ and $\hat{s}_2 = ([\dot{s}_{a2}, \dot{s}_{b2}][\ddot{s}_{c2}, \ddot{s}_{d2}])$ be any two 2DULVs; if $a1 \geq a2$, $b1 \geq b2$, $c1 \geq c2$, $d1 \geq d2$, then $\hat{s}_1 \geq \hat{s}_2$.

In many cases, the above conditions cannot be satisfied. In order to compare two 2DULVs, we can compare them by their expected values, i.e. if $E(\hat{s}_1) \geq E(\hat{s}_2)$, then $\hat{s}_1 \geq \hat{s}_2$, or vice versa.

4. Some normalized weighted geometric Bonferroni mean for 2DULVs

The NWGBM and GNWGBM can only aggregate arguments, which take the form of crisp number, and cannot aggregate the 2DULVs. In this section, we will extend the NWGBM and GNWGBM to aggregate the 2DULVs and propose 2-Dimensional Uncertain Linguistic Normalized Weighted Geometric Bonferroni Mean (2DULNWGBM) and 2-Dimensional Uncertain Linguistic Generalized Normalized Weighted Geometric Bonferroni Mean (2DULGNWGBM) described as follows.

4.1. 2DULNWGBM

Firstly, we give the definition of the 2DULNWGBM operator.

Definition 15. Let $p, q \geq 0$ and $\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$ $(j = 1, 2, \cdots, n)$ be a collection of 2DULVs with the weight vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j > 0$ and $\sum_{j=1}^n w_j = 1$; then, a 2-Dimensional Uncertain Linguistic Normalized Weighted Geometric Bonferroni Mean (2DULNWGBM) of dimension n is a mapping 2DULNWGBM: $\Omega^n \to \Omega$, and has:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) = \frac{1}{p+q} \underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}} (p\hat{s}_i \oplus q\hat{s}_j)^{\frac{w_i w_j}{1-w_i}}, \tag{44}$$

where Ω is the set of all 2DULVs.

Theorem 6. Let $p,q \geq 0$ and $\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$ $(j = 1,2,\cdots,n)$ be a collection of 2DULVs with the weight vector:

$$w = (w_1, w_2, \cdots, w_n)^T$$

such that $w_j > 0$ and $\sum_{j=1}^n w_j = 1$; then, the aggregated value by Eq. (44) can be expressed as:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \left(\left[\dot{s}_{\frac{1}{p+q} \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i+qa_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{\frac{1}{p+q} \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (pb_i+qb_j)^{\frac{w_i w_j}{1-w_i}}} \right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i}) \right] \right). \quad (45)$$

Proof. By the operational rules of the 2DULVs, we have:

$$p\hat{s}_i = ([\dot{s}_{pa_i}, \dot{s}_{pb_i}][\ddot{s}_{c_i}, \ddot{s}_{d_i}]),$$

$$q\hat{s}_j = ([\dot{s}_{qa_j}, \dot{s}_{qb_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}]),$$

$$p\hat{s}_i \oplus q\hat{s}_j = \left([\dot{s}_{pa_i+pa_j}, \dot{s}_{pb_i+qb_j}] [\min(\ddot{s}_{c_i}, \ddot{s}_{c_j}), \min(\ddot{s}_{d_i}, \ddot{s}_{d_j})]\right),$$

$$(p\hat{s}_i \oplus q\hat{s}_j)^{\frac{w_i w_j}{1-w_i}} = \left(\left[\dot{s}_{(pa_i+qa_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{(pb_i+qb_j)^{\frac{w_i w_j}{1-w_i}}} \right], \left[\min(\ddot{s}_{c_i}, \ddot{s}_{c_j}), \min(\ddot{s}_{d_i}, \ddot{s}_{d_j}) \right] \right).$$

