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Numerical solutions of Fourier's law involving fractional derivatives with bi-order

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Abstract. In this paper, we present an alternative representation of the fractional space-time Fourier's law equation using the concept of derivative with two fractional orders α and β. The new definitions are based on the concept of power law and the generalized Mittag-Leffler function, where the first fractional order is incorporated into the power law function, and the second fractional order is the generalized Mittag-Leffler function. The new approach is capable of considering media with two different layers, scales, and properties. The generalization of this equation exhibits different cases of anomalous behaviors and Non-Fourier heat conduction processes. Numerical solutions are obtained using an iterative scheme.

1. Introduction

Fourier's law combined with the energy conservation principle is the basis for the analysis of most conduction processes [1]. However, the Fourier's equation is not adequate to describe certain processes in some cases. Fractional Calculus (FC) is the generalization of ordinary calculus. In recent years, fractional systems have been considered in many publications, e.g., biomedical, electromagnetism, electrical circuits, and transport phenomena [2-10].

The study of non-Fourier heat conduction processes has attracted much interest in recent years. Mainardi et al. [11] presented the interpretation of the corresponding Green function as a probability density and obtained the fundamental space-time fractional diffusion equation from the standard diffusion equation. Qi and Jiang [12] derived the exact solution to the Cattaneo-Vernotte equation by joint Laplace and Fourier transforms. Liu et al. [13] proposed a new time-space fractional Cattaneo-Christov upper-convective derivative flux heat conduction model. In this work, the space fractional derivative was characterized by the weight coefficient of forward versus backward transition probability. Ezzat et al. [14] presented a new mathematical model of heat conduction; they considered the isotropic generalized thermoelasticity with a three-phase lag, and their proposed model was considered as the methodology of FC. Zhao et al. [15] studied the Soret-Dufour effects on Maxwell fluid embedded in Darcy-Boussinesq medium. Other applications of FC in heat conduction are given in [16-21].

The fractional order that appears in the Riemann-Liouville or Liouville-Caputo operators can be used to represent some physical parameters. Nevertheless, it is not possible for these operators to be used in order

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to describe the movement of heat via material with different layers, where each layer possesses different materials. To solve the problem, Abdon Atangana [22] introduced fractional operators with two orders in Riemann-Liouville and Liouville-Caputo sense using the concept of fractional derivative with non-local and non-singular kernel. These operators allow for describing problems that are more complex with different layers and different properties, e.g., the problem in the case of thermal conduction where the heat is flowing within a medium with two different properties. In this context, the aim of this contribution is to present an alternative representation of the fractional-time Fourier's law equation using the concept of derivative with two fractional orders α and β. The fractional orders considered are $n-1<\alpha<n$ and $0<\beta<1$ for the fractional equation.

The paper is organized as follows: Section 2 presents new definitions of fractional operators with bi-order; Section 3 discusses the fractional-time Fourier's law equation; Section 4 concludes the paper.

2. Fractional operators with bi-order

In the following, some definitions of fractional operators with bi-order are presented [22].

The Atangana-Caputo (AC) definition with bi-order (α,β) for $f(t)$ is:

$$ {}_0^{AC}D_t^{\alpha,\beta}f(t)=\frac{B(\beta)}{n-\beta}\cdot\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{d^n}{dt^n}f(t) $$

$$ (t-\tau)^{1-\alpha-n}E_\beta\Big\{-\frac{\beta}{n-\beta}(t-\tau)^{\beta+\alpha}\Big\}d\tau. $$

$$ n-1<\beta<n,\qquad n-1<\alpha<n. \tag{1} $$

The Laplace transform of Eq. (1) produces:

$$ \mathcal{L}[{}_0^{AC}\mathcal{D}_t^{\alpha,\beta}f(t)](s)=\frac{B(\beta)}{1-\beta}\cdot\frac{1}{\Gamma(1-\alpha)}\{sF(s)-f(0)\} $$

$$ \mathcal{L}\Big\{t^{-\alpha}E_\beta\Big(-\frac{\beta}{1-\beta}t^{\beta+\alpha}\Big\}\Big\}, \tag{2} $$

where:

$$ \mathcal{L}\Big\{t^{-\alpha}E_\beta\Big(-\frac{\beta}{1-\beta}t^{\beta+\alpha}\Big)\Big\}=s^{\alpha-1}{}_2\Psi_1 $$

$$ \begin{bmatrix}(1,1),(1-\alpha,\alpha+\beta); & \\ & -\Big(\frac{\beta}{1-\beta}\cdot\frac{1}{s^{\alpha+\beta}}\Big)\\ (1,\beta); & \end{bmatrix}, \tag{3} $$

function ${}_a\Psi_b$ is the Wright's generalized hypergeometric function [23].

