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About a composite fractional relaxation equation via regularized families

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Abstract. This work deals with asymptotic periodicity and compactness for a class of composite fractional relaxation equations. Some difficulties arise when the effect of different kinds of nonhomogeneous terms is taken into consideration. To overcome these, we use methods resulting from regularized families and fixed point techniques, which are an important tool to study nonlinear phenomena. We can cover a large class of nonlinearities.

1. Statement of results

Fractional calculus is a field of mathematical analysis, which deals with the investigation and applications of integration and differentiation of any order, not necessarily integer. This field has in recent years become a powerful tool to investigate various concrete problems of mathematical physics. For this reason, there is much interest in developing the qualitative theory of fractional evolution equation, i.e., evolution equations where the integer derivative with respect to time is replaced by a derivative of fractional order (see [1-3]). We set up our problem and formulate the obtained results precisely. Let X be an arbitrary Banach space. In this work, we study asymptotic periodicity and compactness properties of solutions for a composite fractional relaxation equation in X. Let us start with the linear case (when $\alpha = \frac{1}{2}$ corresponds to the Basset problem a classical in fluid dynamics)(see [4]).):

$$u'(t) - A\,{}^cD_t^\alpha u(t) + u(t) = f(t), \qquad t > 0, \tag{1}$$

with the initial condition:

$$u(0) = x, \tag{2}$$

where $0 < \alpha < 1$, ${}^cD_t^\alpha$ denotes Caputo's fractional derivative. (We recall that the definition of Caputo's fractional derivative of order $\alpha > 0$ of a function f reads as follows: ${}^cD_t^\alpha f(t) = \int_0^t g_{m-\alpha}(t-s) f^{(m)}(s)ds$, $t > 0$, where $m = \lceil \alpha \rceil$, $g_\beta(t) = t^{\beta-1}/\Gamma(\beta)$, $t > 0$, $\beta > 0$ [4].) and A is a closed linear operator, which is the generator of an (a,k)-regularized family (in this setting, we comment that the notion of (a,k)-regularized families of operators, introduced in [5], includes k-convoluted semigroups, r-times integrated cosine families, and integral resolvent [4,6]) $R_\alpha(t)$ of bounded linear operators from X into X (see Definition 2.1), with $k(t) = e^{-t}$ and $a(t) = t^\alpha E_{1,1-\alpha}(-t)$, where $E_{\alpha,\beta}(\cdot)$ denotes the Mittag-Leffler function, which is defined as follows:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{H_a} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu,$$

$$\alpha, \beta > 0, \quad z \in \mathbb{C},$$

where H_a is a Henkel path, i.e. a contour with starts

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and ends at $-\infty$, that encircles the disc $|\mu| \leq |z|^{-1/\alpha}$ counter-clockwise. The Mittag-Leffler function plays the same role for fractional calculus as the exponential function plays for conventional calculus (see [2]). For many properties of Mittag-Leffer function, we refer to Erdélyi et al. [7]. Note that in case of non-integer α, there is no analogue of the Abel semigroup property or cosine functional equation, which plays a crucial role in the developing of the corresponding theories (see [1]).

In [6], the aim of the authors is to obtain the existence of solutions to the stochastic version of (1.1). This was done by using a resolvent family associated with the deterministic version of (1.1). We observe that real systems usually exhibit internal variations or are submitted to perturbations. One may convince oneself that in many situations, we can assume that these variations are approximately periodic in a broad sense (see [8]). The notion of S-asymptotically ω-periodic functions has emerged in the literature, recently, which has been shown to have interesting applications in several branches of evolution equations. This concept was introduced by Henríquez et al. [9]. To date, the research concerning asymptotic periodicity of Eq. (1) is too incipient and should be developed. Due to the rapid evolution of the notion of S-asymptotically ω-periodic functions, in this work, we place a particular emphasis on recent developments of a new type of S-asymptotically ω-periodic functions. We believe that this paper will help to speed up the development of this subject. In [4], the authors studied the existence of an S-asymptotically ω-periodic solution to problems (1) and (2). In [4], the authors also established a general procedure to derive mild solutions to a wide class of fractional equations. They presented how to obtain variations of constant formulae for various classes of fractional equations with Caputo derivative (see [3] for background material). Specifically, they proved the following result.

Theorem 1.1. *If $x \in ker(A)$ and f is a $ker(A)$-valued S-asymptotically ω-periodic function, then every mild solution to Eqs. (1) and (2) is S-asymptotically ω-periodic.*

In [10], a new space of S-asymptotically ω-periodic functions was introduced. It was called the space of pseudo S-asymptotically ω-periodic functions (see Definition 2.4). Some applications of this new type of functions were given in [11] for fractional equations. In [12], the authors studied the existence of pseudo S-asymptotically ω-periodic solutions to models of flexible structures possessing internal material damping and external force. They used the technique of regularized families as a substitute for semigroup. By using this families and Duhamel's principle, they defined mild solutions to such models.

As a starting point, we establish a version of Theorem 1.1 for this new type of functions.

Theorem 1.2. *If $x \in ker(A)$ and f is a $ker(A)$-valued pseudo S-asymptotically ω-periodic function, then every mild solution of Eqs. (1) and (2) is pseudo S-asymptotically ω-periodic.*

Now, we consider the semilinear abstract composite fractional relaxation equation:

$$u'(t) - A^c D_t^\alpha u(t) + u(t) = f(t, u(t)), \qquad t > 0, \quad (3)$$

where A is as above and $\{f(t,y) : t \in \mathbb{R}^+, y \in ker(A)\} \subseteq ker(A)$. We have the following result proved in [4].

Theorem 1.3. *Let $f : [0,\infty) \times ker(A) \to ker(A)$ be continuous function uniformly S-asymptotically ω-periodic on bounded sets of $ker(A)$ (see Definition 2.3) that verifies a Lipschitz condition in the second variable uniformly with respect to the first variable, i.e., there is $L > 0$ such that:*

$$||f(t,u) - f(t,v)|| \leq L||u - v||,$$

$$u, v \in ker(A), \qquad t \geq 0. \quad (4)$$

If $x \in ker(A)$ and $L < 1$, then Problems (2) and (3) have a unique S-asymptotically ω-periodic mild solution.

Note that the preceding theorem is a consequence of the contraction principle. The next result is a refinement of Theorem 1.3 offering an interesting achievement. Indeed, we can get rid of the smallest condition imposed on the constant L, which was instrumental in its proof.