Then:

$$\bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} (p\hat{s}_i \oplus q\hat{s}_j)^{\frac{w_i w_j}{1-w_i}}$$

$$= \left(\left[\dot{s}_{\prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i+qa_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{\prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (pb_i+qb_j)^{\frac{w_i w_j}{1-w_i}}} \right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i}) \right] \right),$$

$$\frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \neq j}}^{n} (p\hat{s}_i \oplus q\hat{s}_j)^{\frac{w_i w_j}{1-w_i}}$$

$$= \left(\left[\dot{s}_{\frac{1}{p+q} \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i+qa_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{\frac{1}{p+q} \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (pb_i+qb_j)^{\frac{w_i w_j}{1-w_i}}} \right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i}) \right] \right).$$

Further:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \left(\left[\dot{s}_{\frac{1}{p+q} \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i+qa_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{\frac{1}{p+q} \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (pb_i+qb_j)^{\frac{w_i w_j}{1-w_i}}} \right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i}) \right] \right).$$

It can be proved that 2DULNWGBM also has the properties shown as follows:

Theorem 7 (Reducibility).
Let $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$; then:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) = \text{2DULGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n). \quad (46)$$

Proof. Since $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, according to Eq. (45), we have the equation shown in Box I.

Theorem 8 (Idempotency).
Let $\hat{s}_j = ([\dot{s}_a, \dot{s}_b][\ddot{s}_c, \ddot{s}_d])(j = 1,2,\cdots,n)$, then:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) = ([\dot{s}_a, \dot{s}_b][\ddot{s}_c, \ddot{s}_d]). \quad (47)$$

Proof. Since $\hat{s}_j = ([\dot{s}_a, \dot{s}_b][\ddot{s}_c, \ddot{s}_d])$, $j = 1,2,\cdots,n$, according to Eq. (45), we have the equation shown in Box II.

Theorem 9 (Commutativity).
Let $\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$ $(j = 1,2,\cdots,n)$ be a collection of 2DULVs and $(\hat{\dot{s}}_1, \hat{\dot{s}}_2, \cdots, \hat{\dot{s}}_n)$ be any permutation of $(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$; then:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) = \text{2DULNWGBM}^{p,q}(\hat{\dot{s}}_1, \hat{\dot{s}}_2, \cdot s, \hat{\dot{s}}_n). \quad (48)$$

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1,\hat{s}_2,\cdots,\hat{s}_n)=\left(\left[\dot{s}_{\frac{1}{p+q}\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(pa_i+qa_j)^{\frac{w_iw_j}{1-w_i}}},\dot{s}_{\frac{1}{p+q}\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(pb_i+qb_j)^{\frac{w_iw_j}{1-w_i}}}\right],\left[\min_i(\ddot{s}_{c_i}),\min_i(\ddot{s}_{d_i})\right]\right)$$

$$=\left(\left[\dot{s}_{\frac{1}{p+q}\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(pa_i+qa_j)^{\frac{1}{n(n-1)}}},\dot{s}_{\frac{1}{p+q}\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(pb_i+qb_j)^{\frac{1}{n(n-1)}}}\right],\left[\min_i(\ddot{s}_{c_i}),\min_i(\ddot{s}_{d_i})\right]\right)$$

$$=\text{2DULGBM}^{p,q}(\hat{s}_1,\hat{s}_2,\cdots,\hat{s}_n).$$

Box I

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1,\hat{s}_2,\cdots,\hat{s}_n)=\left(\left[\dot{s}_{\frac{1}{p+q}\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(pa_i+qa_j)^{\frac{w_iw_j}{1-w_i}}},\dot{s}_{\frac{1}{p+q}\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(pb_i+qb_j)^{\frac{w_iw_j}{1-w_i}}}\right],\left[\min_i(\ddot{s}_{c_i}),\min_i(\ddot{s}_{d_i})\right]\right)$$

$$=\left(\left[\dot{s}_{\frac{1}{p+q}\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(pa+qa)^{\frac{w_iw_j}{1-w_i}}},\dot{s}_{\frac{1}{p+q}\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}(pb+qb)^{\frac{w_iw_j}{1-w_i}}}\right],\left[\min_i(\ddot{s}_{c}),\min_i(\ddot{s}_{d})\right]\right)$$

