Coupling of energy and harmonic balance method for solving a conservative oscillator with strong odd-nonlinearity

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Abstract

In this paper, a new analytical technique, combining the energy balance method (EBM) with harmonic balance method (HBM), is presented to obtain higher-order approximations of a conservative oscillator with strong odd-nonlinearity. To show the accuracy of the present method, one nonlinear oscillator named as cubic-quintic Duffing oscillator is investigated. The results obtained in this paper are compared with those results determined by other methods and exact solutions. The results give high accuracy and also provide better result than other existing results for both small and large amplitudes of oscillation. The main advantage of the present paper is its simplicity, which contains a few harmonic terms with lower order terms and these terms make the solution quickly converges. The present technique can be used to other nonlinear oscillators.

Keywords: Nonlinear oscillation; Odd-nonlinearity; Cubic-quintic Duffing oscillator; Energy balance method; Harmonic balance method.

1. Introduction

The study of nonlinear oscillations is a necessary issue in engineering, physical science, applied mathematics, mechanical structures, nonlinear circuits, chemical oscillation and in many real world applications [1-30]. Nonlinear oscillations are modeled by nonlinear differential equations. Many analytical techniques have been developed to solve these nonlinear differential equations. One of the most widely used techniques is perturbation method [1-4] whereby the nonlinearities are small. But these techniques have many shortcomings, and they can not be used due to strongly nonlinear systems. To overcome these shortcomings, many analytical techniques such as variational iterative method [5-6], homotopy perturbation method [7-9], iterative method [10-12], harmonic balance method [13-17], variational approach [18-19] and coupled method [20] are used to solve strongly nonlinear equations. The energy balance method [21-22] is also another technique to obtain first-order approximation of strongly nonlinear oscillators. Usually, a set of algebraic equations with complex nonlinearities appears when EBM is formulated for determining higher-order approximations. On the contrary, some authors [23-24] extended the energy balance method to obtain higher-order approximations, but the algebraic equations are not solved analytically.

The Duffing equation is a well-known nonlinear differential equation [20, 25-26], which is related to many practical engineering systems, such as the classical nonlinear spring system, with odd nonlinear restoring characteristics [3], and more recently in different physical phenomena [25]. There have been many variations of Duffing equation, for instance, the Duffing–harmonic equation [11-12] and the cubic–quintic Duffing equation. The unperturbed cubic-quintic Duffing equation can be found in the modeling of the free vibration of a restrained uniform beam carrying an intermediate lumped mass and undergoing large amplitudes of oscillation in the unimodel Duffing type temporal problem [26], the nonlinear dynamics of a slender elastica, the generalized Pochhammer–Chree (PC) equation and the compound Korteweg-de Vries (KdV) equation [26]. A differential equation with fifth power nonlinearity is very difficult to handle because of the presence of strong nonlinearity.

Due to the presence of fifth power nonlinearity, the accuracy of approximate analytical methods becomes extremely demanding [20]. Recently, several authors [23-24] have extended the energy balance method to determine higher-order approximations. The limitation of the articles [23-24] is that, they were not analytically solved algebraic equations; they only solved these algebraic equations numerically. On the other hand, Khan et al. [20] used coupled method of He's homotopy perturbation method [7] and variational formulation [18] to obtain higher-order approximations for nonlinear cubic-quintic Duffing equations. But the first, second even third-order approximations are not desired result as compared with the exact solution. Furthermore, Guo et al. [27] obtained the analytical periodic solutions of the oscillator up to third-order approximation.

In this paper, a new analytical technique, combining the energy balance method [23] with harmonic balance method [17], has been presented to obtain higher-order approximations for nonlinear cubic-quintic Duffing equations. Generally, the second-order approximate frequency and the corresponding periodic solution have been determined which contains a few harmonic terms with lower order terms. The algebraic equations are easily analytically solved in this paper. The second-order approximate frequencies (obtained in this paper) give high accuracy for both small and large amplitudes of oscillation and also better than those obtained in [20] (calculated by second, third and fourth-order approximate frequencies). Moreover, the present method gives better result than other second-order approximation obtained by [27].

A cubic-quintic Duffing oscillator of a conservative autonomous system can be described by the following differential equation, with cubic-quintic nonlinearities [20, 26-27]:

$$\ddot{u} + \alpha u + \beta u^3 + \gamma u^5 = 0, \tag{1}$$

with initial conditions

$$u(0) = A, \dot{u}(0) = 0. \tag{2}$$

It is a simple harmonic oscillator if $\alpha \neq 0, \beta = \gamma = 0$, a cubic Duffing oscillator if $\beta \neq 0, \gamma = 0$, and a quintic oscillator if $\gamma \neq 0, \beta = 0$. Otherwise, it is a cubic-quintic oscillator if β and γ do not vanish (see [20, 26-27]).

It is also mentioned that in case of $0 < A < 1$, the system Eq. (1) will represent a small oscillations. On the other hand, in this case $A \geq 1$, the system Eq. (1) will represent a large oscillations (see [20, 26-27]).

2. The basis idea of He's energy balance method

According to the energy balance method [21-22, 28-29], a variational principle for the oscillation is established and then the corresponding Hamiltonian is considered from which the angular frequency can be easily founded by several residual methods.