Theorem 1.4. *Let $f : [0,\infty) \times ker(A) \to ker(A)$ be continuous function uniformly S-asymptotically ω-periodic on bounded sets of $ker(A)$ that verifies the Lip-schitz condition (1.4). In addition, the following conditions are fulfilled.*

- **($\mathbf{S}_\omega$1)**: *There is a continuous nondecreasing function $W : \mathbb{R}^+ \to \mathbb{R}^+$ such that $||f(t,u)|| \leq W(||u||)$ for all $t \geq 0$ and $u \in ker(A)$;*
- **($\mathbf{S}_\omega$2)**: *For each a $a \geq 0$ and $\sigma > 0$, the set $\{f(s,y) : 0 \leq s \leq a,\ y \in ker(A),\ ||y|| \leq \sigma\}$ is relatively compact;*
- **($\mathbf{S}_\omega$3)**: *There is $r > 0$ such that $||x|| + W(r) \leq r$.*

If $x \in ker(A)$, then Problems (2) and (3) have a unique S-asymptotically ω-periodic mild solution.

Remark 1.1. A result similar to Theorem 1.4 was obtained in [13] for obtaining the existence and uniqueness of S-asymptotically ω-periodic mild solution to a class of abstract differential equations.

Next, we are going to focus our presentation in the question of the existence and uniqueness of pseudo S-asymptotically ω-periodic mild solutions for the semilinear composite fractional relaxation equation (1.3) under Lipschitz type hypothesis on the nonlinearity f. We now describe three results, which are not known in the literature.

Theorem 1.5. *Let $f:[0,\infty)\times ker(A)\to ker(A)$ be a continuous function asymptotically bounded on bounded sets of $ker(A)$ and uniformly pseudo S-asymptotically ω-periodic on bounded sets of $ker(A)$ that verifies a Lipschitz condition (4). If $x\in ker(A)$ and $L<1$, then problems (2) and (3) have a unique pseudo S-asymptotically ω-periodic mild solution.*

Theorem 1.6. *Let $f:[0,\infty)\times ker(A)\to ker(A)$ be a continuous function asymptotically bounded on bounded sets of $ker(A)$ and uniformly pseudo S-asymptotically ω-periodic on bounded sets of $ker(A)$ that satisfies a Lipschitz condition:*

$$||f(t,u)-f(t,v)||\leq L(t)||u-v||,\qquad t\geq 0,\qquad (5)$$

where $u,v\in ker(A)$, $L:[0,\infty)\to\mathbb{R}^+$ is an integrable function so that it is bounded on $[N,\infty)$ for some constant $N>0$. If $x\in ker(A)$, then Problems (2) and (3) have a unique pseudo S-asymptotically ω-periodic mild solution.

Remark 1.2. A similar result can be established when f satisfies a local Lipschitz condition. More precisely, let $f:[0,\infty)\times ker(A)\to ker(A)$ be a continuous function asymptotically bounded on bounded sets of $ker(A)$ and uniformly pseudo S-asymptotically ω-periodic on bounded sets of $ker(A)$ so that it satisfies the following condition:

($\mathbf{L_{loc}}$): For each $\sigma\in\mathbb{R}^+$, for all $t\in\mathbb{R}^+$ and all $x,y\in B_\sigma(ker(A))$ we have $||f(t,x)-f(t,y)||\leq L(\sigma)||x-y||$, where $L:[0,\infty)\to\mathbb{R}^+$ is a continuous function. Suppose that $x\in ker(A)$ and there is $\tilde{r}>0$ so that $L(||x||+\tilde{r})+(1/\tilde{r})(L(||x||)||x||+\sup_{s\geq 0}||f(s,0)||)<1$; then, Problems (2) and (3) have a pseudo S-asymptotically ω-periodic mild solution.

Theorem 1.7. *Let $f:[0,\infty)\times ker(A)\to ker(A)$ be a continuous function asymptotically bounded on bounded sets of $ker(A)$ and uniformly pseudo S-asymptotically ω-periodic on bounded sets of $ker(A)$ that satisfies the Lipschitz condition (5), where $L:[0,\infty)\to\mathbb{R}^+$ is locally integrable. Suppose that the following conditions hold:*

($\mathbf{PS}_\omega\mathbf{1}$): $\sup_{t\geq 0}\int_0^t e^{-(t-s)}L(s)ds<1$.

($\mathbf{PS}_\omega\mathbf{2}$): $\lim_{l\to\infty}\frac{1}{l}\int_0^l\int_0^t e^{-(t-s)}L(s)ds=0$.

If $x\in ker(A)$, then Problem (2) and (3) have a pseudo S-asymptotically ω-periodic mild solution.

Remark 1.3 [11]. We observe that:

(i) If the function $t\to\int_0^t e^{-(t-s)}L(s)ds$ is integrable, then ($\mathbf{PS}_\omega\mathbf{2}$) holds;

(ii) If $L(\cdot)$ is integrable with $||L||_1<1$, then ($\mathbf{PS}_\omega\mathbf{1}$) and ($\mathbf{PS}_\omega\mathbf{2}$) hold;

(iii) If $L(\cdot)$ is locally integrable and $\lim_{l\to\infty}\frac{1}{l}\in t_0^l L(s)ds=0$, then ($\mathbf{PS}_\omega\mathbf{2}$) holds.

We next denote the space $\mathcal{C}_{\exp}=\{u\in C([0,\infty);ker(A)):\lim_{t\to\infty}\frac{u(t)}{e^t}=0\}$, endowed with the norm $|||u|||=\sup_{t\geq 0}||u(t)||e^{-t}$ by $\mathcal{C}_{\exp}$. Now, our goal is to investigate Eq. (3) with more general behaviors of the nonlinearities.

