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Coupling of energy and harmonic balance methods for solving a conservative oscillator with strong odd-nonlinearity

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Abstract. In this paper, a new analytical technique, i.e. a combination of the Energy Balance Method (EBM) with Harmonic Balance Method (HBM), is presented to obtain higher-order approximations of a conservative oscillator with strong odd nonlinearity. To show the accuracy of the present method, one nonlinear oscillator, named as cubic-quintic Duffing oscillator, is investigated. The results obtained in this paper are compared with those determined by other methods and exact solutions. The results give high accuracy and also provide better results than other existing results for both small and large amplitudes of oscillation. The main advantage of the present paper is its simplicity, which contains a few harmonic terms with lower order terms, and these terms make the solution converge quickly. The present technique can be used for other nonlinear oscillators.

1. Introduction

The study of nonlinear oscillations is a necessary issue in engineering, physical science, applied mathematics, mechanical structures, nonlinear circuits, chemical oscillation, and many real-world applications [1-30]. Nonlinear oscillations are modeled by nonlinear differential equations. Many analytical techniques have been developed to solve these nonlinear differential equations. One of the most widely used techniques is perturbation method [1-4] whereby the nonlinearities are small. However, these techniques have many shortcomings and cannot be used due to strongly nonlinear systems. To overcome these shortcomings, many analytical techniques, such as variational iterative method [5,6], homotopy perturbation method [7-9], iterative method [10-12], harmonic balance method [13-17], variational approach [18,19], and coupled method [20], are used to solve strongly nonlinear equations. The energy balance method [21,22] is also another technique to obtain a first-order approximation of strongly nonlinear oscillators. Usually, a set of algebraic equations with complex nonlinearities appears when EBM is formulated for determining higher-order approximations. On the contrary, some authors [23,24] have extended the energy balance method to obtain higher-order approximations, but the algebraic equations are not solved analytically.

The Duffing equation is a well-known nonlinear differential equation [20,25,26] which is related to many practical engineering systems, such as the classical nonlinear spring system with odd nonlinear restoring characteristics [3], and has become applicable more recently in different physical phenomena [25]. There have been many variations of Duffing equation, for instance, the Duffing-harmonic equation [11,12] and the cubic-quintic Duffing equation. The unperturbed cubic-

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quintic Duffing equation can be found in the modeling of the free vibration of a restrained uniform beam carrying an intermediate lumped mass and undergoing large amplitudes of oscillation in the unimodel Duffing-type temporal problem [26], the nonlinear dynamics of a slender elastica, the generalized Pochhammer-Chree (PC) equation, and the compound Korteweg-de Vries (KdV) equation [26]. A differential equation with fifth-power nonlinearity is very difficult to handle due to the presence of strong nonlinearity.

Due to the presence of fifth-power nonlinearity, the accuracy of approximate analytical methods becomes extremely demanding [20]. Recently, several authors [23,24] have extended the energy balance method to determine higher-order approximations. The limitation of the articles [23,24] is that they have not analytically solved algebraic equations; instead, they have only solved these algebraic equations numerically. On the other hand, Khan et al. [20] used coupled method of He's homotopy perturbation method [7] and variational formulation [18] to obtain higher-order approximations for nonlinear cubic-quintic Duffing equations. However, the first, second, even third-order approximations bring about unfavorable results, as compared with the exact solution. Furthermore, Guo et al. [27] obtained the analytical periodic solutions of the oscillator up to third-order approximation.

In this paper, a new analytical technique, combining the energy balance method [23] with harmonic balance method [17], has been presented to obtain higher-order approximations for nonlinear cubic-quintic Duffing equations. Generally, the second-order approximate frequency and the corresponding periodic solution have been determined containing a few harmonic terms with lower order terms. The algebraic equations are analytically solved in this paper easily. The second-order approximate frequencies (obtained in this paper) show high accuracy in both small and large amplitudes of oscillation and also better than those obtained in [20] (calculated by the second-, third-, and fourth-order approximate frequencies). Moreover, compared to the other second-order approximation, the present method gives better results obtained by Guo et al. [27].