Let us consider a general form on nonlinear oscillator with initial conditions in the following form [28-29]

$$\ddot{u} + f(u) = 0, u(0) = A, \dot{u}(0) = 0. \tag{3}$$

Its variational can be written as

$$J(u) = \int_0^{T/4} [-\frac{1}{2}\dot{u}^2 + F(u)]\,dt, \tag{4}$$

where $T = \dfrac{2\pi}{\omega}$ is a period of nonlinear oscillation and $F(u) = \int f(u)du$.

The Hamiltonian can be written as

$$H(t) = -\frac{1}{2}\dot{u}^2 + F(u) = F(A) \tag{5}$$

Equation (5) gives the following residual

$$R(t) = -\frac{1}{2}\dot{u}^2 + F(u) - F(A) = 0. \tag{6}$$

We consider the first-order approximate solution in the following form

$$u(t) = A\cos\omega t, \tag{7}$$

Substituting Eq. (7) into Eq. (6) yields the following residual

$$R(t) = -\frac{1}{2}A^2\omega^2\sin^2\omega t + F(A\cos\omega t) - F(A) = 0. \qquad (8)$$

And finally collocation at $\omega t = \frac{\pi}{4}$ gives [28-29]

$$\omega = \frac{2}{A}\sqrt{F(A) - F\left(\frac{\sqrt{2}}{2}A\right)}. \qquad (9)$$

3. Application of the coupled energy and harmonic balance methods

The variational of Eq. (1) can be written as

$$J(u) = \int_0^{T/4}\left[-\frac{1}{2}\dot{u}^2 + \alpha\frac{u^2}{2} + \beta\frac{u^4}{4} + \gamma\frac{u^6}{6}\right]dt \qquad (10)$$

Its Hamiltonian, therefore, can be written in the form

$$H(u) = \frac{\dot{u}^2}{2} + \frac{\alpha u^2}{2} + \frac{\beta u^4}{4} + \frac{\gamma u^6}{6} = \frac{\alpha A^2}{2} + \frac{\beta A^4}{4} + \frac{\gamma A^6}{6}\ . \qquad (11)$$

In order to obtain more accuracy, consider the second-order approximate solution of Eq.(1) in the following form [17]

$$u(t) = A((1-u_3)\cos\omega t + u\ \cos 3\omega t)\,. \qquad (12)$$

Equation (12) must satisfy initial conditions given in Eq. (2).

In order to calculate residual, substituting Eq. (12) into Eq. (11) and then we obtain

$$\begin{aligned} R(t) = {} & \frac{1}{2}\left(-\omega A((1-u_3)\sin\omega t + 3\sin 3\omega t)\right)^2 + \frac{\alpha}{2}\left(A((1-u_3))\cos\omega t + u\ \cos 3\omega t)\right)^2 \\ & + \frac{\beta}{4}\left(A((1-u_3))\cos\omega t + u\ \cos 3\omega t)\right)^4 + \frac{\gamma}{6}\left(A((1-u_3))\cos\omega t + u\ \cos 3\omega t)\right)^6 \\ & - \frac{\alpha A^2}{2} - \frac{\beta A^4}{4} - \frac{\gamma A^6}{6} \end{aligned} \qquad (13)$$

Now, dividing Eq. (13) by the factor $A^2\sec\omega t$ and then equating the coefficients of the terms $\cos\varphi$ and $\cos 3\varphi$ from the integral

$$\int_0^{T/4}\frac{R(t)\cos(2n-1)\omega t}{A^2\sec\omega t}dt, \qquad n = 1,2. \qquad (14)$$

equal to zeros, respectively, we obtain

$$\omega^2(1+4u_3) - \alpha - 4u_3\alpha - 3A^2\beta/4 - 5A^2u_3\beta/2 - 29A^4\gamma/48 - 7A^4u_3\gamma/4 + \cdots = 0\,, \quad (15)$$

$$\omega^2(-1+2u_3) + \alpha + 2u_3\alpha + 5A^2\beta/8 + A^2u_3\beta/2 + 7A^4\gamma/16 + \cdots = 0\,. \qquad (16)$$

It is noted that dividing by the factor $A^2\cos\omega t$ makes the solution rapidly converges and also significantly better result than other existing methods.

For the first approximation, setting $u_3 = 0$ in Eq. (15) we obtain the first approximate frequency as

$$48\alpha + 36A^2\beta + 29A^4\gamma - 48\omega^2 = 0. \tag{17}$$

Solving Eq. (17) for ω, we obtain

$$\omega = \omega_1(A) = \sqrt{\alpha + \frac{3A^2\beta}{4} + \frac{29A^4\gamma}{48}}. \tag{18}$$

Eliminating ω from these two Eqs. (15) and (16), we obtain the equation for u_3 as

$$1 - \frac{3(32\alpha + 20A^2\beta + 15A^4\gamma)}{A^2(3\beta + 4A^2\gamma)} u_3 + 24u_3^2 - 96u_3^3 + \cdots = 0. \tag{19}$$

The Eq. (19) can be written as

$$u_3 = \mu(1 + 24u_3^2 - 96u_3^3 + \cdots), \tag{20}$$

where

$$\mu = \frac{A^2(3\beta + 4A^2\gamma)}{3(32\alpha + 20A^2\beta + 15A^4\gamma)}.$$

Now, u_3 can be obtained in powers of μ of the form $u_3 = l_1\mu + l_2\mu^2 + l_3\mu^3 + \cdots$(see [17] for details) where the unknown coefficients, $l_1, l_2, l_3, \cdots$ to be determined. Therefore, we have obtained the solution of Eq. (20) as

$$u_3 = \mu + 24\mu^3 - 96\mu^4 + \cdots. \tag{21}$$

It is noted that the series of u_3 is converge for all values of A.