Theorem 1.8. *Let $f:[0,\infty)\times ker(A)\to ker(A)$ be a function asymptotically uniformly continuous on bounded sets of $ker(A)$, asymptotically bounded on bounded sets of $ker(A)$, and uniformly pseudo S-asymptotically ω-periodic on bounded sets of $ker(A)$ that satisfy ($\mathbf{S}_\omega\mathbf{1}$) and ($\mathbf{S}_\omega\mathbf{2}$). Assume further that the following properties hold:*

($\mathbf{PS}_\omega\mathbf{3}$): *For each $\xi>0$, $\lim_{t\to\infty}\frac{1}{e^t}\int_0^t e^{-(t-s)}W(\xi e^s)ds=0$.*

($\mathbf{PS}_\omega\mathbf{4}$): *For each $\varepsilon>0$ there is $\delta>0$ such that for every $u,v\in\mathcal{C}_{exp}$, $|||u-v|||<\delta$ implies:*

$$\sup_{t\geq 0}\int_0^t e^{-(t-s)}||f(s,u(s))-f(s,v(s))||ds\leq\varepsilon.$$

Set $\beta(\xi)=\sup_{t\geq 0}\frac{1}{e^t}\int_0^t e^{-(t-s)}W(\xi e^s)ds$. If $\lim inf_{\xi\to\infty}\frac{\beta(\xi)}{\xi}<1$ and $x\in ker(A)$, then Problems (2) and (3) have a pseudo S-asymptotically ω-periodic mild solution.

In contrast with Theorem 1.4, we have the following result.

Theorem 1.9. *Let $f:[0,\infty)\times ker(A)\to ker(A)$ be a continuous function asymptotically bounded on bounded sets of $ker(A)$ and uniformly pseudo S-asymptotically ω-periodic on bounded sets of $ker(A)$ that satisfies the Lipschitz condition (4). Assume that the conditions ($\mathbf{S}_\omega\mathbf{1}$), ($\mathbf{S}_\omega\mathbf{2}$), and ($\mathbf{S}_\omega\mathbf{3}$) of Theorem 1.4 are fulfilled. If $x\in ker(A)$, then Problems (2) and (3) have a unique pseudo S-asymptotically ω-periodic mild solution.*

Now, we study the existence of mild solutions to Problems (2) and (3) in the space $C_0([0,\infty);ker(A))$.

In particular, it is scientifically relevant to know when the set of mild solutions to Problems (2) and (3) enjoy key topological properties. This type of information has not been analyzed in [4]. The last result in this section is the following compactness theorem.

Theorem 1.10. Let x be in $ker(A)$ and assume that $f : [0,\infty) \times ker(A) \to ker(A)$ is a continuous function. In addition, suppose that the following conditions hold:

($\mathbf{C_0 1}$): *For each $R > 0$, there is a positive function $\gamma_R \in C_0(0,\infty)$ so that $sup\{||f(t,x)|| : ||x|| \leq R\} \leq \gamma_R(t)$, $t \geq 0$;*

($\mathbf{C_0 2}$): *For each $s \geq 0$ and $R > 0$, the set $\{f(s,y) : y \in ker(A), ||y|| \leq R\}$ is relatively compact;*

($\mathbf{C_0 3}$): $\liminf_{R\to\infty} \frac{1}{R} \sup_{t\geq 0} \int_0^t e^{-(t-s)}\gamma_R(s)ds < 1.$

Then, there is a mild solution $u \in C_0([0,\infty); ker(A))$ to Problems (2) and (3). Furthermore, if the following condition is fulfilled, then the set $\mathcal{S}$ formed by the mild solutions to Problems (2) and (3) are compact in $C_0([0,\infty); ker(A))$.

($\mathbf{C_0 4}$): $\limsup_{R\to\infty} \frac{1}{R} \sup_{t\geq 0} \int_0^t e^{-(t-s)}\gamma_R(s)ds < 1.$

Before ending the section, we make a brief comment on the framework of the proof of Theorem 1.10. This uses two basic ingredients: The Ascoli-Arzela characterization of compact subsets in $C_0([0,\infty); ker(A))$ and the Schauder's fixed point theorem. (One may wonder about the connectedness of the set of mild solutions. Unfortunately, we do not know the answer.)

We will now present a summary of this work, which is arranged in three sections and each of these is divided into numbered subsections. Section 2 provides the definitions and preliminary results to be used in the theorems stated and proved in the subsequent sections. In particular, we review some of the standard definitions like (a,k)-regularized resolvent family, S-asymptotically ω-periodic function, and pseudo S-asymptotically ω-periodic function. In Section 3, we will prove all our results. We have tried to make the presentation almost self-contained, seeking to attract the attention of nonspecialist researchers in the field.

2. Technical tools

In this section, our aim is to introduce notations, definitions, and preliminary facts, which are used throughout this work. Let X be a Banach space; we denote the space of bounded linear operators from X into X, endowed with the norm of operators, by $\mathcal{B}(X)$. In this work, $\widehat{R}(\lambda)$ denotes the Laplace transformation of the function $R(t)$. Let $C_0([0,\infty); X)$ be the subspace of all continuous functions $x(t)$ such that $\lim_{t\to\infty} ||x(t)|| = 0$.: We denote the space of locally integrable functions by L^1_{loc}. Let A be a closed linear operator defined in $D(A) \subseteq X$. We set $B_\sigma(X)$ for the closed ball with center at 0 and radius σ in the space X.

2.1. Regularized families

In this subsection, we review the notion of (a,k)-regularized families. The definition can be stated as follows.

Definition 2.1 [5]. Let X be a Banach space, $k \in C(\mathbb{R}^+)$, $k \neq 0$, and let a be in $L^1_{\text{loc}}(\mathbb{R}^+)$ with $a \neq 0$. Assume that A is a closed linear operator with domain $D(A)$. A strongly continuous family $\{R(t)\}_{t\geq 0}$ of bounded linear operators from X into X is called an (a,k)-regularized resolvent family on X (or simply (a,k)-regularized family) having A as a generator if the following holds:

(RF1): $R(0) = k(0)I$;

(RF2): $R(t)x \in D(A)$ and $R(t)Ax = AR(t)x$ for all $x \in D(A)$ and $t \geq 0$;

(RF3): $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)xds$; $t \geq 0$, $x \in D(A)$.

Remark 2.1 [4]. Let A be closed linear operator and let $\{R(t)\}_{t\geq 0}$ be an exponentially bounded and strongly continuous operator family in $\mathcal{B}(X)$ such that the Laplace transformation $\widehat{R}(\lambda)$ exists for $\lambda > \omega$. $R(t)$ is an (a,k)-regularized family with generator A if and only if for every $\lambda > \omega$, $(I - \hat{a}(\lambda)A)^{-1}$ exists in $\mathcal{B}(X)$ and:

$$\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)}I - A\right)^{-1} x = \int_0^\infty e^{-\lambda s}R(s)xds, \qquad x \in X.$$

2.2. S-asymptotically ω-periodic functions

Let Y be an arbitrary Banach space. In this work, $C_b([0,\infty); Y)$ denotes the space consisting of the continuous and bounded functions from $[0,\infty)$ into Y, endowed with the norm of the uniform convergence.