$$=\left(\left[\dot{s}_{\frac{1}{p+q}a^{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{w_iw_j}{1-w_i}}(p+q)^{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{w_iw_j}{1-w_i}}},\dot{s}_{\frac{1}{p+q}b^{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{w_iw_j}{1-w_i}}(p+q)^{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{w_iw_j}{1-w_i}}}\right],[\ddot{s}_c,\ddot{s}_d]\right)$$

$$=\left(\left[\dot{s}_{\frac{1}{p+q}a(p+q)},\dot{s}_{\frac{1}{p+q}b(p+q)}\right],[\ddot{s}_c,\ddot{s}_d]\right)=\left([\dot{s}_a,\dot{s}_b],[\ddot{s}_c,\ddot{s}_d]\right).$$

Box II

Proof. Since $(\dot{\hat{s}}_1,\dot{\hat{s}}_2,\cdots,\dot{\hat{s}}_n)$ is any permutation of $(\hat{s}_1,\hat{s}_2,\cdots,\hat{s}_n)$, then:

$$\frac{1}{p+q}\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}(p\hat{s}_i\oplus q\hat{s}_j)^{\frac{w_iw_j}{1-w_i}}$$

$$=\frac{1}{p+q}\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\dot{\hat{s}}_i\oplus q\dot{\hat{s}}_j\right)^{\frac{w_iw_j}{1-w_i}}.$$

Therefore:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1,\hat{s}_2,\cdots,\hat{s}_n)$$

$$=\text{2DULNWGBM}^{p,q}\left(\dot{\hat{s}}_1,\dot{\hat{s}}_2,\cdots,\dot{\hat{s}}_n\right).$$

Theorem 10 (Monotonicity).
Let $\hat{s}_j=([\dot{s}_{a_j},\dot{s}_{b_j}][\ddot{s}_{c_j},\ddot{s}_{d_j}])$ and $\dot{\hat{s}}_j=([\dot{s}_{\dot{a}_j},\dot{s}_{\dot{b}_j}][\ddot{s}_{\dot{c}_j},\ddot{s}_{\dot{d}_j}])$ $(j=1,2,\cdots,n)$ be two collections of 2DULVs. If $a_j\geq\dot{a}_j$, $b_j\geq\dot{b}_j$, $c_j\geq\dot{c}_j$, $d_j\geq\dot{d}_j$ for all j, then:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1,\hat{s}_2,\cdots,\hat{s}_n)$$

$$\geq\text{2DULNWGBM}^{p,q}\left(\dot{\hat{s}}_1,\dot{\hat{s}}_2,\cdots,\dot{\hat{s}}_n\right).\qquad(49)$$

Proof. Since $a_j\geq\dot{a}_j$, $b_j\geq\dot{b}_j$, $c_j\geq\dot{c}_j$, $d_j\geq\dot{d}_j$ for all j and $p,q>0$, we have:

$$pa_i+qa_j\geq p\dot{a}_i+q\dot{a}_j,$$

$$(pa_i + qa_j)^{\frac{w_i w_j}{1-w_i}} \geq (p\dot{a}_i + q\dot{a}_j)^{\frac{w_i w_j}{1-w_i}}.$$

Further, we have:

$$\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(pa_i+qa_j)^{\frac{w_i w_j}{1-w_i}} \geq \frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(p\dot{a}_i+q\dot{a}_j)^{\frac{w_i w_j}{1-w_i}}.$$

Similarly, we have:

$$\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(pb_i+qb_j)^{\frac{w_i w_j}{1-w_i}} \geq \frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\dot{b}_i+q\dot{b}_j\right)^{\frac{w_i w_j}{1-w_i}}.$$

Therefore, we have:

$$\left(\left[\dot{s}_{\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(pa_i+qa_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(pb_i+qb_j)^{\frac{w_i w_j}{1-w_i}}}\right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i})\right]\right)$$

$$\geq \left(\left[\dot{s}_{\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(p\dot{a}_i+q\dot{a}_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(p\dot{b}_i+q\dot{b}_j)^{\frac{w_i w_j}{1-w_i}}}\right], \left[\min_i(\ddot{s}_{\dot{c}_i}), \min_i(\ddot{s}_{\dot{d}_i})\right]\right),$$

i.e.:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) \geq \text{2DULNWGBM}^{p,q}(\hat{\dot{s}}_1, \hat{\dot{s}}_2, \cdots, \hat{\dot{s}}_n).$$