Solving Eq. (15) for ω, we obtain second approximate frequency as

$$\omega = \omega_2(A) = \sqrt{\frac{\alpha + 4u_3\alpha + \frac{3A^2\beta}{4} + \frac{5A^2u_3\beta}{2} + \frac{29A^4\gamma}{48} + \frac{7A^4u_3\gamma}{4}}{1 + u_3}}, \tag{22}$$

where, u_3 is given in Eq. (21).

Therefore, the second-order approximation becomes

$$u(t) = A((1 - u_3))\cos\omega t + u_3 \cos 3\omega t) \tag{24}$$

where, u_3 and ω are given in Eq. (21) and Eq. (22), respectively.

4. Results and discussion

A new analytical technique coupled by the energy and harmonic balance methods has been presented to determine the approximate frequency and the corresponding solution of cubic-quintic Duffing oscillator. The method is valid for both small ($0 < A < 1$) and large ($A \geq 1$) amplitudes of oscillation. Recently, Khan et al. [20, 27] has investigated the same oscillator by coupling of homotopy and variational approach and obtained first, second, third and forth-order approximate frequencies. But the determinations of third and fourth-order approximations are laborious process. In this situation, the determinations of first (given in Eq. (18) and second-order (given in Eq. (22)) approximations obtained in this paper are easy and straightforward.

To verify the efficiency and accuracy of the present method for cubic-quintic Duffing oscillator, in comparison with other results and the exact result, three cases are given: $\alpha = \beta = \gamma = 1$, $\alpha = 5, \beta = 3, \gamma = 1$ and $\alpha = 1, \beta = 10, \gamma = 100$ (see [20]). The relative errors of frequencies are defined [26]

$$Error(\%) = \frac{|\omega_i - \omega_{Exact}|}{\omega_{Exact}}, \; i = 1,2,3,4,\cdots. \qquad (21)$$

The relative errors of the first and second-order analytical approximations obtained in this paper are compared with the exact solution, provide results less than 4.077% and 0.102%, respectively in the case $A \geq 1$ (i.e. large amplitudes). In all Tables 1–3, the relative errors for the approximate frequencies for different parameters have been presented.

On the other hand, the relative errors of the first and second, third and fourth-order analytical approximations obtained by [20] are compared with the exact solution provide results less than 25.149%, 15.519%, 7.050% and 0.154% respectively.

Furthermore, the relative errors of the second-order analytical approximations obtained by [27] are compared with the exact solution goes less than 1.078%.

Another from these three Tables, we also see that the present method gives better result than those obtained in [20, 27] for small values of amplitude, $0 < A < 1$. The convergence rate of the present method is faster than [20, 27]. Therefore, the present method is suitable for solving Eq. (1) than [20, 27].

Furthermore, we have determined the second-order approximate solutions of Eq. (1) for different values of parameters A, α, β, γ and including all results with corresponding numerical solutions obtained by fourth-order Runge-Kutta method. All results have been presented in figures 1-6. From these figures, we see that the present method solutions are nicely agreed with the corresponding numerical result for all values of parameters A, α, β, γ.

5. Conclusion

In this paper, a new simple analytical technique coupled by energy and harmonic balance methods has been presented to solve of cubic-quintic Duffing oscillator. The second-order approximation is determined in this paper. The solution contains a few harmonic terms and also contains a lower order terms. These terms make the solution rapidly converges. The present method gives better result than other existing results for both small and large amplitudes of oscillation. It has been proved that the present method is very effective, convenient and also gives more precise accuracy for solving strongly nonlinear oscillators.