Definition 2.2 [9]. A function $f \in C_b([0,\infty); Y)$ is called S-asymptotically ω-periodic if $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$.

The notation $\text{SAP}_\omega(Y)$ stands for the space formed by the Y-valued S-asymptotically ω-periodic functions endowed with the norm of the uniform convergence. It is clear that $\text{SAP}_\omega(Y)$ is a Banach space.

Definition 2.3 [9]. A continuous function $f : [0,\infty) \times Y \to Y$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if $f(\cdot, x)$ is bounded for each $x \in Y$, and for every $\varepsilon > 0$ and all bounded set $K \subseteq Y$, there is $\tau(K,\varepsilon) \geq 0$ such that $||f(t,y) - f(t+\omega, y)|| \leq \varepsilon$ for all $t \geq \tau(K,\varepsilon)$ and all $y \in K$.

Remark 2.2. We observe that $AP_\omega(Y) \hookrightarrow \text{SAP}_\omega(Y)$. We use the notation $AP_\omega(Y)$ to represent the subspace of $C_b([0,\infty);Y)$ formed by all functions f, which are asymptotically ω-periodic, that is $f = g + h$, where g is an ω-periodic function and $h \in C_0([0,\infty);Y)$ (see [11]).

2.3. Pseudo S-asymptotically ω-periodic functions

Definition 2.4 [10]. A function $f \in C_b([0,\infty);Y)$ is called pseudo S-asymptotically periodic if there is $\omega > 0$ such that:

$$\lim_{h\to\infty} \frac{1}{h}\int_0^h ||f(s+\omega) - f(s)||ds = 0. \tag{6}$$

In this case, we say that f is S-asymptotically ω-periodic.

We use the notation $\text{PSAP}_\omega(Y)$ to represent the subspace of $C_b([0,\infty);Y)$ formed by all pseudo S-asymptotically ω-periodic functions. We note that $\text{PSAP}_\omega(Y)$ endowed with the norm of uniform convergence is a Banach space. We observe that $\text{SAP}_\omega(Y) \hookrightarrow \text{PSAP}_\omega(Y)$ and $\text{PSAP}_\omega(Y) \neq \text{SAP}_\omega(Y)$ [12].

Definition 2.5 [11]. We say that a continuous function $f : [0,\infty) \times Y \to Y$ is uniformly pseudo S-asymptotically ω-periodic on bounded sets of Y if for every bounded subset $K \subseteq Y$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \sup_{x\in K} ||f(s+\omega, x) - f(s,x)||ds = 0. \tag{7}$$

Definition 2.6 [11]. A continuous function $f : [0,\infty) \times Y \to Y$, is said to be asymptotically bounded on bounded sets of Y if for every bounded subset $K \subseteq Y$, there is $T_K > 0$ so that the set $\{f(t,x) : t \geq T_K, x \in K\}$ is bounded.

Definition 2.7 [9]. A function $f : [0,\infty) \times Y \to Y$ is said to be asymptotically uniformly continuous on bounded sets of Y if for every $\varepsilon > 0$ and all bounded set $K \subset Y$; there are constants $T = T_{\epsilon,K} \geq 0$ and $\delta = \delta_{\epsilon,K} > 0$ such that $\|f(t,x) - f(t,y)\| \leq \varepsilon$ for all $t \geq T$ and $x, y \in K$ with $\|x - y\| \leq \delta$.

3. Proofs of the results

3.1. Proof of Theorem 1.2

Let $u(t)$ be a mild solution to Problems (1) and (2). Taking into account that $x \in ker(A)$ and that f is a $ker(A)$-valued function, we have $u(t) = e^{-t}x + \int_0^t e^{-(t-s)} f(s)ds$ (see [4]). It is clear that the function $t \to e^{-t}x$ is pseudo S-asymptotically ω-periodic. We can verify that the function $v : t \to \int_0^t e^{-(t-s)} f(s)ds$ is pseudo S-asymptotically ω-periodic. In fact, we observe that $||v||_\infty \leq ||f||_\infty$ and we have the following identity:

$$v(\tau+\omega) - v(\tau) = \int_0^\omega e^{-(\tau+\omega-s)} f(s)ds + \int_0^\tau e^{-(\tau-s)}(f(s+\omega) - f(s))ds,$$

where:

$$\frac{1}{t}\int_0^t ||v(\tau+\omega) - v(\tau)||d\tau \leq \frac{1}{t}\int_0^t \int_\tau^{\tau+\omega} e^{-s}||f(\tau+\omega-s)||dsd\tau + \frac{1}{t}\int_0^t \left(\int_0^{t-s} e^{-\tau}d\tau\right) ||f(s+\omega) - f(s)||ds \leq \frac{||f||_\infty}{t} + \frac{1}{t}\int_0^t ||f(s+\omega) - f(s)||ds,$$

which shows that v is pseudo S-asymptotically ω-periodic.

3.2. Proof of Theorem 1.4

We consider the Fréchet space $C([0,\infty); ker(A))$ endowed with the topology of uniform convergence on compact sets τ_C. We define the map Υ on the space $C([0,\infty); ker(A))$ by the expression:

$$\Upsilon(u)(t) = e^{-t}x + \int_0^t e^{-(t-s)} f(s, u(s))ds. \tag{8}$$

- **Step 1.** The map Υ is continuous from $C([0,\infty); ker(A))$ into itself. If $(u_n)_n$ is a sequence in $C([0,\infty); ker(A))$ that converges to u, then $(\Upsilon u_n)_n$ converges to Υu. Indeed, for each $a > 0$ we get:

$$\sup_{t\in[0,a]} ||\Upsilon(u_n)(t) - \Upsilon(u)(t)|| \leq L \sup_{t\in[0,a]} ||u_n(t) - u(t)||.$$