A cubic-quintic Duffing oscillator of a conservative autonomous system can be described by the following differential equation with cubic-quintic nonlinearities [20,26,27]:

$$\ddot{u} + \alpha\, u + \beta\, u^3 + \gamma\, u^5 = 0, \tag{1}$$

with initial conditions:

$$u(0) = A, \qquad \dot{u}(0) = 0. \tag{2}$$

It is a simple harmonic oscillator if $\alpha \neq 0, \beta = \gamma = 0$; it is a cubic Duffing oscillator if $\beta \neq 0, \gamma = 0$; further, it is a quintic oscillator if $\gamma \neq 0, \beta = 0$. Otherwise, it is a cubic-quintic oscillator if β and γ do not vanish (see [20,26,27]).

It should be noted that, in the case of $0 < A < 1$, system Eq. (1) will present small oscillations. On the other hand, in the case of $A \geq 1$, system Eq. (1) will present large oscillations (see [20,26,27]).

2. The basic idea of He's energy balance method

According to the energy balance method [21,22,28,29], a variational principle for the oscillation is established, and then the corresponding Hamiltonian is considered from which the angular frequency can be easily founded by several residual methods.

Let us consider a general form of a nonlinear oscillator with initial conditions in the following form [28,29]:

$$\ddot{u} + f(u) = 0, \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{3}$$

Its variational principle can be written as follows:

$$J(u) = \int_0^{T/4} \left[-\frac{1}{2}\dot{u}^2 + F(u) \right] dt, \tag{4}$$

where $T = \frac{2\pi}{\omega}$ is a period of nonlinear oscillation and $F(u) = \int f(u)du$.

The Hamiltonian can be written as follows:

$$H(t) = -\frac{1}{2}\dot{u}^2 + F(u) = F(A). \tag{5}$$

Eq. (5) gives the following residual:

$$R(t) = -\frac{1}{2}\dot{u}^2 + F(u) - F(A) = 0. \tag{6}$$

We consider the first-order approximate solution in the following form:

$$u(t) = A\,\cos\omega t, \tag{7}$$

Substituting Eq. (7) into Eq. (6) yields the following residual:

$$R(t) = -\frac{1}{2}A^2\omega^2\sin^2\omega t + F(A\,\cos\omega t) - F(A) = 0. \tag{8}$$

Finally, collocation at $\omega t = \frac{\pi}{4}$ gives [28,29]:

$$\omega = \frac{2}{A}\sqrt{F(A) - F\left(\frac{\sqrt{2}}{2}A\right)}. \tag{9}$$

3. Application of the coupled energy and harmonic balance methods

The variational principle of Eq. (1) can be written as follows:

$$J(u)=\int_0^{T/4}\left[-\frac{1}{2}\dot{u}^2+\alpha\frac{u^2}{2}+\beta\frac{u^4}{4}+\gamma\frac{u^6}{6}\right]dt. \qquad (10)$$

Its Hamiltonian, therefore, can be written in the following form:

$$H(u)=\frac{\dot{u}^2}{2}+\frac{\alpha u^2}{2}+\frac{\beta u^4}{4}+\frac{\gamma u^6}{6}=\frac{\alpha A^2}{2}+\frac{\beta A^4}{4}+\frac{\gamma A^6}{6}. \qquad (11)$$

In order to obtain more accuracy, consider the second-order approximate solution of Eq. (1) in the following form [17]:

$$u(t)=A((1-u_3)\cos\omega t+u\ \ \cos 3\omega t). \qquad (12)$$

Eq. (12) must satisfy initial conditions given in Eq. (2).