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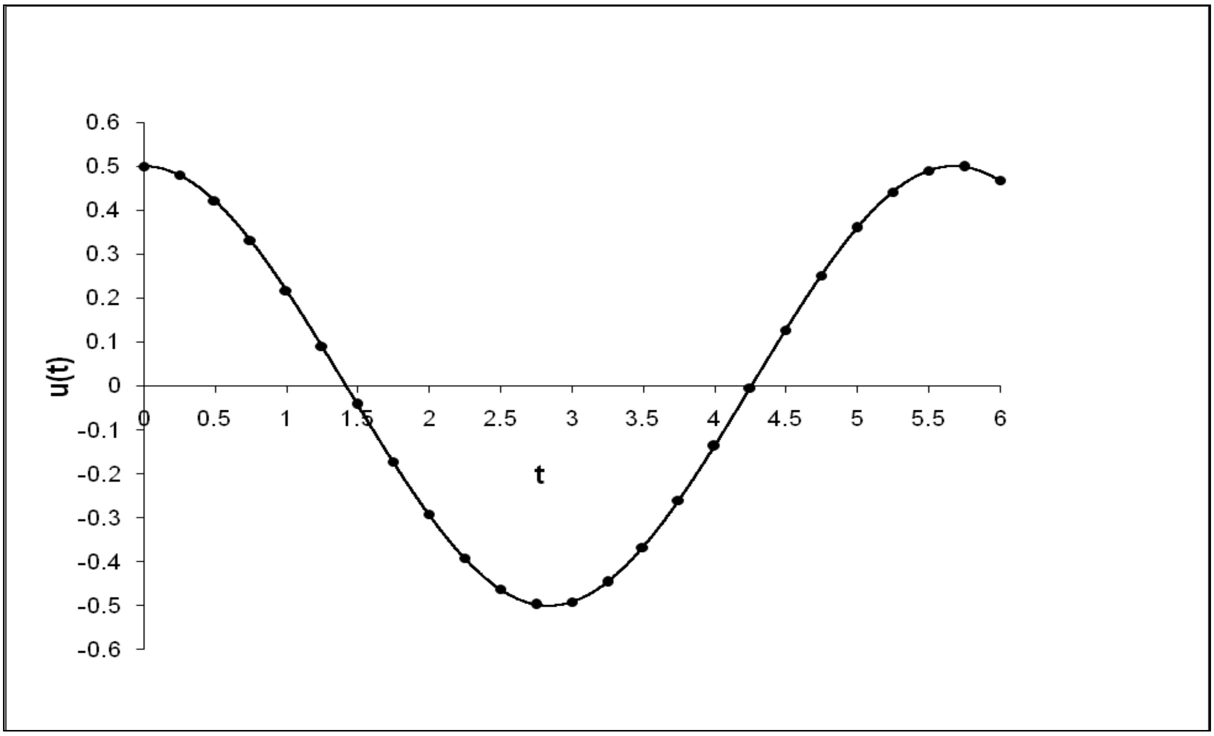

Figure 1. Comparison of approximate periodic solution obtained by present method (denoting by circles) with numerical solution obtained by fourth-order Runge-Kutta method (denoting by solid line) for the cubic-quintic Duffing oscillator (Eq. (1) for $\alpha = 1, \beta = 1, \gamma = 1, A = 0.5$.

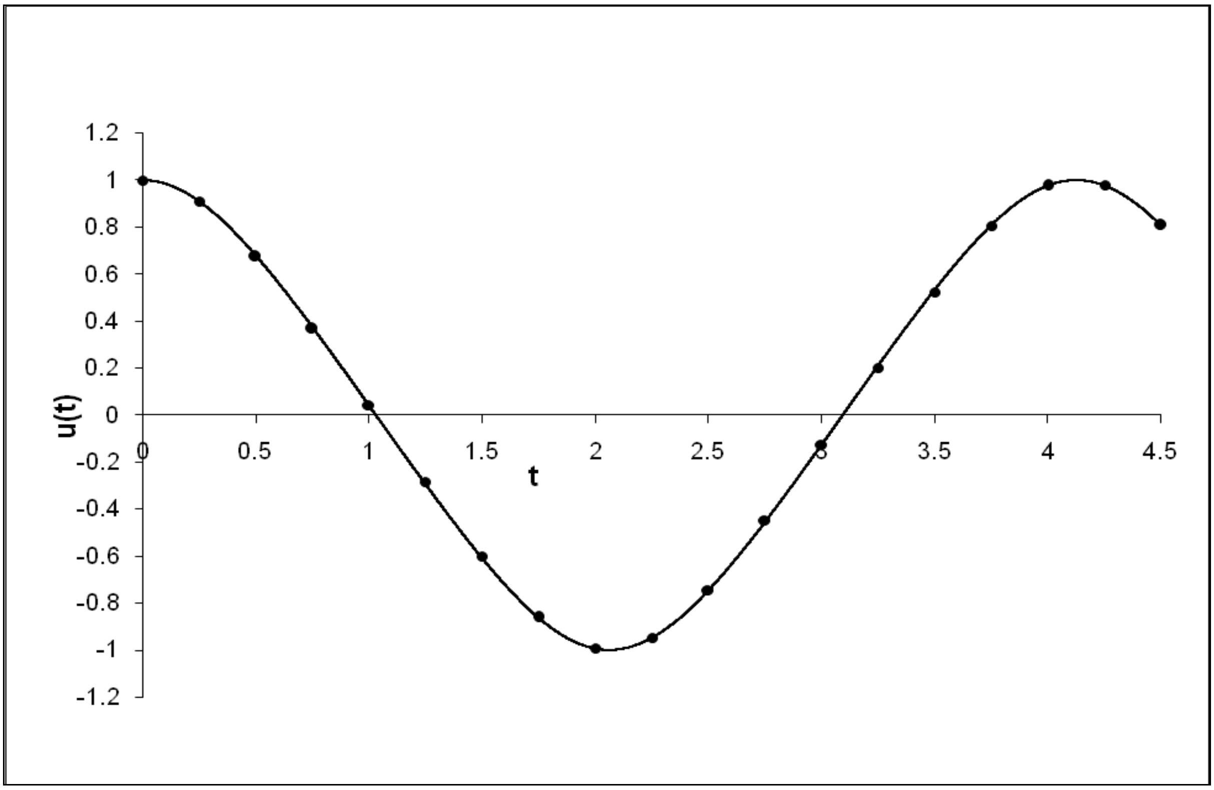

Figure 2. Comparison of approximate periodic solution obtained by present method (denoting by circles) with numerical solution obtained by fourth-order Runge-Kutta method (denoting by solid line) for the cubic-quintic Duffing oscillator (Eq. (1) for $\alpha = 1, \beta = 1, \gamma = 1, A = 1.0$.