- **Step 2.** Put $B_r = \{u \in C([0,\infty); ker(A)) : ||u||_\infty \leq r\}$, where r is given by $(S_\omega 3)$. It is clear that B_r is a closed convex subset of $C([0,\infty); ker(A))$. From conditions $(\mathbf{S_\omega 1})$ and $(\mathbf{S_\omega 3})$, we deduce that B_r is invariant under Υ. Note that $\Upsilon(B_r)$ is a relatively compact set in $C([0,\infty); ker(A))$. In fact, we first note that $\Upsilon(B_r)(t) \subseteq e^{-t}x + \overline{tco(K_r)}$, where $K_r = \{e^{-(t-s)} f(s,x) : 0 \leq s \leq t, ||x|| \leq r\}$. Taking $(\mathbf{S_\omega 2})$ into account, we infer that $\Upsilon(B_r)(t)$

is relatively compact. Let u be in B_r and $h \geq 0$. We have the following decomposition:

$$\Upsilon(u)(t+h) - \Upsilon(u)(t) = e^{-(t+h)}x - e^{-t}x + \int_t^{t+h} e^{-(t+h-s)} f(s,u(s))ds + \int_0^t \left(e^{-(t+h-s)} - e^{-(t-s)}\right) f(s,u(s))ds.$$

It follows that the set $\Upsilon(B_r)$ is equicontinuous on $[0,a]$ for all $a \geq 0$. We get as consequence of the Ascoli's theorem that $\Upsilon(B_r)$ is a relatively compact set in $C([0,\infty); ker(A))$. Applying [13, Lemma 2.4] and [14, Lemma 3.1], we can affirm that:

$$\Upsilon(\overline{\mathrm{SAP}_\omega(ker(A))}^{\tau_C}) \subseteq \overline{\mathrm{SAP}_\omega(ker(A))}^{\tau_C}.$$

We define the Υ-invariant set $C := B_r \cap \overline{\mathrm{SAP}_\omega(ker(A))}^{\tau_C}$. From the Schauder-Tychonoff theorem, we infer that Υ has a fixed point $\tilde{u} \in C$.

- **Step 3.** $v(t) = \tilde{u}(t+\omega)$. A simple analysis shows that:

$$\Upsilon v - v \in C_0([0,\infty); ker(A)).$$

Indeed, since f is a continuous function uniformly S-asymptotically ω-periodic on bounded sets, for each $\varepsilon > 0$, there is $T_\varepsilon > 0$ such that $||f(t+\omega, \tilde{u}(t+\omega)) - f(t, \tilde{u}(t+\omega))|| \leq \varepsilon$, for all $t \geq T_\varepsilon$. Using $(S_\omega 1)$ we get:

$$||\Upsilon v(t) - v(t)|| \leq e^{-t}((1+e^{-\omega})||x|| + W(r)) + \varepsilon.$$

Therefore, $\Upsilon v(t) - v(t) \to 0$ and $t \to \infty$.

- **Step 4.** Set $\varphi(t) = \|\Upsilon v(t) - v(t)\|$, $t \geq 0$. We next show that there is a positive continuous function $\nu : [0,\infty) \to [0,\infty)$ that vanishes at infinity such that:

$$\nu(t) = \varphi(t) + L\int_0^t e^{-(t-s)}\nu(s)ds, \qquad t \geq 0. \tag{9}$$

In fact, let $r(t)$ be the solution to equation:

$$r(t) = -Le^{-t} + L\int_0^t e^{-(t-s)} r(s)ds. \tag{10}$$

By [15, Theorem IV.6.2], Eq. (10) has a solution $r(\cdot) \in L^1(\mathbb{R}^+)$. Through defining $\nu(t) = \varphi(t) - \int_0^t r(t-s)\varphi(s)ds$, we have:

$$L\int_0^t e^{-(t-s)}\nu(s)ds = L\int_0^t e^{-(t-s)}\varphi(s)ds - L\int_0^t \left(\int_\tau^t e^{-(t-s)} r(s-\tau)ds\right)\varphi(\tau)d\tau = -\int_0^t r(t-\tau)\varphi(\tau)d\tau = \nu(t) - \varphi(t).$$

Therefore, the function $\nu(\cdot)$ is solution to Eq. (9). On the other hand, since $r(\cdot) \in L^1(\mathbb{R}^+)$ and $\varphi \in C_0([0,\infty))$, we infer that the function $t \to \int_0^t r(t-s)\varphi(s)ds$ belongs to $C_0([0,\infty))$, where ν vanishes at infinity.

- **Step 5.** We consider the set $C^\sharp = v + \{u \in C_0([0,\infty); ker(A)) : \|u(t)\| \leq \nu(t),\ t \in \mathbb{R}^+\}$, where $v(\cdot)$ and $\nu(\cdot)$ are the functions given in Steps 3 and 4, respectively. Note that $C^\sharp$ is a τ_C-closed convex subset of $C_b([0,\infty); ker(A))$. Let u be in $C_0([0,\infty); ker(A))$. We observe that $\Upsilon(v+u) - \Upsilon(v) \in C_0([0,\infty); ker(A))$. In fact, we get:

$$||\Upsilon(v+u)(t) - \Upsilon(v)(t)|| \leq L\int_0^t e^{-(t-s)}||u(s)||ds \leq L\int_0^t e^{-(t-s)}\nu(s)ds = L(\nu(t) - \varphi(t)).$$

Next, taking into account that ν and φ vanish at infinity, we deduce that $\Upsilon(v+u) - \Upsilon(v)$ vanishes at infinity. Now, using Step 3, we may conclude that $\Upsilon(v+u) - v$ vanishes at infinity. On the other hand, we get:

$$||\Upsilon(v+u)(t) - v(t)|| \leq L\int_0^t e^{-(t-s)}\nu(s)ds + \varphi(t) = \nu(t),$$

which implies that $\Upsilon(v+u) \in C^\sharp$; hence, we infer that $C^\sharp$ is invariant under the operator Υ.