Theorem 11 (Boundedness).
Let $\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$ $(j = 1, 2, \cdots, n)$ be a collection of 2DULVs and:

$$\hat{s}^- = \left(\left[\dot{s}_{\min_j(a_j)}, \dot{s}_{\min_j(b_j)}\right], \left[\ddot{s}_{\min_j(c_j)}, \ddot{s}_{\min_j(d_j)}\right]\right),$$

$$\hat{s}^+ = \left(\left[\dot{s}_{\max_j(a_j)}, \dot{s}_{\max_j(b_j)}\right], \left[\ddot{s}_{\min_j(c_j)}, \ddot{s}_{\min_j(d_j)}\right]\right),$$

then:

$$\hat{s}^- \leq \text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) \leq \hat{s}^+. \quad (50)$$

Proof. Since $\hat{s}_j \geq \hat{s}^-$, based on Theorems 8 and 10, we have:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) \geq \text{2DULNWGBM}^{p,q}(\hat{s}^-, \hat{s}^-, \cdots, \hat{s}^-) = \hat{s}^-.$$

Likewise, we can get:

$$\text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) \leq \text{2DULNWGBM}^{p,q}(\hat{s}^+, \hat{s}^+, \cdots, \hat{s}^+) = \hat{s}^+.$$

Then:

$$\hat{s}^- \leq \text{2DULNWGBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) \leq \hat{s}^+.$$

Further, we will explore some special cases of the 2DULNWGBM for parameters p and q.

If $q = 0$, then Eq. (45) reduces to the 2-dimensional uncertain linguistic weighted geometric mean operator as follows:

$$\text{2DULNWGBM}^{p,0}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) = \left(\left[\dot{s}_{\frac{1}{p}\prod_{i=1}^{n}(pa_i)^{w_i}}, \dot{s}_{\frac{1}{p}\prod_{i=1}^{n}(pb_i)^{w_i}}\right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i})\right]\right)$$

$$= \left(\left[\dot{s}_{\prod_{i=1}^{n} a_i^{w_i}}, \dot{s}_{\prod_{i=1}^{n} b_i^{w_i}}\right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i})\right]\right). \quad (51)$$

Obviously, when $q = 0$, 2DULNWGBM^{p,0} does not have any relationship with p.

If $p = q$, then Eq. (45) reduces to the following form:

$$\text{2DULNWGBM}^{p,p}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) = \left(\left[\dot{s}_{\frac{1}{2}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(a_i+a_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{\frac{1}{2}\prod_{\substack{i,j=1\\ i\neq j}}^{n}(b_i+b_j)^{\frac{w_i w_j}{1-w_i}}}\right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i})\right]\right). \quad (52)$$

Likewise, when $p = q$, 2DULNWGBM^{p,p} does not have any relationship with p.

4.2. 2DULGNWGBM

In this section, we will further propose a new definition, i.e. 2DULGNWGBM operator, which is shown as follows.

Definition 16. Let $p,q,r \geq 0$ and $\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$ $(j = 1,2,\cdots,n)$ be a collection of 2DULVs with the weight vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j > 0$ and $\sum_{j=1}^n w_j = 1$; then, a 2-Dimensional Uncertain Linguistic Generalized Normalized Weighted Geometric Bonferroni Mean (2DULGNWGBM) of dimension n is a mapping 2DULGNWGBM: $\Omega^n \to \Omega$, and has:

$$\text{2DULGNWGBM}^{p,q,r}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \frac{1}{p+q+r} \underset{\substack{i,j,k=1 \\ i \neq j \neq k}}{\overset{n}{\otimes}} (p\hat{s}_i \oplus q\hat{s}_j \oplus r\hat{s}_k)^{\frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)}}, \tag{53}$$

where Ω is the set of all the 2DULVs.