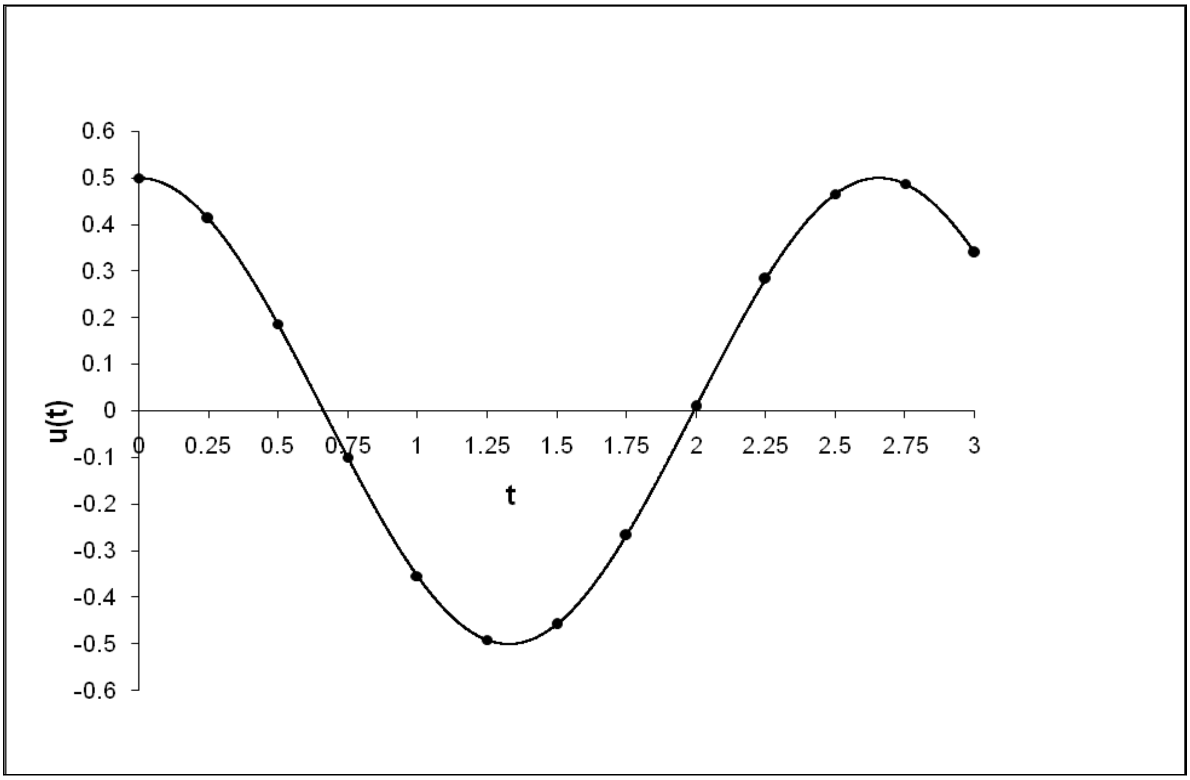

Figure 3. Comparison of approximate periodic solution obtained by present method (denoting by circles) with numerical solution obtained by fourth-order Runge-Kutta method (denoting by solid line) for the cubic-quintic Duffing oscillator (Eq. (1) for $\alpha = 5, \beta = 3, \gamma = 1, A = 0.5$.