- **Step 6.** Finally, proceeding as in Step 2, we get that Υ has a fixed point $\tilde{u}_0 \in C^\sharp$. Therefore, there is $u_0 \in C_0([0,\infty); ker(A))$ such that $\tilde{u}_0 = v + u_0$. Using the fact that Υ has a unique fixed point in $C_b([0,\infty); ker(A))$, we conclude that $\tilde{u}_0 = \tilde{u}$ which implies that $\tilde{u} - v$ vanishes at infinity. Therefore, $\tilde{u}$ is a function S-asymptotically ω-periodic. □

3.3. Proof of Theorem 1.5

We define the operator Π on the space $\mathrm{PSAP}_\omega(ker(A))$ by the expression:

$$\Pi u(t)=e^{-t}x+\int_0^t e^{-(t-s)}f(s,u(s))ds, \quad t\geq 0. \qquad (11)$$

We initially show that Πu is in $\mathrm{PSAP}_\omega(ker(A))$ for $u\in\mathrm{PSAP}_\omega(ker(A))$. It is easy to see that $f(\cdot,u(\cdot))$ is a bounded function. Hence, we obtain the following estimate: $||\Pi u||_\infty \leq ||x|| + ||f(\cdot,u(\cdot))||_\infty$. From [12, Lemma 2.1], we get that the function $s\to f(s,u(s))$ is pseudo S-asymptotically ω-periodic; then, using the proof of Theorem 1.2, we infer that $s\to\int_0^t e^{-(t-s)}f(s,u(s))ds$ belongs to $\mathrm{PSAP}_\omega(ker(A))$. Furthermore, Π is an L-contraction on the space $\mathrm{PSAP}_\omega(ker(A))$; from this, we conclude that Π has a unique fixed point $u\in\mathrm{PSAP}_\omega(ker(A))$. □

3.4. Proof of Theorem 1.6

We use the same notations as those in the proof of Theorem 1.5. Let u be in $\mathrm{PSAP}_\omega(ker(A))$; taking into account that $\{L(t):t\geq N\}$ is bounded, it follows from [12, Lemma 2.2] that the function $s\to f(s,u(s))$ is pseudo S-asymptotically ω-periodic; then, $\Pi u\in\mathrm{PSAP}_\omega(ker(A))$. Hence, Π is well defined. On the other hand, for $u,v\in\mathrm{PSAP}_\omega(ker(A))$, we get $||\Pi^n u-\Pi^n v||_\infty\leq\frac{||L||_1^n}{n!}||u-v||_\infty$. Since $\frac{||L||_1^n}{n!}<1$, for a sufficiently large value of n, by the fixed point iteration, method Π has a unique fixed point $u\in\mathrm{PSAP}_\omega(ker(A))$. □

3.5. Proof of Theorem 1.7

We define the operator Π on the space $\mathrm{PSAP}_\omega(ker(A))$ by Expression (11). We prove that Π is well defined. Let u be in $\mathrm{PSAP}_\omega(ker(A))$; we take $\varepsilon>0$; $T=T(Im(u))\in\mathbb{R}^+$ is big enough so that $\{f(t,u(t)):t\geq T\}$ is bounded and $\frac{1}{t}\int_0^t\sup_{x\in Im(u)}||f(s+\omega,x)-f(s,x)||ds\geq\frac{\varepsilon}{2}$ for $t>T$. We observe that $f(\cdot,u(\cdot))$ is a bounded function in $\mathbb{R}^+$; hence, Πu is bounded in $[0,\infty)$. It only remains to show that the function $v(t)=\int_0^t e^{-(t-s)}f(s,u(s))ds$ is pseudo S-asymptotically ω-periodic. For $l>T$, we get:

$$\frac{1}{l}\int_0^l(v(t+\omega)-v(t))dt$$

$$=\frac{1}{l}\int_0^T\int_0^t e^{-(t-s)}(f(s+\omega,u(s+\omega))-f(s,u(s)))dsdt$$

$$+\frac{1}{l}\int_T^l\int_0^T e^{-(t-s)}(f(s+\omega,u(s+\omega))-f(s,u(s+\omega)))dsdt$$

$$+\frac{1}{l}\int_T^l\int_T^t e^{-(t-s)}(f(s+\omega,u(s+\omega))$$

$$-f(s,u(s+\omega)))dsdt+\frac{1}{l}\int_T^l\int_0^t e^{-(t-s)}(f(s,u(s+\omega))$$

$$-f(s,u(s)))dsdt+\frac{1}{l}\int_0^l\int_t^{t+\omega}e^{-s}f(t+\omega-s,u(t+\omega$$

$$-s))dsdt:=\sum_{i=1}^5 I_i(l).$$

We have the following estimates for the terms I_i, $1\leq i\leq 5$.

$$||I_1(l)||\leq\frac{2}{l}(T+1)\sup\{||f(t,x)||:t\geq 0,x\in Im(u)\},$$

$$\|I_2(l)\|\leq\frac{1}{l}\int_T^l te^{-(t-T)}\left(\frac{1}{t}\int_0^t\sup_{x\in Im(u)}\|f(s+\omega,x)\right.$$

$$\left.-f(s,x)\|ds\right)dt\leq\frac{\epsilon}{2},$$

$$||I_3(l)||\leq\frac{1}{l}\int_0^l\sup_{x\in Im(u)}\|f(s+\omega,x)-f(s,x)\|ds\leq\frac{\epsilon}{2},$$

$$\|I_4(l)\|\leq\frac{1}{l}\int_T^l\int_0^t e^{-(t-s)}L(s)||u(s+\omega)-u(s)||dsdt$$

$$\leq\frac{2}{l}||u||\int_T^l\int_0^t e^{-(t-s)}L(s)dsdt,$$

$$||I_5(l)||\leq\frac{\omega}{l}\sup\{||f(t,x)||:t\geq 0,x\in Im(u)\}.$$

From the above estimates, we get that v is pseudo S-asymptotically ω-periodic. Finally, we observe that Π is a $\sup_{t\geq 0}\int_0^t e^{-(t-s)}L(s)ds$-contraction on the space $\mathrm{PSAP}_\omega(ker(A))$. This completes the proof of Theorem 1.7. □