In order to calculate the residual, by substituting Eq. (12) into Eq. (11), we obtain:

$$\begin{aligned}R(t)=&\frac{1}{2}(-\omega A((1-u_3)\sin\omega t+3\sin 3\omega t))^2\\&+\frac{\alpha}{2}(A((1-u_3))\cos\omega t+u\ \ \cos 3\omega t))^2\\&+\frac{\beta}{4}(A((1-u_3))\cos\omega t+u\ \ \cos 3\omega t))^4\\&+\frac{\gamma}{6}(A((1-u_3))\cos\omega t+u\ \ \cos 3\omega t))^6\\&-\frac{\alpha A^2}{2}-\frac{\beta A^4}{4}-\frac{\gamma A^6}{6}.\end{aligned} \qquad (13)$$

Now, through dividing Eq. (13) by factor $A^2\sec\omega t$ and then equating the coefficients of the terms $\cos\varphi$ and $\cos 3\varphi$ from the integral:

$$\int_0^{T/4}\frac{R(t)\cos(2n-1)\omega t}{A^2\sec\omega t}dt, \qquad n=1,2. \qquad (14)$$

Respective zeros are obtained as follows:

$$\omega^2(1+4u_3)-\alpha-4u_3\alpha-3A^2\beta/4-5A^2u_3\beta/2-29A^4\gamma/48-7A^4u_3\gamma/4+\cdots=0, \qquad (15)$$

$$\omega^2(-1+2u_3)+\alpha+2u_3\alpha+5A^2\beta/8+A^2u_3\beta/2+7A^4\gamma/16+\cdots=0. \qquad (16)$$

It is noted that dividing factor $A^2\cos\omega t$ makes the solution converge rapidly and also gives significantly better results than other existing methods do.

For the first approximation, by setting $u_3=0$ in Eq. (15), the first approximate frequency is obtained as follows:

$$48\alpha+36A^2\beta+29A^4\gamma-48\omega^2=0. \qquad (17)$$

Solving Eq. (17) for ω, the following is obtained:

$$\omega=\omega_1(A)=\sqrt{\alpha+\frac{3A^2\beta}{4}+\frac{29A^4\gamma}{48}}. \qquad (18)$$

Eliminating ω from these two Eqs. (15) and (16), the equation for u_3 is obtained as follows:

$$1-\frac{3(32\alpha+20A^2\beta+15A^4\gamma)}{A^2(3\beta+4A^2\gamma)}u_3+24u_3^2-96u_3^3+\cdots=0. \qquad (19)$$

Eq. (19) can be written as follows:

$$u_3=\mu(1+24u_3^2-96u_3^3+\cdots), \qquad (20)$$

where:

$$\mu=\frac{A^2(3\beta+4A^2\gamma)}{3(32\alpha+20A^2\beta+15A^4\gamma)}.$$

Now, u_3 can be obtained in powers of μ of the form $u_3=l_1\mu+l_2\mu^2+l_3\mu^3+...$ (see [17] for details) where unknown coefficients $l_1,l_2,l_3,...$ are to be determined. Therefore, we have obtained the solution of Eq. (20) as follows:

$$u_3=\mu+24\mu^3-96\mu^4+\cdots. \qquad (21)$$

It is noted that the series of u_3 converge to all values of A.

By solving Eq. (15) for ω, the second approximate frequency is obtained as follows:

$$\omega=\omega_2(A)=\sqrt{\frac{\alpha+4u_3\alpha+\frac{3A^2\beta}{4}+\frac{5A^2u_3\beta}{2}+\frac{29A^4\gamma}{48}+\frac{7A^4u_3\gamma}{4}}{1+u_3}}, \qquad (22)$$

where u_3 is given in Eq. (21).

Therefore, the second-order approximation becomes:

$$u(t)=A((1-u_3))\cos\omega t+u_3\ \ \cos 3\omega t), \qquad (23)$$

where u_3 and ω are given in Eqs. (21) and (22), respectively.

4. Results and discussion

A new analytical technique coupled by the energy and harmonic balance methods has been presented to determine the approximate frequency and the corresponding solution to cubic-quintic Duffing oscillator. The method is valid for both small ($0 < A < 1$) and large ($A \geq 1$) amplitudes of oscillation. Recently, Khan et al. [20,27] have investigated the same oscillator by coupling homotopy with variational approaches and obtained the first-, second-, third-, and forth-order approximate frequencies. However, the determination of the third- and fourth-order approximations is a laborious process. In this situation, the determination of (first-order (given in Eq. (18) and second-order (given in Eq. (22)) approximations obtained in this paper is an easy and straightforward process.