Substituting Wright's function (3) into Eq. (2), we obtain:

$$ \mathcal{L}[{}_0^{AC}\mathcal{D}_t^{\alpha,\beta}f(t)](s)=\frac{B(\beta)}{1-\beta}\cdot\frac{s}{\Gamma(1-\alpha)}{}_2\Psi_1 $$

$$ \begin{bmatrix}(1,1),(1-\alpha,\alpha+\beta); & \\ & -\Big(\frac{\beta}{1-\beta}\cdot\frac{1}{s^{\alpha+\beta}}\Big)\\ (1,\beta); & \end{bmatrix}F(s) $$

$$ -\frac{B(\beta)}{1-\beta}\cdot\frac{s^{\alpha-1}}{\Gamma(1-\alpha)}{}_2\Psi_1 $$

$$ \begin{bmatrix}(1,1),(1-\alpha,\alpha+\beta); & \\ & -\Big(\frac{\beta}{1-\beta}\cdot\frac{1}{s^{\alpha+\beta}}\Big)\\ (1,\beta); & \end{bmatrix}f(0). \tag{4} $$

For this operator, the inverse Laplace transform is defined by:

$$ {}_0^{AC}\mathcal{D}_t^{\alpha,\beta}f(t)={}_0^{AR}\mathcal{D}_t^{\alpha,\beta}f(t)-\frac{B(\beta)}{1-\beta}\cdot\frac{1}{\Gamma(1-\alpha)} $$

$$ \cdot f(0)\ t^{-\alpha}E_\beta\{-t^{\beta+\alpha}\}. \tag{5} $$

For the fractional operator with bi-order (1), when $\alpha=0$, we recover Atangana-Baleanu fractional derivative in Liouville-Caputo sense (ABC). This operator is defined as follows [24]:

$$ {}_0^{\mathrm{ABC}}\mathcal{D}_t^{\beta}f(t)=\frac{B(\beta)}{1-\beta}\int_0^t\dot{f}(\tau)E_\beta\Big[-\frac{\beta}{1-\beta}(t-\tau)^\beta\Big]d\tau, $$

$$ 0<\beta\le 1, \tag{6} $$

where $\frac{d^\beta}{dt^\beta}={}_0^{\mathrm{ABC}}\mathcal{D}_t^{\beta}$ is an ABC fractional derivative with respect to t, and $B(\beta)$ is a normalization function that has the same properties as in Caputo and Caputo-Fabrizio case.

The Laplace transform of Eq. (6) is defined as follows:

$$ \mathcal{L}[{}_0^{\mathrm{ABC}}\mathcal{D}_t^{\beta}f(t)](s)=\frac{B(\beta)}{1-\beta}\mathcal{L}\Big[\int_a^t\dot{f}(\tau)E_\beta $$

$$ \Big[-\frac{\beta}{1-\beta}(t-\tau)^\beta\Big]d\tau\Big] $$

$$ =\frac{B(\beta)}{1-\beta}\frac{s^\beta\mathcal{L}[f(t)](s)-s^{\beta-1}f(0)}{s^\beta+\frac{\beta}{1-\beta}}. \tag{7} $$

For the fractional operator with bi-order Eq. (1), when $\beta=0$, we recover Liouville-Caputo fractional derivative (C). This operator is defined as follows:

$$ {}_0^{C}\mathcal{D}_t^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\frac{d}{dt}f(\tau)d\tau, $$

$$ 0<\alpha\le 1]. \tag{8} $$

3. Fourier's law equation

To keep the dimensionality of the differential equation, new parameters σ_t and σ_x are introduced [25]. For the AC fractional derivative with bi-order Eq. (1), we have:

$$\frac{d}{dt} \to \frac{1}{\sigma_t^{1-\alpha,\beta}} \cdot {}_0^{AC}\mathcal{D}_t^{\alpha,\beta}, \qquad 0<\alpha,\beta\leq 1, \tag{9}$$

$$\frac{d^2}{dx^2} \to \frac{1}{\sigma_x^{2(1-\alpha,\beta)}} \cdot {}_0^{AC}\mathcal{D}_x^{2(\alpha,\beta)}, \qquad 0<\alpha,\beta\leq 1, \tag{10}$$

when $\alpha = 0$ in Eqs. (9) and (10), we obtain the fractional operator of type ABC (6) as follows:

$$\frac{d}{dt} \to \frac{1}{\sigma_t^{1-\beta}} \cdot {}_0^{\mathrm{ABC}}\mathcal{D}_t^{\beta}, \qquad 0<\beta\leq 1, \tag{11}$$

$$\frac{d^2}{dx^2} \to \frac{1}{\sigma_x^{2(1-\beta)}} \cdot {}_0^{\mathrm{ABC}}\mathcal{D}_x^{2\beta}, \qquad 0<\beta\leq 1. \tag{12}$$

For all cases, σ_t has the dimension of time, and σ_x has the dimension of length. These parameters are associated with the temporal and spatial components in the system [25], when ($\alpha = \beta = 1$) and ($\beta = 1$), respectively; the expressions above are recovered in the traditional sense. From now on, this idea will be appliedto the fractional Fourier's law.

The Fourier's law is described by the classical parabolic equation:

$$\chi\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0, \tag{13}$$

where $\chi = \frac{k}{\rho C_p}$, χ is the thermal diffusivity, k is the thermal conductivity, ρ is density, C_p is the specific heat capacity, and T is the temperature. conduction in a planar medium with constant properties.

3.1. Fractional time Fourier's law equation

Considering Eq. (13) and assuming that the time derivative is fractional and the space derivative is ordinary, the temporal fractional equation will be as follows:

$${}_0^{AC}\mathcal{D}_t^{\alpha,\beta} T(x,t) - \chi\frac{\partial^2 T(x,t)}{\partial x^2} = 0. \tag{14}$$

A particular solution to Eq. (13) can be found in the following form:

$$T(x,t) = T_0 \cdot e^{-i\tilde{k}x} u(t). \tag{15}$$

Substituting Eq. (15) into Eq. (14) and considering Eq. (9), we obtain:

$${}_0^{AC}\mathcal{D}_t^{\alpha,\beta} u(t) + \tilde{\omega} u(t) = 0, \tag{16}$$

where $\tilde{\omega} = \omega\sigma_t^{1-\alpha,\beta}$ and $\omega = \chi\tilde{k}^2$ are the angular frequencies.