3.6. Proof of Theorem 1.8

Let $\mathcal{C}^0_{\exp}$ be the subspace of $\mathcal{C}_{\exp}$ consisting of the functions u such that $u(0)=0$. We define the operator Π^0 on $\mathcal{C}^0_{\exp}$ by:

$$\Pi^0u(t)=\int_0^t e^{-(t-s)}f(s,e^{-s}x+u(s))ds. \qquad (12)$$

It follows from condition ($\mathbf{PS}_\omega\mathbf{3}$) that the operator Π^0 is well defined. We observe that the map Π^0 is continuous from $\mathcal{C}_{\exp}$ into itself. This assertion is a direct consequence of the condition ($\mathbf{PS}_\omega\mathbf{4}$). We claim that Π^0 is a completely continuous map. We take $r>0$ and define the sets:

$$V = \Pi^0(B_r(\mathcal{C}^0_{\exp})),$$

and:

$$V(t) = \{\Pi^0 u(t) : u \in B_r(\mathcal{C}^0_{\exp})\}.$$

Taking into account the mean value theorem for the Bochner's integral and condition ($\mathbf{S}_\omega\mathbf{2}$), we infer that $V(t)$ is relatively compact. On the other hand, following a similar argument to the proof of Theorem 1.4, one can easily conclude that V is equicontinuous on $[0,a]$ for all $a \geq 0$. For $u \in \mathcal{C}^0_{\exp}$, with $|||u||| \leq r$, we observe that $\frac{||\Pi^0 u(t)||}{e^t} \leq \frac{1}{e^t}\int_0^t e^{-(t-s)}W(e^s(||x|| + r))ds$. From ($\mathbf{PS}_\omega\mathbf{3}$), we infer that $\frac{||\Pi^0 u(t)||}{e^t} \to 0$ as $t \to \infty$ which is independent of $u \in B_r(\mathcal{C}^0_{\exp})$. Using [16, Lemma 2.8], we get that V is a relatively compact set. This proves that Π^0 is completely continuous. We next observe that the operator Π^0 maps $B_\rho(\mathcal{C}^0_{\exp})$ into itself for some $\rho > 0$. In fact, if we assume that the assertion is false, then for all $\rho > 0$, we can choose $u^\rho \in B_\rho(\mathcal{C}^0_{\exp})$ such that $|||\Pi^0 u^\rho||| > \rho$; this fact implies that $\rho < \beta(||x|| + \rho)$; hence, $\liminf_{\xi\to\infty} \frac{1}{\xi}\beta(\xi) \geq 1$ is absurd. The closed vector subspace of all $u \in \mathrm{PSAP}_\omega(ker(A))$ can be denoted with $u(0) = 0$ by PSAP^0_ω. Taking into account [12, Lemma 2.1] and the proof of Theorem 1.2, we obtain that the space PSAP^0_ω is invariant under the map Π^0; hence, we infer that the closure of $B_\rho(\mathcal{C}^0_{\exp}) \cap \mathrm{PSAP}^0_\omega$, $\overline{B_\rho(\mathcal{C}^0_{\exp}) \cap \mathrm{PSAP}^0_\omega}$ is invariant under the map Π^0. Applying the Schauder fixed point theorem, we deduce that the map Π^0 has a fixed point $u \in \overline{B_\rho(\mathcal{C}^0_{\exp}) \cap \mathrm{PSAP}^0_\omega}$. Therefore, there is a sequence $(u^n)_n$ in $B_\rho(\mathcal{C}^0_{\exp}) \cap \mathrm{PSAP}^0_\omega$ that converges to u in the norm of $\mathcal{C}_{\exp}$. From condition ($\mathbf{PS}_\omega\mathbf{4}$), we get $\Pi^0 u^n \to u$ as $n \to \infty$ uniformly in $[0,\infty)$. Hence, $u \in \mathrm{PSAP}^0_\omega$, which completes the proof of Theorem 1.8.□

3.7. Proof of Theorem 1.9

In what follows, we consider the Frechet space $C([0,\infty); Ker(A))$ endowed with the topology of uniform convergence on compact sets τ_C. We define a continuous map Υ on the space $C([0,\infty); Ker(A))$ by Eq. (8). From Step 2 of the proof of Theorem 1.4, we have B_r, where r is given by ($\mathbf{S}_\omega\mathbf{3}$), that is invariant under Υ, and $\Upsilon(B_r)$ is a relatively compact set in $C([0,\infty); Ker(A))$. Let u be in $\mathrm{PSAP}_\omega(ker(A))$ applying [11, Lemma 2.3] and consider the fact that the function $s \to \int_0^t e^{-(t-s)}f(s,u(s))ds$ is pseudo S-asymptotically ω-periodic (see the proof of Theorem 1.2), we obtain $\Upsilon u \in \mathrm{PSAP}_\omega(ker(A))$, where $\mathrm{PSAP}_\omega(ker(A))$ is invariant under Υ. Next, we define $\mathcal{H} = B_r \cap \overline{\mathrm{PSAP}_\omega(ker(A))}^{\tau_C}$ from the Schauder-Tychonoff theorem; we get that Υ has a fixed point $\tilde{u}$ in $\mathcal{H}$. Setting $v(t) = \tilde{u}(t+\omega)$, $t \geq 0$. We claim that $\Upsilon v - v$ is an ergodic function. In fact, this is a consequence of the following estimate:

$$\frac{1}{t}\int_0^t \|\Upsilon v(s) - v(s)\|ds \leq (2\|x\| + W(r))\frac{1}{t}$$

$$+ \frac{1}{t}\int_0^t \sup_{\|x\|\leq r} \|f(\tau+\omega, x) - f(\tau, x)\|d\tau.$$

Set $\varphi(t) = ||\Upsilon v(t) - v(t)||$. We observe that there is a unique positive ergodic function $\tilde{\nu} : \mathbb{R}^+ \to \mathbb{R}^+$ so that $\tilde{\nu}(t) = \varphi(t) + L\int_0^t e^{-(t-s)}\tilde{\nu}(s)ds$. Indeed, by [15, Theorem IV.6.2], the resolvent $r(t)$ of Le^{-t} exists as an element of $L^1(\mathbb{R}^+)$ and is unique in this class, where $\tilde{\nu}(t)$ is given by $\tilde{\nu}(t) = \varphi(t) - \int_0^t r(t-\tau)\varphi(\tau)d\tau$. Since φ is an ergodic function, we obtain that the function $t \to \int_0^t r(t-\tau)\varphi(\tau)d\tau$ is an ergodic function. We consider the ergodic. Therefore, $\tilde{\nu}$ is an ergodic function. We consider the set $\mathcal{H}_\sharp = v + \{u \in P_0(\mathbb{R}^+; ker(A)) : ||u(t)|| \leq \tilde{\nu}(t), t \geq 0\}$. It is easy to check whether $\mathcal{H}_\sharp$ is a τ_C-closed convex subset of $C_b([0,\infty); Ker(A))$. We next show that $\mathcal{H}_\sharp$ is invariant under Υ. Let u be in $P_0(\mathbb{R}^+; ker(A))$, we observe that $\Upsilon(v+u) - \Upsilon v$ is an ergodic function. In fact, we get:

$$\frac{1}{t}\int_0^t \|\Upsilon(v+u)(s) - (\Upsilon v)(s)\|ds \leq \frac{L}{t}\int_0^t ||u(s)||ds.$$