Theorem 12.
Let $p,q,r \geq 0$ and $\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$ $(j = 1,2,\cdots,n)$ be a collection of 2DULVs with the weight vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j > 0$ and $\sum_{j=1}^n w_j = 1$; then, the aggregated value by Eq. (53) can be expressed as shown in Box III. Similar to Theorem 6, the proof of Theorem 12 is omitted here.

Similar to the proofs of Theorems 7-11, it is easy to prove that 2DULGNWGBM also has the properties as follows.

Theorem 13 (Reducibility).
Let $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$; then:

$$\text{2DULGNWGBM}^{p,q,r}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \text{2DUGLGBM}^{p,q,r}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n). \tag{55}$$

Theorem 14 (Idempotency).
Let $\hat{s}_j = ([\dot{s}_a, \dot{s}_b][\ddot{s}_c, \ddot{s}_d])$ $(j = 1,2,\cdots,n)$; then:

$$\text{2DULGNWGBM}^{p,q,r}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= ([\dot{s}_a, \dot{s}_b][\ddot{s}_c, \ddot{s}_d]). \tag{56}$$

Theorem 15 (Commutativity).
Let $\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$ $(j = 1,2,\cdots,n)$ be a collection of 2DULVs and $(\hat{\dot{s}}_1, \hat{\dot{s}}_2, \cdots, \hat{\dot{s}}_n)$ be any permutation of $(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$; then:

$$\text{2DULGNWGBM}^{p,q,r}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \text{2DULGNWGBM}^{p,q,r}\left(\hat{\dot{s}}_1, \hat{\dot{s}}_2, \cdots, \hat{\dot{s}}_n\right). \tag{57}$$

Theorem 16 (Monotonicity).
Let:

$$\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$$

and:

$$\hat{\dot{s}}_j = ([\dot{s}_{\dot{a}_j}, \dot{s}_{\dot{b}_j}][\ddot{s}_{\dot{c}_j}, \ddot{s}_{\dot{d}_j}]) \qquad (j = 1,2,\cdots,n),$$

be two collections of 2DULVs. If $a_j \geq \dot{a}_j$, $b_j \geq \dot{b}_j$, $c_j \geq \dot{c}_j$, and $d_j \geq \dot{d}_j$ for all j, then:

$$\text{2DULGNWGBM}^{p,q,r}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$\geq \text{2DULGNWGBM}^{p,q,r}\left(\hat{\dot{s}}_1, \hat{\dot{s}}_2, \cdots, \hat{\dot{s}}_n\right). \tag{58}$$

Theorem 17 (Boundedness).
Let $\hat{s}_j = ([\dot{s}_{a_j}, \dot{s}_{b_j}][\ddot{s}_{c_j}, \ddot{s}_{d_j}])$ $(j = 1,2,\cdots,n)$ be a collection of 2DULVs, and:

$$\hat{s}^- = \left(\left[\dot{s}_{\min_j(a_j)}, \dot{s}_{\min_j(b_j)}\right], \left[\ddot{s}_{\min_j(c_j)}, \ddot{s}_{\min_j(d_j)}\right]\right),$$

$$\hat{s}^+ = \left(\left[\dot{s}_{\max_j(a_j)}, \dot{s}_{\max_j(b_j)}\right], \left[\ddot{s}_{\min_j(c_j)}, \ddot{s}_{\min_j(d_j)}\right]\right),$$

then:

$$\hat{s}^- \leq \text{2DULGNWGBM}^{p,q,r}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) \leq \hat{s}^+. \tag{59}$$

Some special cases of the 2DULGNWGBM for the parameters p and q can be discussed as follows.