The numerical approximation of Eq. (16) is given by:

$$\frac{B(\beta)}{1-\beta}\cdot\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{du(\tau)}{d\tau}(t-\tau)^{-\alpha} E_\beta\Big\{-\frac{\beta}{1-\beta}(t-\tau)^{\beta+\alpha}\Big\}d\tau + \tilde{\omega}u(t) = 0,$$

$$\frac{B(\beta)}{1-\beta}\cdot\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u(t)-u(t+\Delta t)}{2\Delta t}(t-\tau)^{-\alpha} E_\beta \Big\{-\frac{\beta}{1-\beta}(t-\tau)^{\beta+\alpha}\Big\}d\tau + \tilde{\omega}u(t) = 0,$$

$$\frac{B(\beta)}{1-\beta}\cdot\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n}\int_{t_i}^{t_{i+1}} \frac{u(t_{i+1})-u(t_i)}{2\Delta t} (t_n-\tau)^{-\alpha} E_\beta\Big\{-\frac{\beta}{1-\beta}(t_n-\tau)^{\beta+\alpha}\Big\}d\tau + \tilde{\omega}\Big[\frac{u(t_{n+1})-u(t_n)}{2}\Big] = 0,$$

$$\frac{B(\beta)}{1-\beta}\cdot\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n}\frac{u(t_{i+1})-u(t_i)}{2\Delta t}\int_{t_n-t_i}^{t_n-t_i+1}\nu^{-\alpha} E_\beta\Big\{-\frac{\beta}{1-\beta}\nu^{\beta+\alpha}\Big\}d\nu + \tilde{\omega}\Big[\frac{u(t_{n+1})-u(t_n)}{2}\Big] = 0, \tag{17}$$

thus:

$$u(t) = \frac{B(\beta)}{1-\beta}\cdot\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n}\frac{u^{i+1}-u^i}{2\Delta t} \Big[(t_n-t_{i+1})^{1-\alpha}E_{\beta,2-\alpha}\Big\{-\frac{\beta}{1-\beta}(t_n-t_{i+1})^{\beta+\alpha}\Big\} - (t_n-t_i)^{1-\alpha}E_{\beta,2-\alpha}\Big\{-\frac{\beta}{1-\beta}(t_n-t_{i+1})^{\beta+\alpha}\Big\}\Big] + \tilde{\omega}\Big[\frac{u(t_{n+1})-u(t_n)}{2}\Big], \tag{18}$$

where E_β is the Mittag-Leffler function.

The particular solution to Eq. (16) is written as follows:

$$T(x,t) = T_0 \cdot e^{-i\tilde{k}x}\Bigg[\frac{B(\beta)}{1-\beta}\cdot\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n}\frac{u^{i+1}-u^i}{2\Delta t}$$

$$\Big[(t_n - t_{i+1})^{1-\alpha}\cdot E_{\beta,2-\alpha}$$

$$\Big\{-\frac{\beta}{1-\beta}(t_n-t_{i+1})^{\beta+\alpha}\Big\}$$

$$-(t_n-t_i)^{1-\alpha}E_{\beta,2-\alpha}$$

$$\Big\{-\frac{\beta}{1-\beta}(t_n-t_{i+1})^{\beta+\alpha}\Big\}\Big]$$

$$+\tilde{\omega}\Big[\frac{u(t_{n+1})-u(t_n)}{2}\Big]\Bigg]. \tag{19}$$

Eq. (19) represents the time thermal diffusion.

If $\alpha = 0$ in Eq. (16), we recover ABC fractional derivative Eq. (6); from Eq. (14), we have:

$${}_0^{\mathrm{ABC}}\mathcal{D}_t^{\beta}T(x,t) - \chi\frac{\partial^2 T(x,t)}{\partial x^2} = 0. \tag{20}$$

A particular solution to this equation can be found in the form of Eq. (15), and by substituting Eq. (15) into Eq. (20) and considering Eq. (11), we obtain:

$${}_0^{\mathrm{ABC}}\mathcal{D}_t^{\alpha,\beta}u(t) + \tilde{\omega}u(t) = 0, \tag{21}$$

where $\tilde{\omega} = \omega\sigma_t^{1-\beta}$ and $\omega = \chi\tilde{k}^2$.

Applying Laplace transforms (7) to (21) and considering $U(0) = u_0$ yield the following expression:

$$U(s) = \frac{(1-\beta)s^{\beta-1}}{s^{\beta}[B(\beta)+\tilde{\omega}(1-\beta)]} + \frac{B(\beta)u_0}{B(\beta)+\tilde{\omega}(1-\beta)}$$

$$\cdot\frac{s^{\beta-1}}{s^{\beta}+\frac{\tilde{\omega}\beta}{B(\beta)+\tilde{\omega}(1-\beta)}}. \tag{22}$$

Taking the inverse Laplace transform of Eq. (22), we obtain the following particular solution of the Eq. (21):

$$T(x,t) = T_0\cdot e^{-i\tilde{k}x}\Bigg\{\frac{1-\beta}{B(\beta)+\tilde{\omega}(1-\beta)}$$

$$\cdot E_{\beta,1}\Bigg[-\Big(\frac{\tilde{\omega}\beta}{B(\beta)+\tilde{\omega}(1-\beta)}\Big)t^{\beta}\Bigg]$$

$$+\frac{B(\beta)u_0}{B(\beta)+\tilde{\omega}(1-\beta)}$$

$$\cdot E_{\beta,1}\Bigg[-\Big(\frac{\tilde{\omega}\beta}{B(\beta)+\tilde{\omega}(1-\beta)}\Big)t^{\beta}\Bigg]\Bigg\}, \tag{23}$$

where E_{β} is the Mittag-Leffler function. In this case, when ($\beta = 1$), we have:

$$\tilde{T}(x,t) = \Re[\tilde{T}_0\cdot e^{-\omega t - i\tilde{k}x}], \tag{24}$$

where $\Re$ indicates the real part. Equation (24) represents the classical case for the time Fourier's law equation.