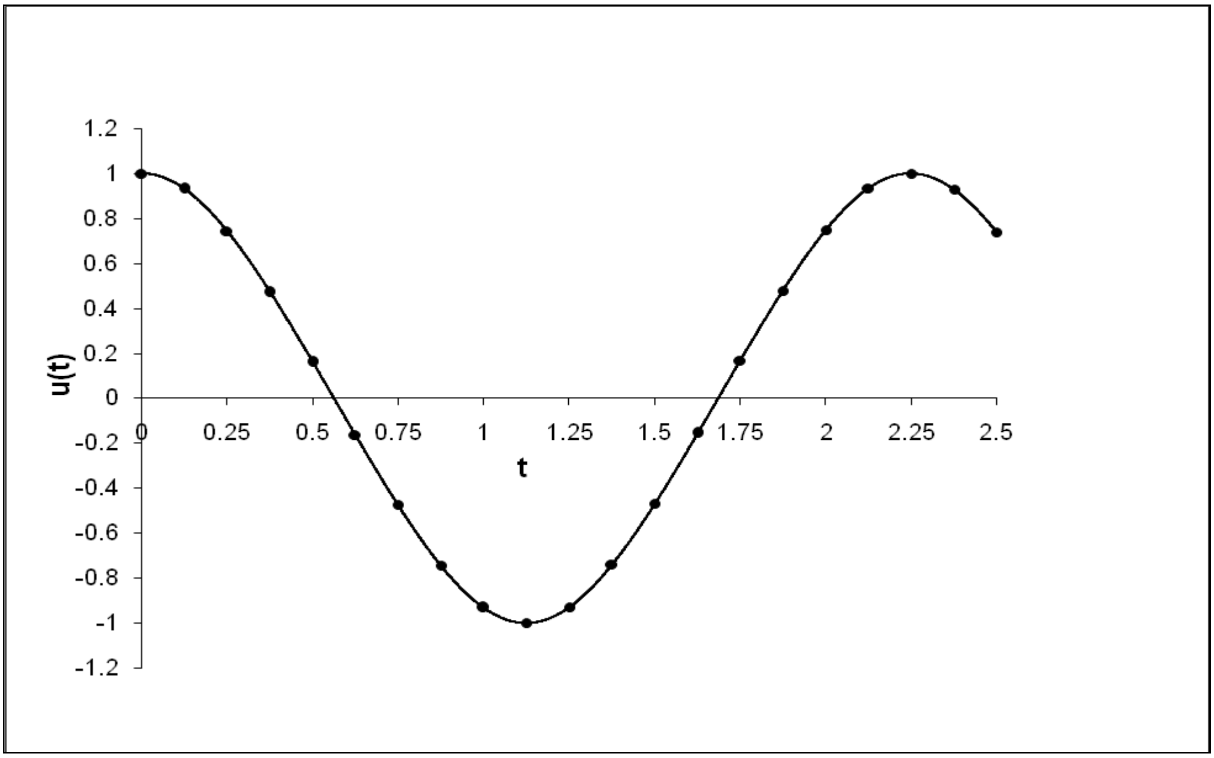

Figure 4. Comparison of approximate periodic solution obtained by present method (denoting by circles) with numerical solution obtained by fourth-order Runge-Kutta method (denoting by solid line) for the cubic-quintic Duffing oscillator (Eq. (1) for $\alpha = 5, \beta = 3, \gamma = 1, A = 1.0$.

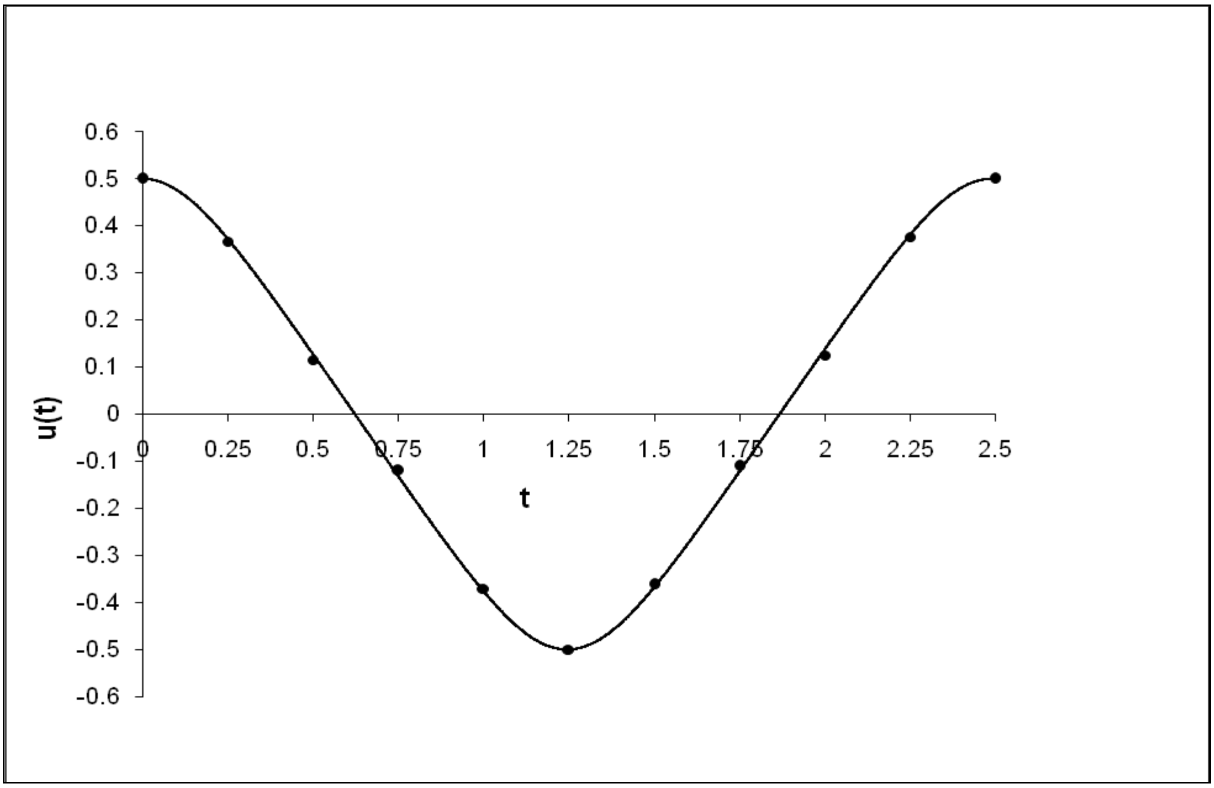

Figure 5. Comparison of approximate periodic solution obtained by present method (denoting by circles) with numerical solution obtained by fourth-order Runge-Kutta method (denoting by solid line) for the cubic-quintic Duffing oscillator (Eq. (1) for $\alpha = 1, \beta = 10, \gamma = 100, A = 0.5$.