If $r = 0$, then Eq. (54) reduces to 2DULGNWGBM as follows:

$$\text{2DUGLNWGBM}^{p,q,r}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \left(\left[\dot{s}_{\frac{1}{p+q+r} \prod\limits_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} (pa_i + qa_j + ra_k)^{\frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)}}}, \dot{s}_{\frac{1}{p+q+r} \prod\limits_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} (pb_i + qb_j + rb_k)^{\frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)}}}\right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i})\right]\right). \tag{54}$$

Box III

$$\text{2DULGNWGBM}^{p,q,0}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \left(\left[\dot{s}_{\frac{1}{p+q} \prod_{\substack{i,j=1 \\ i\neq j}}^{n} (pa_i+qa_j)^{\frac{w_i w_j}{1-w_i}}}, \dot{s}_{\frac{1}{p+q} \prod_{\substack{i,j=1 \\ i\neq j}}^{n} (pb_i+qb_j)^{\frac{w_i w_j}{1-w_i}}} \right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i}) \right] \right). \quad (60)$$

If $q = 0$ and $r = 0$, then Eq. (54) reduces to the 2-dimensional uncertain linguistic weighted geometric mean operator as follows:

$$\text{2DULGNWGBM}^{p,0,0}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \left(\left[\dot{s}_{\prod_{i=1}^{n} a_i^{w_i}}, \dot{s}_{\prod_{i=1}^{n} b_i^{w_i}} \right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i}) \right] \right). \quad (61)$$

If $p = q = r$, then Eq. (54) reduces to the form shown in Box IV.

Likewise, when $p = q = r$, $\text{2DULGNWGBM}^{p,p,p}$ does not have any relationship with p.

5. The MADM method based on the 2DULGNWGBM operator

In what follows, we will apply the 2DULGNWGBM operator to the MADM problems in which the attributes take the form of 2DULVs.

5.1. The description of the MADM problems based on the 2DULVs

Let $A = (a_1, a_2, \cdots, a_m)$ be the set of alternatives, $C = \{c_1, c_2, \cdots, c_n\}$ be the set of attributes, and the attribute weight vector be $W = (w_1, w_2, \cdots, w_n)$, satisfying $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$, where w_j denotes the important degree of the attribute c_j. Suppose that 2DULV $\hat{s}_{ij} = ([x_{ij}^L, x_{ij}^U], [g_{ij}^L, g_{ij}^U])$ is the attribute value in the attribute c_j with respect to the alternative a_i, where $[x_{ij}^L, x_{ij}^U]$ is the I class of ULV and $x_{ij}^L, x_{ij}^U \in S_I$, $S_I = (\dot{s}_0, \dot{s}_1, \cdots, \dot{s}_{l-1})$ is the predefined linguistic set. $[g_{ij}^L, g_{ij}^U]$ is the II class of ULV and $g_{ij}^L, g_{ij}^U \in S_{\text{II}}$, $S_{\text{II}} = (\ddot{s}_0, \ddot{s}_1, \cdots, \ddot{s}_{t-1})$ is the predefined linguistic set. The goal of this decision making is to rank the alternatives.

5.2. Decision making steps

- **Step 1.** Calculate the comprehensive value of each alternative. Based on the 2DULGNWGBM operator, we can calculate the comprehensive value of each alternative:

$$\hat{z}_i = ([x_i^L, x_i^U], [g_i^L, g_i^U]) = \text{2DULGNWGBM}^{p,q,r}(\hat{s}_{i1}, \hat{s}_{i2}, \cdots, \hat{s}_{in}).$$

- **Step 2.** Rank the alternatives. Because $\hat{z}_i$ is a 2DULV, we can get the expected value $E(\hat{z}_i)$. The larger the value of $E(\hat{z}_i)$, the better the alternative would be.

6. Illustrative example

This is an example about the enterprise technological innovation ability evaluation (Cited from [22]). There are four enterprises $\{a_1, a_2, a_3, a_4\}$, which are evaluated by the following four attributes: resources input (c_1), innovation management (c_2), innovation tendency (c_3), and the research and development (c_4). Suppose the attribute values are expressed by 2DULVs shown in Table 1. The I class and II class of the linguistic evaluation sets are $S_I = (\dot{s}_0, \dot{s}_1, \dot{s}_2, \dot{s}_3, \dot{s}_4, \dot{s}_5, \dot{s}_6)$ and $S_{\text{II}} = (\ddot{s}_0, \ddot{s}_1, \ddot{s}_2, \ddot{s}_3, \ddot{s}_4)$, respectively. $W = (0.25, 0.27, 0.25, 0.23)$ is the weight vector of the attributes. Please rank the four enterprises according to their technological innovation ability.