Figure 1 shows numerical simulations for temperature, $T(x,t)$, considering different values of α and β, in Eq. (19), arbitrarily chosen.

Figure 2 shows numerical simulations for temperature, $T(x,t)$, considering different values of β, in Eq. (23), arbitrarily chosen.

Considering, $\tilde{\omega} = \chi\tilde{k}^2\sigma_t^{1-\beta}$ and $1/\chi$ as the reciprocal of the time constant or thermal diffusion coefficient, we have:

$$\tilde{\omega} = \tilde{k}^2\Big(\frac{\sigma_t^{1-\beta}}{\chi}\Big) = \tilde{k}^2\Big(\frac{1}{\tau_\beta}\Big), \tag{25}$$

where if $\tau_\beta = \chi\sigma_t^{\beta-1}$, it can be called fractional time constant, and when $\beta = 1$, it is the classical time constant; $\tilde{\omega}$ is the angular frequency in the medium in the presence of fractional time components, $\tilde{k}$ is the wave number, and $1/\chi$ is the time constant of the system. Substituting Eq. (25) into Eq. (23), we obtain:

$$T(x,t) = T_0\cdot e^{-ikx}\Bigg\{\frac{1-\beta}{B(\beta)+(\frac{\tilde{k}^2}{\tau_\beta})(1-\beta)}$$

$$\cdot E_{\beta,1}\Bigg[-\Big(\frac{(\frac{\tilde{k}^2}{\tau_\beta})\beta}{B(\beta)+(\frac{\tilde{k}^2}{\tau_\beta})(1-\beta)}\Big)t^{\beta}\Bigg]$$

$$+\frac{B(\beta)u_0}{B(\beta)+(\frac{\tilde{k}^2}{\tau_\beta})(1-\beta)}$$

$$\cdot E_{\beta,1}\Bigg[-\Big(\frac{(\frac{\tilde{k}^2}{\tau_\beta})\beta}{B(\beta)+(\frac{\tilde{k}^2}{\tau_\beta})(1-\beta)}\Big)t^{\beta}\Bigg]\Bigg\}. \tag{26}$$

Figure 3 shows the simulation results of Eq. (26) using fractional exponents $\beta = 1$, $\beta = 0.95$, $\beta = 0.9$, $\beta = 0.85$, and $\beta = 0.80$, respectively.

Table 1 shows different values of the thermal diffusion when β changes from $\beta = 1$, $\beta = 0.95$, $\beta = 0.90$ to $\beta = 0.85$, respectively, when $\beta < 1$, the thermal

Table 1. Thermal diffusion versus Constant time τ_β.

β	Constant time (τ_β)	Thermal diffusion
1	1	0.63
0.95	0.9376	0.63
0.90	0.9216	0.63
0.85	0.9143	0.63

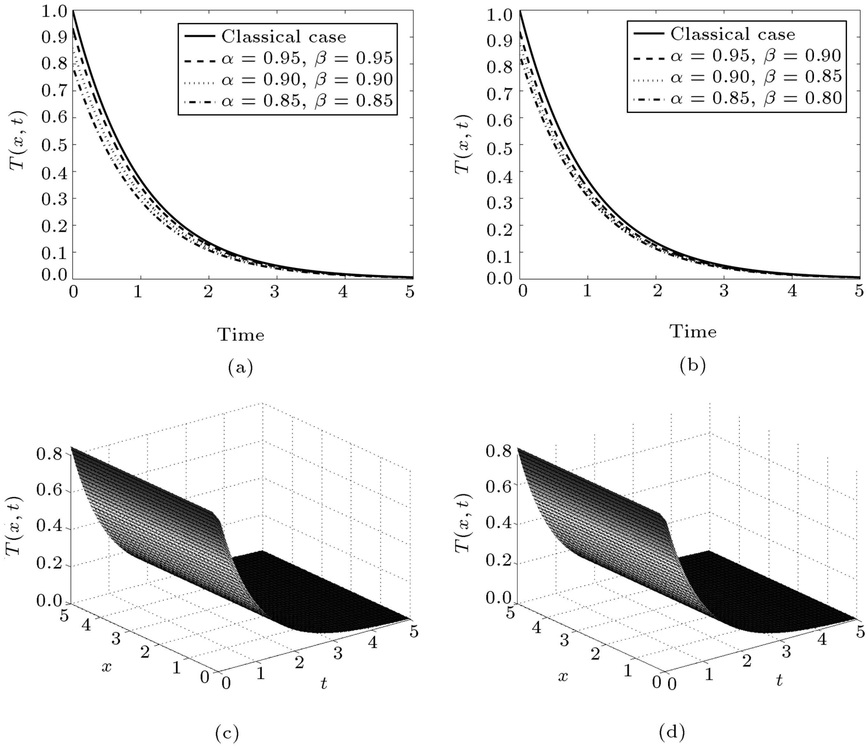

Figure 1. Thermal diffusion for different values of α and β arbitrarily chosen, using numerical evaluation of Eq. (19): (a) α and β take the same value, (b) α and β take different values, (c) $\alpha = 0.90$ and $\beta = 0.85$, and (d) $\alpha = 0.85$ and $\beta = 0.80$.