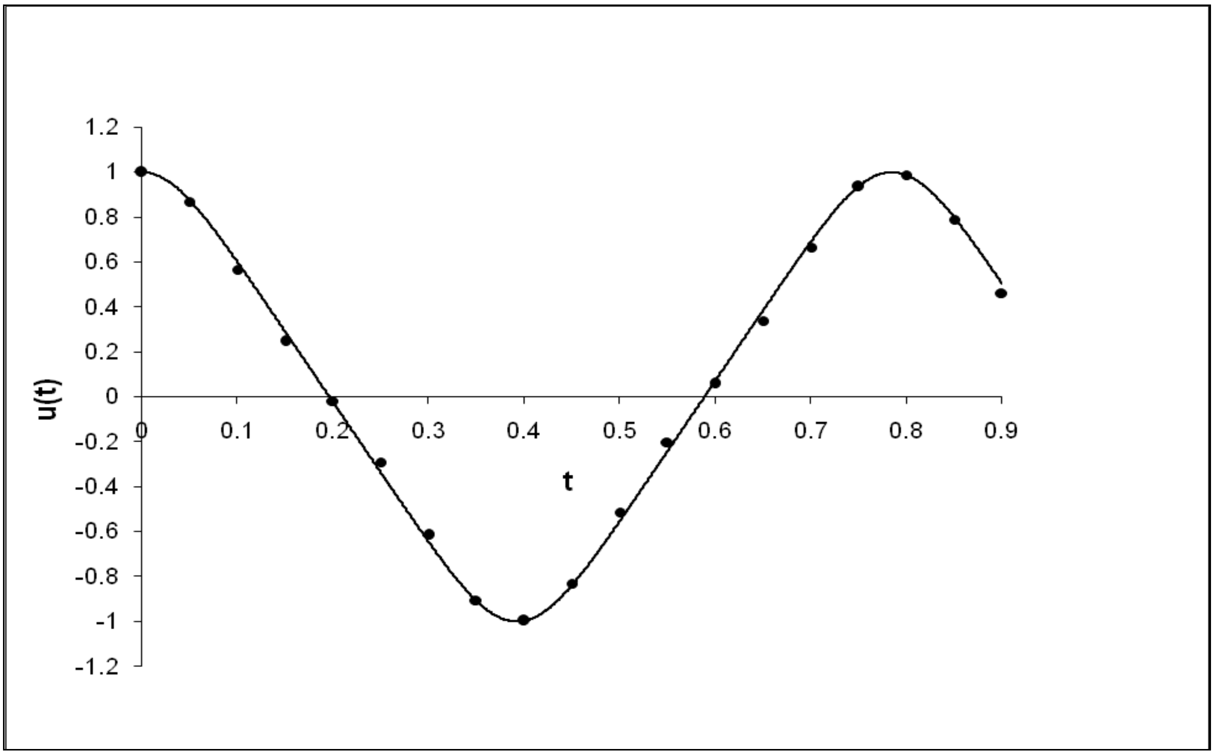

Figure 6. Comparison of approximate periodic solution obtained by present method (denoting by circles) with numerical solution obtained by fourth-order Runge-Kutta method (denoting by solid line) for the cubic-quintic Duffing oscillator (Eq. (1) for $\alpha = 1, \beta = 10, \gamma = 100, A = 1.0$.

Table 1. Comparison of present frequency and existing results for cubic–quintic Duffing oscillator when $\alpha = \beta = \gamma = 1$.

A	ω_e	Khan et al. [20]				Guo et al. [27]	Present study	
		ω_1	ω_2	ω_3	ω_4	ω_2	ω_1	ω_2
		Er(%)	Er(%)	Er(%)	Er(%)	Er(%)	Er(%)	Er(%)
0.1	1.0037770	1.0025125	1.0028276	1.0031009	1.0034276	1.0037730	1.0037732	1.0037722
		0.126	0.095	0.067	0.035	0.000	0.000	0.000
0.5	1.1065487	1.0698277	1.0792589	1.0877056	1.0974873	1.1065755	1.1069148	1.1062745
		3.319	2.467	1.703	0.819	0.003	0.033	0.025
1	1.5235914	1.3462912	1.3984287	1.4456576	1.4951413	1.5250736	1.5343294	1.5224068
		11.637	8.215	5.115	1.867	0.097	0.705	0.078
5	19.1815720	14.4503460	16.2514248	17.8276787	19.1806374	19.3735477	19.9337444	19.1676301
		24.666	15.276	7.058	0.005	1.001	3.921	0.073
10	75.1776276	56.3560104	63.5547600	69.8760834	75.2651825	75.9737510	78.2155142	75.1074146
		25.036	15.461	7.052	0.115	1.060	4.041	0.093
20	299.22427	224.05580	252.83170	278.12708	299.65771	302.43540	311.39632	298.92664
		25.121	15.504	7.051	0.145	1.073	4.068	0.099
50	1867.5796	1397.9900	1577.7996	1735.9103	1870.4300	1887.6949	1943.6866	1865.6890
		25.145	15.516	7.050	0.153	1.077	4.075	0.101
100	7468.8525	5590.6172	6309.8315	6942.2827	7480.3364	7549.3401	7773.2984	7461.2727
		25.148	15.518	7.050	0.154	1.078	4.076	0.102
500	186709.59	139754.70	157734.86	173546.20	186997.34	188721.99	194320.88	186519.96
		25.149	15.519	7.050	0.154	1.078	4.077	0.102
1000	746836.94	559017.44	630938.13	694183.44	747988.00	754886.53	777282.07	746078.35
		25.149	15.519	7.050	0.154	1.078	4.077	0.102

where Er(%) denotes the absolute percentage error.