6.1. The evaluation steps by 2DULGNWGBM operator

- **Step 1.** Calculate the comprehensive value $\hat{z}_i$ of each alternative by the 2DULGNWGBM operator (suppose $p = q = r = 1$); we can get:

$$\hat{z}_1 = ([\dot{s}_{3.473}, \dot{s}_{4.233}], [\ddot{s}_2, \ddot{s}_3]),$$

$$\hat{z}_2 = ([\dot{s}_{3.749}, \dot{s}_{4.002}], [\ddot{s}_1, \ddot{s}_2]),$$

$$\hat{z}_3 = ([\dot{s}_{2.980}, \dot{s}_{3.982}], [\ddot{s}_1, \ddot{s}_2]),$$

$$\text{2DULGNWGBM}^{p,p,p}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \left(\left[\dot{s}_{\frac{1}{2} \prod_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} (a_i+a_j+a_k)^{\frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)}}}, \dot{s}_{\frac{1}{2} \prod_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} (b_i+b_j+b_k)^{\frac{w_i w_j w_k}{(1-w_i)(1-w_i-w_j)}}} \right], \left[\min_i(\ddot{s}_{c_i}), \min_i(\ddot{s}_{d_i}) \right] \right). \quad (62)$$

Box IV

Table 1. The ratings of the enterprises with respect to multiple criteria.

Enterprises	Attribute (c_1)	Attribute (c_2)	Attribute (c_3)	Attribute (c_4)
a_1	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$
a_2	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_1, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_3], [\ddot{s}_2, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_2, \ddot{s}_2])$
a_3	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_1, \ddot{s}_2])$
a_4	$([\dot{s}_5, \dot{s}_6], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_2], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_4, \ddot{s}_4])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_2])$

Table 2. The ranking results by the different values of p, q, and r in 2DULGNWGBM operator.

p, q, r	Expected values $E(\hat{z}_i)$ $(i = 1, 2, 3, 4)$	Ranking
$p = 0.0001$, $q = r = 0$	$E(\hat{z}_1) = 0.387$, $E(\hat{z}_2) = 0.239$ $E(\hat{z}_3) = 0.213$, $E(\hat{z}_4) = 0.257$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = q = 1$, r=0	$E(\hat{z}_1) = 0.398$, $E(\hat{z}_2) = 0.242$ $E(\hat{z}_3) = 0.216$, $E(\hat{z}_4) = 0.272$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = 2$, $q = 1$, $r = 0$	$E(\hat{z}_1) = 0.396$, $E(\hat{z}_2) = 0.241$ $E(\hat{z}_3) = 0.216$, $E(\hat{z}_4) = 0.270$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = 10$, $q = 1$ $r = 0$	$E(\hat{z}_1) = 0.390$, $E(\hat{z}_2) = 0.240$ $E(\hat{z}_3) = 0.214$, $E(\hat{z}_4) = 0.262$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = q = 1$, $r = 1$	$E(\hat{z}_1) = 0.401$, $E(\hat{z}_2) = 0.242$ $E(\hat{z}_3) = 0.218$, $E(\hat{z}_4) = 0.278$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = 2$, $q = 1$, $r = 1$	$E(\hat{z}_1) = 0.400$, $E(\hat{z}_2) = 0.242$ $E(\hat{z}_3) = 0.217$, $E(\hat{z}_4) = 0.276$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = 10$, $q = 1$, $r = 1$	$E(\hat{z}_1) = 0.393$, $E(\hat{z}_2) = 0.240$ $E(\hat{z}_3) = 0.215$, $E(\hat{z}_4) = 0.266$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = 1$, $q = 2$, $r = 1$	$E(\hat{z}_1) = 0.401$, $E(\hat{z}_2) = 0.242$ $E(\hat{z}_3) = 0.217$, $E(\hat{z}_4) = 0.266$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = 1$, $q = 10$, $r = 1$	$E(\hat{z}_1) = 0.394$, $E(\hat{z}_2) = 0.240$ $E(\hat{z}_3) = 0.215$, $E(\hat{z}_4) = 0.267$	$a_1 \succ a_4 \succ a_2 \succ a_3$
$p = 10$, $q = 10$, $r = 1$	$E(\hat{z}_1) = 0.398$, $E(\hat{z}_2) = 0.242$ $E(\hat{z}_3) = 0.217$, $E(\hat{z}_4) = 0.273$	$a_1 \succ a_4 \succ a_2 \succ a_3$