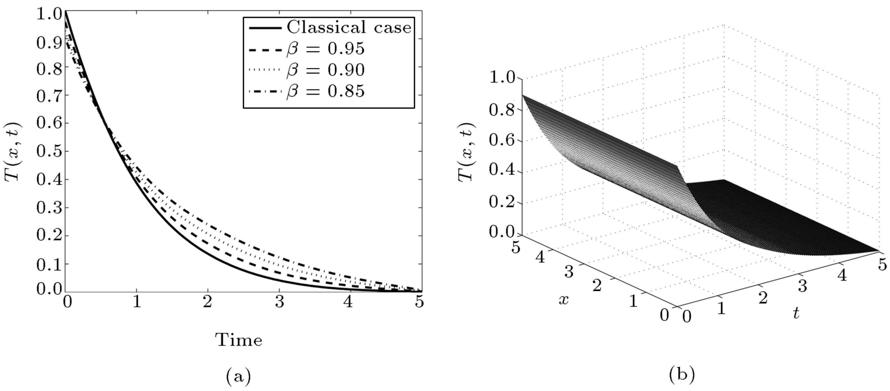

Figure 2. Thermal diffusion for different values of β arbitrarily chosen, using numerical evaluation of Eq. (23): (a) β takes different values, and (b) $\beta = 0.85$.

diffusion occurs in less time than the ordinary thermal diffusion. This phenomenon indicates the change of the medium properties, and the system presents dissipative effects [11].

3.2. Fractional space Fourier's law equation

Considering Eq. (13) and assuming that the space derivative is fractional and the time derivative is ordinary, the spatial fractional equation will be as follows:

$$ {}_0^{AC}\mathcal{D}_x^{2(\alpha,\beta)}T(x,t) - \frac{1}{\chi}\frac{\partial T(x,t)}{\partial t} = 0. \tag{27}$$

A particular solution to Eq. (27) can be found in the following form:

$$T(x,t) = T_0 \cdot e^{-\omega t}u(x). \tag{28}$$

Substituting Eq. (28) into Eq. (27) and considering Eq. (10), we obtain:

$$ {}_0^{AC}\mathcal{D}_x^{2(\alpha,\beta)}u(x) + \tilde{k}^2 u(x) = 0, \tag{29}$$

where $\tilde{k}^2 = k^2\sigma_x^{2(1-\alpha,\beta)}$ is the fractional wave number with bi-order and $k = \frac{\omega}{\chi}$ is the classical wave number.

The numerical approximation of Eq. (29) is given by:

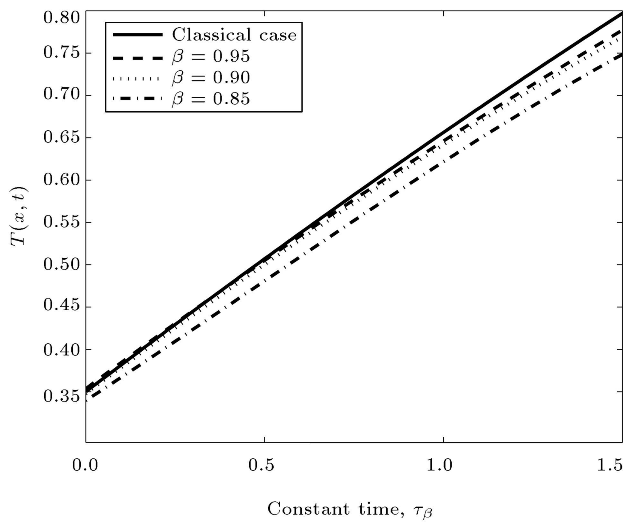

Figure 3. Thermal diffusion versus constant time, using exponent $\beta = 1$, $\tau_\beta = 0.63$ located in $t = 1$ s, $\beta = 0.95$; $\tau_\beta = 0.63$ located in $t = 1.033$ s, $\beta = 0.90$; $\tau_\beta = 0.63$ located in $t = 1.037$ s, $\beta = 0.85$; $\tau_\beta = 0.63$ located in $t = 1.041$ s and $\beta = 0.80$; $\tau_\beta = 0.63$ located in $t = 1.044$ s.

$$\frac{B(\beta)}{2-\beta}\cdot\frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{d^2u(\eta)}{d\eta^2}(x-\eta)^{-\alpha}E_{2\beta}\left\{-\frac{\beta}{2-\beta}(x-\eta)^{2(\beta+\alpha)}\right\}d\eta+\tilde{k}^2u(x)=0,$$

$$\frac{B(\beta)}{2-\beta}\cdot\frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{u(x)-u(x+\Delta x)}{2\Delta x}(x-\eta)^{-\alpha}E_{2\beta}\left\{-\frac{\beta}{1-\beta}(x-\eta)^{2(\beta+\alpha)}\right\}d\eta+\tilde{k}^2u(x)=0,$$