To verify the efficiency and accuracy of the present method for cubic-quintic Duffing oscillator, in comparison with other results and the exact result, three cases are given: $\alpha = \beta = \gamma = 1$, $\alpha = 5$, $\beta = 3$, $\gamma = 1$ and $\alpha = 1$, $\beta = 10$, $\gamma = 100$ (see [20]). The relative errors of frequencies are defined as follows [26]:

$$\text{Error } (\%) = \frac{|\omega_i - \omega_{\text{Exact}}|}{\omega_{\text{Exact}}}, \ i = 1, 2, 3, 4, \cdots . \tag{24}$$

The relative errors of the first- and second-order analytical approximations obtained in this paper are compared with the exact solution, providing results less than 4.077% and 0.102%, respectively, in the case of $A \geq 1$ (i.e., large amplitudes). In Tables 1-3, the relative errors for the approximate frequencies of different parameters are presented.

On the other hand, the relative errors of the first-, second-, third-, and fourth-order analytical approximations obtained by [20] are compared with the exact solution, providing results less than 25.149%, 15.519%, 7.050%, and 0.154%, respectively.

Furthermore, the relative errors of the second-order analytical approximations obtained by [27] are compared with the exact solution which are less than 1.078%.

Based on these three tables, we also see that the present method gives better results than those obtained in [20,27] for small values of amplitude, $0 < A < 1$. The convergence rate of the present method is faster

Table 1. Comparison of the present frequency with existing results for cubic-quintic Duffing oscillator when $\alpha = \beta = \gamma = 1$.

A	ω_e	Khan et al. [20]			Guo et al. [27]		Present study	
		ω_1 Er (%)*	ω_2 Er (%)	ω_3 Er (%)	ω_4 Er (%)	ω_2 Er (%)	ω_1 Er (%)	ω_2 Er (%)
0.1	1.0037770	1.0025125 0.126	1.0028276 0.095	1.0031009 0.067	1.0034276 0.035	1.0037730 0.000	1.0037732 0.000	1.0037722 0.000
0.5	1.1065487	1.0698277 3.319	1.0792589 2.467	1.0877056 1.703	1.0974873 0.819	1.1065755 0.003	1.1069148 0.033	1.1062745 0.025
1	1.5235914	1.3462912 11.637	1.3984287 8.215	1.4456576 5.115	1.4951413 1.867	1.5250736 0.097	1.5343294 0.705	1.5224068 0.078
5	19.1815720	14.4503460 24.666	16.2514248 15.276	17.8276787 7.058	19.1806374 0.005	19.3735477 1.001	19.9337444 3.921	19.1676301 0.073
10	75.1776276	56.3560104 25.036	63.5547600 15.461	69.8760834 7.052	75.2651825 0.115	75.9737510 1.060	78.2155142 4.041	75.1074146 0.093
20	299.22427	224.05580 25.121	252.83170 15.504	278.12708 7.051	299.65771 0.145	302.43540 1.073	311.39632 4.068	298.92664 0.099
50	1867.5796	1397.9900 25.145	1577.7996 15.516	1735.9103 7.050	1870.4300 0.153	1887.6949 1.077	1943.6866 4.075	1865.6890 0.101
100	7468.8525	5590.6172 25.148	6309.8315 15.518	6942.2827 7.050	7480.3364 0.154	7549.3401 1.078	7773.2984 4.076	7461.2727 0.102
500	186709.59	139754.70 25.149	157734.86 15.519	173546.20 7.050	186997.34 0.154	188721.99 1.078	194320.88 4.077	186519.96 0.102
1000	746836.94	559017.44 25.149	630938.13 15.519	694183.44 7.050	747988.00 0.154	754886.53 1.078	777282.07 4.077	746078.35 0.102

*: Er (%) denotes the absolute percentage error.

Table 2. Comparison of the present frequency with existing results for cubic-quintic Duffing oscillator when $\alpha = 5, \beta = 3$, and $\gamma = 1$.