$\hat{z}_4 = ([\dot{s}_{2.962}, \dot{s}_{3.699}], [\ddot{s}_2, \ddot{s}_2])$.

- **Step 2.** Calculate the expected values:

$$E(\hat{z}_1) = 0.401, \qquad E(\hat{z}_2) = 0.242,$$

$$E(\hat{z}_3) = 0.218, \qquad E(\hat{z}_4) = 0.278.$$

- **Step 3.** Rank the alternatives. Based on the expected values, the order of the four enterprises $\{a_1, a_2, a_3, a_4\}$ is $a_1 \succ a_4 \succ a_2 \succ a_3$.

6.2. The influence of the parameters p, q, and r on decision making

The influences of the parameters p, q, and r on evaluation results are shown in Table 2.

From Table 2, we find that parameters p, q, and r will influence the ranking results in the 2DULGN-WGBM operator. Generally speaking, we can set $p = q = r = 1$, and this is the simplest situation by considering the interrelationships between two arguments.

6.3. Comparison with the existing methods

In order to verify the validity of the developed method in this paper, we can solve this illustrative example by the method proposed by Liu and Zhang [16]; we will obtain the same ranking results. However, the proposed method can consider the interrelationships between two arguments, a function that the method by Liu and Zhang [16] lacks.

Similarly, this method has the same ranking results with the method proposed by Liu et al. [22]; however, both have their advantages. The method proposed by Liu et al. [22] is the generalization of Weighted Arithmetic (WA) operator, Weighted Geometric (WG) operator, Ordered WA (OWA), and Ordered WG (OWG), while the method in this paper can take the interrelationships between two arguments into account.

Compared with the methods proposed by Wei et al. [43,45] and the one proposed by Tan et al. [49], these two methods can consider the interrelationships between two arguments based on BM; however, the method proposed by Wei et al. [43] can only deal with the uncertain or simplified neutrosophic linguistic information and the method proposed by Wei et al. [45] can process the 2-tupe linguistic information, while the proposed method in this paper can process the 2-dimensional uncertain linguistic information. Obviously, these two methods are applied to solve different decision making methods with different attributes. However, the methods proposed by Wei et al. [43,45] are only special cases of our proposed methods.

In a word, 2DULVs can better express the fuzzy or uncertain information by adding the second dimensional information and BM can take the interrelationships between two arguments into account. The proposed 2DULNWGBM and 2DULGNWGBM operators for 2DULVs in this paper can generalize the existing BM operators for linguistic information and they can overcome the shortcomings with no reducibility.

7. Conclusions

Bonferroni mean can capture the interrelationships between two arguments and 2DULVs can easily describe the fuzzy information. Based on the NWBM and GNWBM, the NWGBM and GNWGBM operators were proposed, and their characteristics were discussed. Furthermore, for 2-dimensional uncertain linguistic information, we proposed the 2-Dimensional Uncertain Linguistic Normalized Weighted Geometric Bonferroni Mean (2DULNWGBM) and the 2-Dimensional Uncertain Linguistic Generalized Normalized Weighted Geometric Bonferroni Mean (2DULGNWGBM), and some desirable properties and special cases of 2DULNWGBM and 2DULGNWGBM were investigated in detail; also, a multi-criteria decision making method with 2DULVs was developed based on these operators. In further research, it is important to apply these operators in other domains such as fuzzy cluster analysis, pattern recognition, uncertain programming, etc., or extend the proposed operators and methods to other uncertain information such as interval neutrosophic uncertain linguistic variables or neutrosophic soft sets [51,52].

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Biography

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