$$\frac{B(\beta)}{2-\beta}\cdot\frac{1}{\Gamma(2-\alpha)}\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}}\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{2(\Delta x)^2}(x_n-\eta)^{-\alpha}\cdot E_{2\beta}\left\{-\frac{\beta}{2-\beta}(x_n-\eta)^{2(\beta+\alpha)}\right\}d\eta+\tilde{k}^2\left[\frac{u(x_{n+1})-u(x_n)}{2}\right]=0,$$

$$\frac{B(\beta)}{2-\beta}\cdot\frac{1}{\Gamma(2-\alpha)}\sum_{i=0}^{n}\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{2(\Delta x)^2}\int_{x_n-x_i}^{x_n-x_i+1}\nu^{-\alpha}(x_n-\eta)^{-\alpha}\cdot E_{2\beta}\left\{-\frac{\beta}{2-\beta}(x_n-x_{i+1})\right\}d\eta+\tilde{k}^2\left[\frac{u(x_{n+1})-u(x_n)}{2}\right]=0,$$

thus:

$$u(x)=\frac{B(\beta)}{2-\beta}\cdot\frac{1}{\Gamma(2-\alpha)}\sum_{i=0}^{n}\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{2(\Delta x)^2}\Big[(x_n-x_{i+1})^{1-\alpha}\cdot E_{2\beta,2-\alpha}\left\{-\frac{\beta}{2-\beta}(x_n-x_{i+1})^{2(\beta+\alpha)}\right\}-(x_n-x_i)^{1-\alpha}E_{2\beta,2-\alpha}\left\{-\frac{\beta}{2-\beta}(x_n-x_{i+1})^{2(\beta+\alpha)}\right\}\Big]+\tilde{k}^2\left[\frac{u(x_{n+1})-u(x_n)}{2}\right], \tag{30}$$

where $E_{2\beta}$ is the Mittag-Leffler function.

The particular solution to Eq. (28) is written as follows:

$$T(x,t)=T_0\cdot e^{-\omega t}\Bigg[\frac{B(\beta)}{2-\beta}\cdot\frac{1}{\Gamma(2-\alpha)}\sum_{i=0}^{n}\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{2(\Delta x)^2}\Big[(x_n-x_{i+1})^{1-\alpha}\cdot E_{2\beta,2-\alpha}\left\{-\frac{\beta}{2-\beta}(x_n-x_{i+1})^{2(\beta+\alpha)}\right\}-(x_n-x_i)^{1-\alpha}\cdot E_{2\beta,2-\alpha}\left\{-\frac{\beta}{2-\beta}(x_n-x_{i+1})^{2(\beta+\alpha)}\right\}\Big]+\tilde{k}^2\left[\frac{u(x_{n+1})-u(x_n)}{2}\right]\Bigg]. \tag{31}$$

Eq. (31) represents the thermal diffusion with spatial-decaying amplitude with respect to space x.

If $\alpha = 0$ in Eq. (29), we recover ABC fractional derivative (6); from Eq. (27), we have:

$${}_0^{\mathrm{ABC}}\mathcal{D}_t^{\beta}T(x,t)-\chi\frac{\partial^2 T(x,t)}{\partial x^2}=0. \tag{32}$$

A particular solution to this equation can be found in the form of Eq. (28); substituting Eq. (28) into Eq. (32) and considering Eq. (12), we obtain:

$$ {}_0^{\mathrm{ABC}}\mathcal{D}_x^{2\beta}u(x) + \tilde{k}^2 u(x) = 0, \tag{33} $$

where, $\tilde{k}^2 = k^2\sigma_x^{2(1-\beta)}$ is the fractional wave number and $k = \frac{\omega}{\chi}$ is the classical wave number.

Applying Laplace transform (8) to (33) and considering $u(0) = u_0$ and $\dot{u}(0) = 0$ yield the following expression:

$$ \begin{aligned} U(s) =& \frac{1}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \\ & \left[\frac{(1-\beta)^2 s^{2\beta-1}}{\left(s^\beta + \frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta - \frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)}\right. \\ & + \frac{2\beta(1-\beta)s^{\beta-1}}{\left(s^\beta + \frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta - \frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)} \\ & \left. + \frac{\beta^2 s^{-1}}{\left(s^\beta + \frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta - \frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)}\right] \\ & + \frac{B(\beta)^2 u_0}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \\ & \left[\frac{s^{2\beta-1}}{\left(s^\beta + \frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta - \frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)}\right]. \end{aligned} \tag{34} $$

Taking the inverse Laplace transform of Eq. (34), we obtain the following particular solution to Eq. (33):

$$ \begin{aligned} T(x,t) =& T_0 \cdot e^{-\omega t}\left[\frac{(1-\beta)^2}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2}\right. \\ & \cdot \frac{aE_{\beta,1}(ax^\beta) + bE_{\beta,1}(-bx^\beta)}{a+b} \\ & + \frac{2\beta(1-\beta)}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \\ & \cdot \frac{aE_{\beta,\beta+1}(ax^\beta) + bE_{\beta,\beta+1}(-bx^\beta)}{a+b} \cdot x^\beta \\ & + \frac{\beta^2}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \\ & \cdot \frac{aE_{\beta,2\beta+1}(ax^\beta) + bE_{\beta,2\beta+1}(-bx^\beta)}{a+b} \cdot x^{2\beta} \\ & + \frac{B(\beta)u_0}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \\ & \left. \cdot \frac{aE_{\beta,1}(ax^\beta) + bE_{\beta,1}(-bx^\beta)}{a+b}\right], \end{aligned} \tag{35} $$

where $a = \frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}$, $b = \frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}$ and E_β is the Mittag-Leffler function.