A	ω_e	Khan et al. [20]			Guo et al. [27]		Present study	
		ω_1 Er (%)*	ω_2 Er (%)	ω_3 Er (%)	ω_4 Er (%)	ω_2 Er (%)	ω_1 Er (%)	ω_2 Er (%)
0.1	2.2411156	2.2394266 0.075	2.2398469 0.057	2.2402105 0.041	2.2406456 0.021	2.2411070 0.001	2.2411070 0.000	2.2411063 0.000
0.5	2.3661575	2.3226130 1.840	2.3337476 1.370	2.3434565 0.960	2.3548751 0.477	2.3661560 0.014	2.3664870 0.003	2.3660775
1	2.7962794	2.6100767 6.659	2.6612775 4.828	2.7063482 3.216	2.7566507 1.417	2.7966959 0.015	2.8025286 0.223	2.7958356 0.016
5	20.2164536	15.4211702 23.720	17.2236977 14.804	18.7895069 7.058	20.1565380 0.296	20.3911022 0.864	20.9488464 3.623	20.2109616 0.027
10	76.1700134	57.2712860 24.811	64.4817429 15.345	70.7962723 7.055	76.2022247 0.042	76.9486769 1.022	79.1938550 3.970	76.1106133 0.078
50	1868.5568	1398.8853 25.136	1578.7106 15.512	1736.8159 7.051	1871.3540 0.150	1888.6547 1.076	1944.6521 4.073	1866.6778 0.101
100	7469.8296	5591.5117 25.146	6310.7422 15.517	6943.1880 7.050	7481.2598 0.153	7550.2994 1.078	7774.2634 4.076	7462.2611 0.101
500	186710.58	139755.59 25.149	157735.78 15.519	173547.11 7.050	186998.27 0.154	188722.95 1.078	194321.85 4.077	186520.95 0.102
1000	746837.94	559018.31 25.149	630939.00 15.519	694184.38 7.050	747988.88 0.154	754887.48 1.078	777283.04 4.077	746079.34 0.102

*: Er (%) denotes the absolute percentage error.

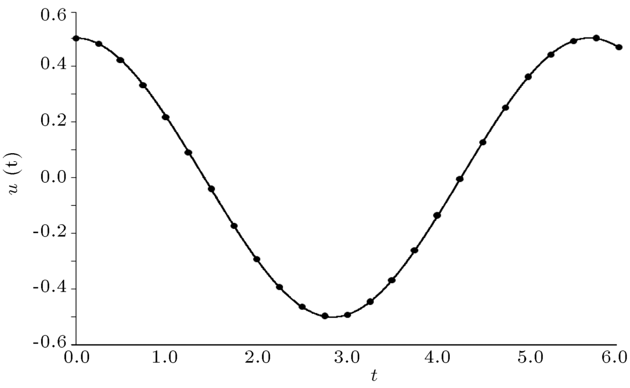

Figure 1. Comparison of approximate periodic solution obtained by the present method (denoted by circles) with the numerical solution obtained by the fourth-order Runge-Kutta method (denoted by solid line) for the cubic-quintic Duffing oscillator (Eq. (1)) for $\alpha = 1$, $\beta = 1$, $\gamma = 1$, and $A = 0.5$).

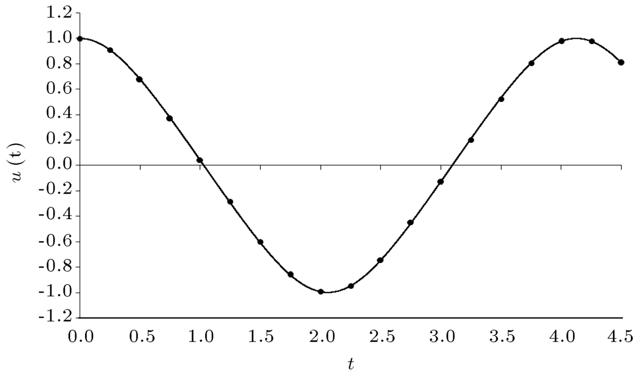

Figure 2. Comparison of approximate periodic solution obtained by the present method (denoted by circles) with the numerical solution obtained by the fourth-order Runge-Kutta method (denoted by solid line) for the cubic-quintic Duffing oscillator (Eq. (1)) for $\alpha = 1$, $\beta = 1$, $\gamma = 1$, and $A = 1.0$).

than [20,27]. Therefore, the present method is suitable for solving Eq. (1), as compared to [20,27].