On the other hand, if $||u(t)|| \leq \tilde{\nu}(t)$ for all $t \geq 0$, then:

$$||\Upsilon(v+u)(t) - v(t)|| \leq ||\Upsilon(v+u)(t) - (\Upsilon v)(t)||$$

$$+ ||(\Upsilon v)(t) - v(t)|| \leq \tilde{\nu}(t) - \varphi(t) + \varphi(t) = \tilde{\nu}(t),$$

which implies that $\Upsilon(v+u) \in \mathcal{H}_\sharp$; hence, we prove that $\mathcal{H}_\sharp$ is invariant under the operator Υ. Therefore, we can affirm that Υ has a fixed point $\tilde{u}_0 \in \mathcal{H}_\sharp$. Using the fact that Υ has a unique fixed point in $C_b([0,\infty); Ker(A))$, we conclude that $\tilde{u} = \tilde{u}_0$, which implies $\tilde{u} - v$ is an ergodic function. Hence, we infer that $\tilde{u}$ is pseudo S-asymptotically ω-periodic.

3.8. Proof of Theorem 1.10

We define the operator Υ on the space $C_0([0,\infty); ker(A))$ by Eq. (8). Let u be in $C_0([0,\infty); ker(A))$, since $\|f(t,u(t))\|_X \leq \gamma_R(t) \to 0$, $t \to \infty$ for $R > 0$ such that $\|u(t)\|_X \leq R$. Then, $\int_0^t e^{-(t-s)}f(s,u(s))ds \to 0$, $t \to \infty$. Indeed, we fix $a > 0$; our assertion follows from the next inequality:

$$\left\|\int_0^t e^{-(t-s)}f(s,u(s))ds\right\|$$

$$\leq e^{-t}\int_0^a e^s\gamma_R(s)ds + \sup_{\sigma\geq a}\gamma_R(\sigma). \tag{13}$$

Hence, we conclude that $\Upsilon u \in C_0([0,\infty); ker(A))$. Now, let $(u_n)_n$ be a sequence in $C_0([0,\infty); ker(A))$ that converges to $u \in C_0([0,\infty); ker(A))$. Then, $R > 0$ such that $\|u_n(t)\|, \|u(t)\| \leq R$ for all $t \geq 0$ and all $n \in \mathbb{N}$. We fix $a > 0$, then:

$$\left\| \int_a^t e^{-(t-s)}[f(s,u_n(s)) - f(s,u(s))]ds \right\| \leq 2 \sup_{\sigma \geq a} \gamma_R(\sigma). \tag{14}$$

Furthermore, there is a compact set $K \subseteq ker(A)$ such that $u_n(t), u(t) \in K$ for all $n \in \mathbb{N}$ and all $0 \leq t \leq a$. The function $f : [0,a] \times K \to ker(A)$ is uniformly continuous. Hence, $\|f(s,u_n(s)) - f(s,u(s))\| \to 0$, $n \to \infty$, uniformly for $0 \leq s \leq a$. This implies that:

$$\int_0^t e^{-(t-s)}\|f(s,u_n(s)) - f(s,u(s))\|ds \to 0,$$

$$n \to \infty, \tag{15}$$

uniformly for $0 \leq t \leq a$. Combining Eqs. (14) and (15), we deduce that Υ is continuous.

Next, we show that there is $\rho > 0$, such that $B_\rho(C_0([0,\infty); ker(A)))$ is invariant under Υ. Indeed, assuming the opposite, for each $R > 0$, there is a function u^R such that $\|u^R\| \leq R$ and $\|\Upsilon(u^R)\| > R$. Hence, $1 \leq \frac{\|x\|}{R} + \frac{1}{R}\sup_{t\geq 0}\int_0^t e^{-(t-s)}\gamma_R(s)ds$, which contradicts the condition ($\mathbf{C_0 3}$). To prove that Υ is a completely continuous map, we apply the Ascoli-Arzelá characterization of compact subsets in $C_0([0,\infty); ker(A))$. We consider $R > 0$ using [17, Corollary 2.10] and ($\mathbf{C_0 2}$); we can affirm that $\Upsilon(B_R(C_0([0,\infty); ker(A))))$ is relatively compact in $C([0,a]; ker(A))$ for all $a > 0$. Moreover, using ($\mathbf{C_0 1}$) and Eq. (13), we obtain $||\Upsilon(u)(t)|| \leq e^{-t}||x|| + e^{-t}\int_0^a e^s \gamma_R(s)ds + \sup_{\sigma \geq a}\gamma_R(\sigma)$, where $\Upsilon(u)(t) \to 0$ as $t \to \infty$ uniformly for $u \in B_R(C_0([0,\infty); ker(A)))$. Combining these assertions, we deduce that Υ is completely continuous. Applying the Schauder's fixed point theorem, we infer that Υ has a fixed point in $B_\rho(C_0([0,\infty); ker(A)))$. Moreover, the continuity of Υ implies that the set $\mathcal{S}$ consisting of mild solutions of Eqs. (2) and (3) is closed. On the other hand, if condition ($\mathbf{C_0 4}$) holds, then $\mathcal{S}$ is bounded. In fact, if we assume that $\mathcal{S}$ is not bounded, then there is a sequence of functions $u_k \in \mathcal{S}$ such that $R_k = ||u_k|| \geq k$. Hence, one gets $||u_k(t)|| \leq ||x|| + \sup_{t\geq 0}\int_0^t e^{-(t-s)}\gamma_{R_k}(s)ds$; this yields $1 \leq \limsup_{k\to\infty}\frac{1}{R_k}\sup_{t\geq 0}\int_0^t e^{-(t-s)}\gamma_{R_k}(s)ds$, which is contrary to ($\mathbf{C_0 4}$). Finally, taking into account that Υ is completely continuous, we deduce that $\mathcal{S}$ is compact.

4. Conclusions

In this work, we studied the existence and uniqueness of S-asymptotically pseudo S-asymptotically ω-periodic mild solutions for a class of fractional relaxation equations. We also considered compactness properties for the set of mild solutions. The main ingredients to achieve our results were the regularized families and fixed point techniques. Our results are new and contribute to the development of the asymptotic periodicity of fractional equations.

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