In this case, when ($\beta = 1$), we have:

$$ \tilde{T}(x,t) = \tilde{T}_0 \cdot e^{-\omega t}\cos(kx). \tag{36} $$

Eq. (36) represents the classical case for the space Fourier's law equation.

Figure 4 shows the numerical simulations for temperature, $T(x,t)$, considering different values of α and β, in Eq. (31), arbitrarily chosen.

Figure 5 shows numerical simulations for temperature, $T(x,t)$, considering different values of β, in Eq. (23), arbitrarily chosen.

When fractional orders α,β change from $\alpha,\beta \in [0.85;1]$, the simulated fractional diffusion occurs in greater time than the simulated ordinary diffusion. This phenomenon indicates the change of the medium properties, different from the ideal properties presented in Eq. (13). The velocity of the concentration wave through a medium is determined by the inertia and the elasticity of the medium. Usually this dissipation is known as internal friction [26].

3.3. Fractional space-time Fourier's law equation

Now, considering Eq. (13) and assuming the space time derivatives as fractional, the order of space-time fractional differential equation is $0 < \alpha,\beta \leq 1$ and $0 < \gamma,\delta \leq 1$; the space-time fractional equation is as follows:

$$ {}_0^{AC}\mathcal{D}_x^{2(\alpha,\beta)}T(x,t) - \frac{1}{\chi}{}_0^{AC}\mathcal{D}_t^{(\gamma,\delta)}T(x,t) = 0. \tag{37} $$

The numerical approximation of Eq. (37) is given by:

$$ \begin{aligned} T(x,t) = A \cdot & \left[\frac{B(\beta)}{2-\beta}\right. \\ & \cdot \frac{1}{\Gamma(2-\alpha)}\sum_{i=0}^{n}\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{2(\Delta x)^2} \\ & \Big[(x_n - x_{i+1})^{1-\alpha} \cdot E_{2\beta,2-\alpha} \\ & \left\{-\frac{\beta}{2-\beta}(x_n - x_{i+1})^{2(\beta+\alpha)}\right\} \\ & -(x_n - x_i)^{1-\alpha} \cdot E_{2\beta,2-\alpha} \end{aligned} $$

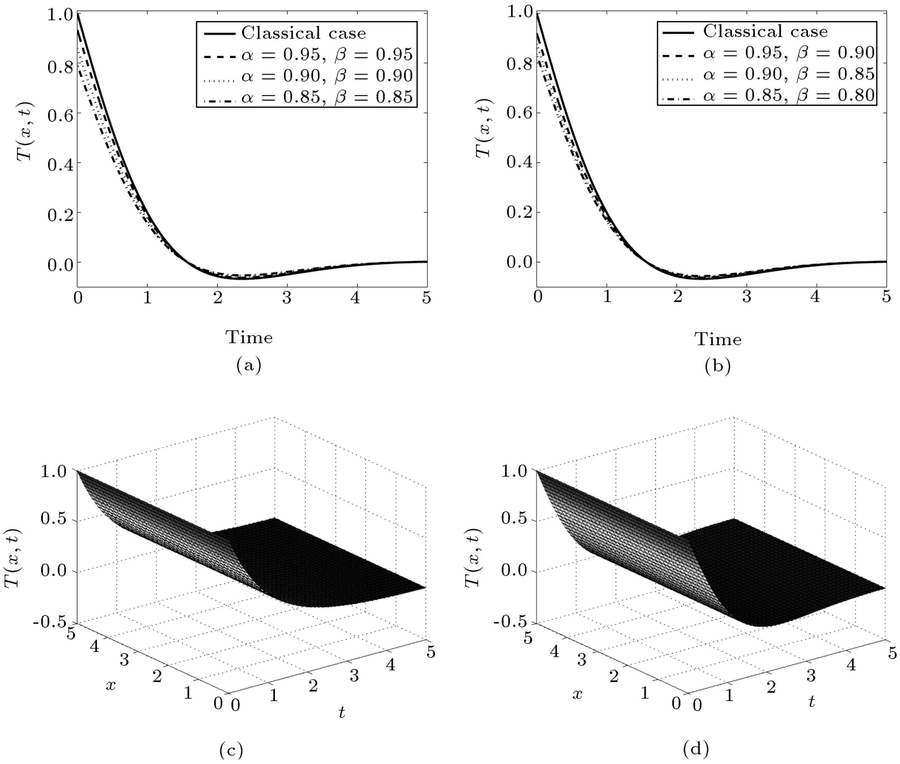

Figure 4. Thermal diffusion for different values of α and β arbitrarily chosen, using numerical evaluation of Eq. (32): (a) α and β take the same value, (b) α and β take different values, (c) $\alpha = 0.90$ and $\beta = 0.85$, and (d) $\alpha = 0.85$ and $\beta = 0.80$.