Furthermore, we have determined the second-order approximate solutions to Eq. (1) for different values of parameters A, α, β, and γ, including all results with the corresponding numerical solutions obtained by fourth-order Runge-Kutta method. All results are presented in Figures 1-6. Based on these figures, we

Table 3. Comparison of the present frequency with existing results for cubic-quintic Duffing oscillator when $\alpha = 1, \beta = 10$, and $\gamma = 100$.

A	ω_e	Khan et al. [20]			Guo et al. [27]		Present study	
		ω_1 Er (%)*	ω_2 Er (%)	ω_3 Er (%)	ω_4 Er (%)	ω_2 Er (%)	ω_1 Er (%)	ω_2 Er (%)
0.1	1.0397019	1.0262188	1.0296117	1.0325994	1.0361322	1.0397000	1.0397315	1.0396423
		1.297	0.971	0.683	0.343	0.000	0.003	0.006
0.5	2.5247023	2.0501525	2.2114758	2.3542032	2.4890764	2.5350468	2.5789614	2.5236147
		18.796	12.407	6.753	1.411	0.410	2.150	0.043
1	8.0100698	6.1032777	6.8193183	7.4440041	7.9883609	8.0806905	8.3016063	8.0064163
		23.805	14.866	7.067	0.271	0.882	3.640	0.046
5	187.19966	140.20432	158.19193	174.00040	187.46034	189.20333	194.80482	187.01580
		25.105	15.496	7.051	0.139	1.070	4.063	0.098
10	747.32526	559.46490	631.39343	694.63605	748.44946	755.36618	777.76453	746.57252
		25.138	15.513	7.050	0.151	1.076	4.073	0.100
50	18671.400	13975.872	15773.896	17355.027	18700.148	18872.631	19432.522	18652.440
		25.148	15.519	7.050	0.154	1.078	4.075	0.102
100	74684.133	55902.148	63094.219	69418.750	74799.211	75489.084	77728.641	74608.280
		25.149	15.519	7.050	0.154	1.078	4.077	0.102
500	1867091.6	1397542.9	1577344.5	1735457.9	1869969.3	1887215.6	1943200.0	1865200.0
		25.149	15.519	7.050	0.154	1.078	4.076	0.101
1000	7468365.0	5590170.5	6309377.0	6941830.5	7479875.5	7548860.9	7772820.0	7460780.0
		25.149	15.519	7.050	0.154	1.078	4.077	0.102

*: Er (%) denotes the absolute percentage error.

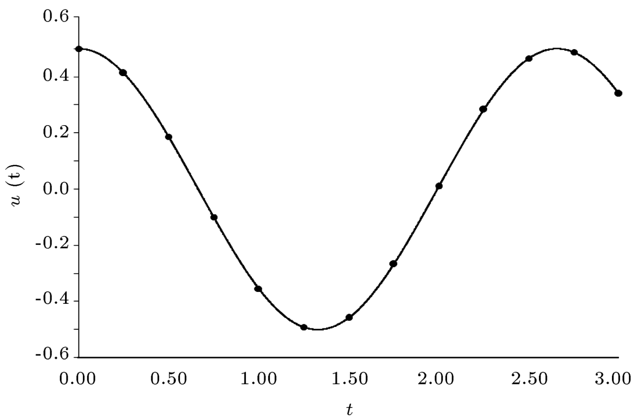

Figure 3. Comparison of approximate periodic solution obtained by the present method (denoted by circles) with the numerical solution obtained by the fourth-order Runge-Kutta method (denoted by solid line) for the cubic-quintic Duffing oscillator (Eq. (1)) for $\alpha = 5$, $\beta = 3$, $\gamma = 1$, and $A = 0.5$).

see that the present method's solutions are nicely in agreement with the corresponding numerical results for all values of parameters A, α, β, and γ.