Table 2. Comparison of present frequency and existing results for cubic–quintic Duffing oscillator when $\alpha = 5, \beta = 3, \gamma = 1$.

A	ω_e	Khan et al. [20]				Guo et al. [27]	Present study	
		ω_1	ω_2	ω_3	ω_4	ω_2	ω_1	ω_2
		Er(%)	Er(%)	Er(%)	Er(%)	Er(%)	Er(%)	Er(%)
0.1	2.2411156	2.2394266	2.2398469	2.2402105	2.2406456	2.2411070	2.2411070	2.2411063
		0.075	0.057	0.041	0.021	0.001	0.000	0.000
0.5	2.3661575	2.3226130	2.3337476	2.3434565	2.3548751	2.3661560	2.3664870	2.3660775
		1.840	1.370	0.960	0.477		0.014	0.003
1	2.7962794	2.6100767	2.6612775	2.7063482	2.7566507	2.7966959	2.8025286	2.7958356
		6.659	4.828	3.216	1.417	0.015	0.223	0.016
5	20.2164536	15.4211702	17.2236977	18.7895069	20.1565380	20.3911022	20.9488464	20.2109616
		23.720	14.804	7.058	0.296	0.864	3.623	0.027
10	76.1700134	57.2712860	64.4817429	70.7962723	76.2022247	76.9486769	79.1938550	76.1106133
		24.811	15.345	7.055	0.042	1.022	3.970	0.078
50	1868.5568	1398.8853	1578.7106	1736.8159	1871.3540	1888.6547	1944.6521	1866.6778
		25.136	15.512	7.051	0.150	1.076	4.073	0.101
100	7469.8296	5591.5117	6310.7422	6943.1880	7481.2598	7550.2994	7774.2634	7462.2611
		25.146	15.517	7.050	0.153	1.078	4.076	0.101
500	186710.58	139755.59	157735.78	173547.11	186998.27	188722.95	194321.85	186520.95
		25.149	15.519	7.050	0.154	1.078	4.077	0.102
1000	746837.94	559018.31	630939.00	694184.38	747988.88	754887.48	777283.04	746079.34
		25.149	15.519	7.050	0.154	1.078	4.077	0.102

where Er(%) denotes the absolute percentage error.

Table 3. Comparison of present frequency and existing results for cubic–quintic Duffing oscillator when $\alpha = 1, \beta = 10, \gamma = 100$.

A	ω_e	Khan et al. [20]				Guo et al. [27]	Present study	
		ω_1 $Er(\%)$	ω_2 $Er(\%)$	ω_3 $Er(\%)$	ω_4 $Er(\%)$	ω_2 $Er(\%)$	ω_1 $Er(\%)$	ω_2 $Er(\%)$
0.1	1.0397019	1.0262188 1.297	1.0296117 0.971	1.0325994 0.683	1.0361322 0.343	1.0397000 0.000	1.0397315 0.003	1.0396423 0.006
0.5	2.5247023	2.0501525 18.796	2.2114758 12.407	2.3542032 6.753	2.4890764 1.411	2.5350468 0.410	2.5789614 2.150	2.5236147 0.043
1	8.0100698	6.1032777 23.805	6.8193183 14.866	7.4440041 7.067	7.9883609 0.271	8.0806905 0.882	8.3016063 3.640	8.0064163 0.046
5	187.19966	140.20432 25.105	158.19193 15.496	174.00040 7.051	187.46034 0.139	189.20333 1.070	194.80482 4.063	187.01580 0.098
10	747.32526	559.46490 25.138	631.39343 15.513	694.63605 7.050	748.44946 0.151	755.36618 1.076	777.76453 4.073	746.57252 0.100
50	18671.400	13975.872 25.148	15773.896 15.519	17355.027 7.050	18700.148 0.154	18872.631 1.078	19432.522 4.075	18652.440 0.102
100	74684.133	55902.148 25.149	63094.219 15.519	69418.750 7.050	74799.211 0.154	75489.084 1.078	77728.641 4.077	74608.280 0.102
500	1867091.6	1397542.9 25.149	1577344.5 15.519	1735457.9 7.050	1869969.3 0.154	1887215.6 1.078	1943200.0 4.076	1865200.0 0.101
1000	7468365.0	5590170.5 25.149	6309377.0 15.519	6941830.5 7.050	7479875.5 0.154	7548860.9 1.078	7772820.0 4.077	7460780.0 0.102

where $Er(\%)$ denotes the absolute percentage error.

Biography

Md. Abdur Razzak completed his both M.Sc. and B.Sc. degrees in the Department of Applied Mathematics from Rajshahi University, Bangladesh. At present, he is an assistant Professor at the Department of Mathematics, Rajshahi University of Engineering and Technology (RUET), and a M.Phil fellow of RUET, Bangladesh. He is a lifetime member of Bangladesh Mathematical Society. He is the author of nine research articles in Mathematics. His research area is Nonlinear Dynamics.