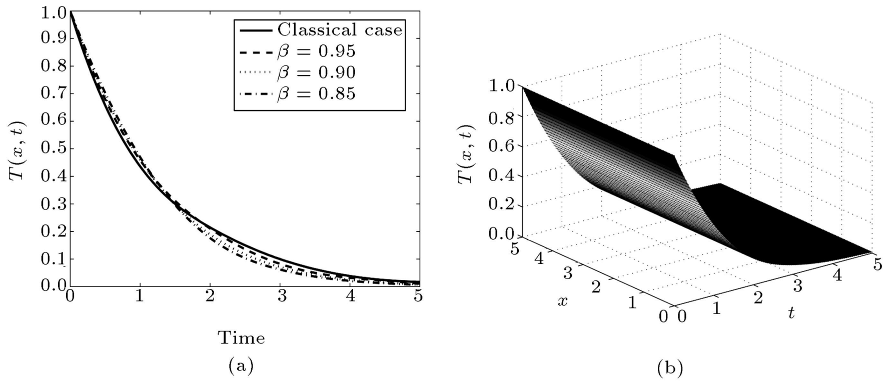

Figure 5. Thermal diffusion for different values of β arbitrarily chosen, using numerical evaluation of Eq. (23): (a) β takes different values, and (b) $\beta = 0.85$.

$$\left\{-\frac{\beta}{2-\beta}(x_n - x_{i+1})^{2(\beta+\alpha)}\right\}\Big]\Bigg]$$

$$\cdot\left[\frac{B(\beta)}{1-\delta}\cdot\frac{1}{\Gamma(1-\gamma)}\right.$$

$$\sum_{k=0}^{n}\frac{u^{k+1}-u^{k}}{2\Delta t}\Big[(t_n - t_{k+1})^{1-\gamma}E_{\delta,2-\gamma}$$

$$\left\{-\frac{\delta}{1-\delta}(t_n - t_{k+1})^{\delta+\gamma}\right\}$$

$$-(t_n - t_i)^{1-\gamma}E_{\delta,2-\gamma}$$

$$\left.\left\{-\frac{\delta}{1-\delta}(t_n - t_{k+1})^{\delta+\gamma}\right\}\Big]\right], \qquad (38)$$

where A is a constant.

Figure 6 shows the simulations where the fractional time and the spatial fractional derivatives are taken at the same time for different particular cases of (α,β) and (γ,δ). These figures show different behaviors of anomalous thermal diffusion; when the fractional orders are less than 1, the thermal diffusion is slower

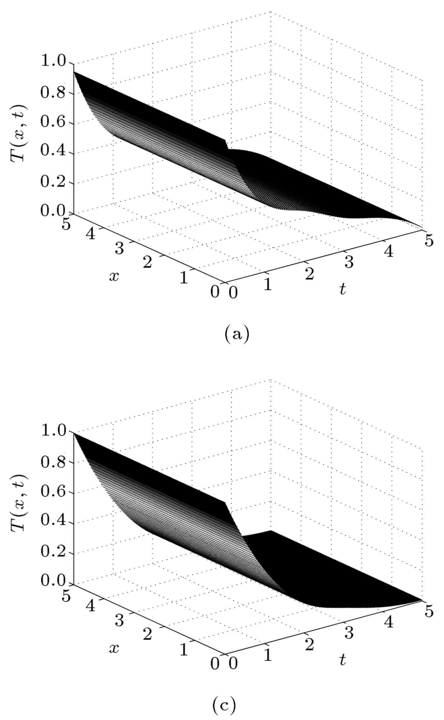

Figure 6. Thermal diffusion for different values of (α,β) and (γ,δ) arbitrarily chosen, using numerical evaluation of Eq. (38): (a) $\alpha=1,\beta=0.9$ and $\gamma=0.95,\delta=0.9$, (b) $\alpha=0.9,\beta=1$ and $\gamma=0.9,\delta=0.95$, and (c) $\alpha=0.95,\beta=0.85$ and $\gamma=0.85,\delta=0.9$, and (d) $\alpha=0.85,\beta=0.9$ and $\gamma=0.8,\delta=0.85$.

(thermal subdiffusion). These cases can represent a medium with two different properties represented by fractional bi-order α and β.

4. Conclusion

In this paper, we presented an alternative representation of the fractional-time Fourier's law equation using the concept of derivative with two fractional orders α and β. The new approach is capable of considering media with different properties represented by fractional bi-order α and β. This is the case, for instance, in thermal conduction for a reaction diffusion within a medium with two different layers and properties. These novel fractional operators allow studying the heat transfer through a material with different scales or heterogeneous media. Our results indicate that fractional bi-order α and β have an important influence on the temperature.

The motivation of this study comes from the fact that it is possible to find in nature some systems with different material layers, where each layer possesses different materials, e.g., in thermal conduction where the heat is flowing within a medium with two different properties. These types of problems cannot be portrayed with the existing derivatives and fractional order, which are based on the power law. For this reason, the Atangana derivative uses a kernel that is more powerful than $x^{-\beta}$, and the generalized Mittag-Leffler function is combined with a power law.

In the case where the space-time fractional differential equation is considered, Eq. (38) shows the numerical solution to the Fourier's law equation using the concept of derivative with two fractional orders (α,β) and (γ,δ) for space and time, respectively. Different values of α and β or γ and δ represent intermediate states between conservative and dissipative systems, which present anomalous relaxations. This combination of stored and dissipated energies is conveniently based on the representation of linear thermoviscoelasticity theory [27].

It is suggested that this derivative can be used to model more complex problems found in nature, groundwater studies, the phenomena of heat transfer from continuous media to discontinuous media, thermal convection of non-Fourier fluids, the non-Newtonian effects in thermal convection, relaxing gas dynamics, irreversible thermodynamics, thermoelasticity, orthodox viscoelastic materials and for the study of systems with heterogeneous media.

Nomenclature

T	Temperature (°C)
α	Thermal diffusivity (m^2/s)

χ Thermal conductivity (W/mK)

ρ Density (kg/m^3)

C_p Specific heat capacity (J/kgK)

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