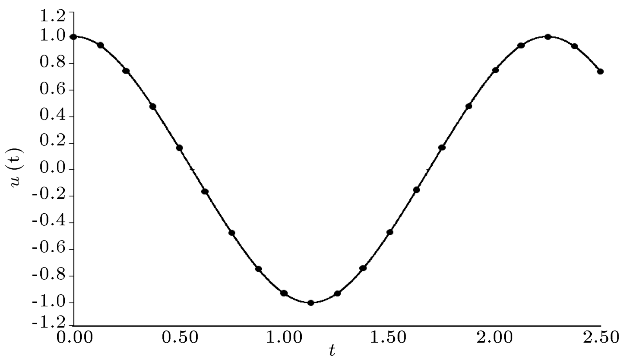

Figure 4. Comparison of approximate periodic solution obtained by the present method (denoted by circles) with the numerical solution obtained by the fourth-order Runge-Kutta method (denoted by solid line) for the cubic-quintic Duffing oscillator (Eq. (1)) for $\alpha = 5$, $\beta = 3$, $\gamma = 1$, and $A = 1.0$.

5. Conclusion

In this paper, a new simple analytical technique coupled by energy and harmonic balance methods was presented to solve the cubic-quintic Duffing oscillator. Next, The second-order approximation was

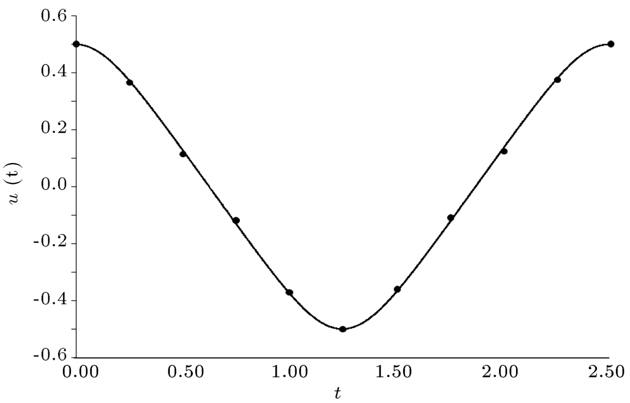

Figure 5. Comparison of approximate periodic solution obtained by the present method (denoted by circles) with the numerical solution obtained by the fourth-order Runge-Kutta method (denoted by solid line) for the cubic-quintic Duffing oscillator (Eq. (1)) for $\alpha = 1$, $\beta = 10$, $\gamma = 100$, and $A = 0.5$.

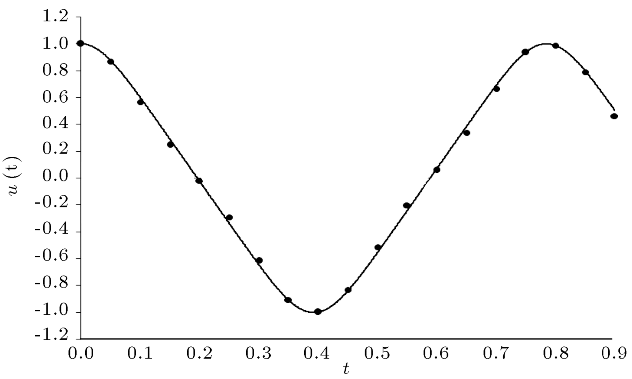

Figure 6. Comparison of approximate periodic solution obtained by the present method (denoted by circles) with the numerical solution obtained by the fourth-order Runge-Kutta method (denoted by solid line) for the cubic-quintic Duffing oscillator (Eq. (1)) for $\alpha = 1$, $\beta = 10$, $\gamma = 100$, and $A = 1.0$).

determined. The solution contains a few harmonic terms and also a lower-order term. These terms make the solution converge rapidly. It was observed that the present method gives better results than other existing results do, for both small and large amplitudes of oscillation. It was proved that the present method is very effective, convenient and also gives more precise accuracy for solving strongly nonlinear oscillators.

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