



Sharif University of Technology

Scientia Iranica

Transactions A: Civil Engineering

<http://scientiairanica.sharif.edu>



# A new Fourier series solution for free vibration of non-uniform beams, resting on variable elastic foundation

S.E. Motaghian<sup>a</sup>, M. Mofid<sup>b,\*</sup>, and J.E. Akin<sup>c</sup>

a. Department of Civil Engineering, University of Tehran, Tehran, Iran.

b. Department of Civil Engineering, Sharif University of Technology, Azadi St., Tehran, Iran.

c. Department of Mechanical Engineering, Rice University, Houston, TX, USA.

Received 9 May 2016; received in revised form 12 November 2016; accepted 8 July 2017

## KEYWORDS

Fourier series solution;  
Free vibration;  
Non-uniform beam;  
Variable elastic  
foundation;  
General boundary  
conditions.

**Abstract.** In this research, the combination of Fourier sine and cosine series is employed to develop an analytical method to conduct the free vibration analysis of an Euler-Bernoulli beam with varying cross-sections, fully or partially supported by a variable elastic foundation. The foundation stiffness and cross-section of the beam are considered as arbitrary functions in the beam's length direction. The idea behind the proposed method is to superpose Fourier sine and cosine series to satisfy the general elastically end constraints; therefore, no auxiliary functions are required to supplement the Fourier series. This method provides a simple, accurate and flexible solution to various beam problems and is also able to be extended to other cases whose governing differential equations are nonlinear. Moreover, this method is applicable to plate problems with different boundary conditions if two-dimensional Fourier sine and cosine series are taken as a displacement function. Numerical examples are carried out, illustrating the accuracy and efficiency of the presented approach.

© 2018 Sharif University of Technology. All rights reserved.

## 1. Introduction

The application of non-uniform beams resting on an elastic foundation in civil and mechanical engineering has resulted in a number of extensive studies. Researchers have proposed various analytical and numerical methods to analyze vibrations of Euler-Bernoulli beams with varying cross-sectional properties. Jategaonkar and Chehil [1] calculated the natural frequencies of beams with inertia, area, and mass varying

in a general manner. Katsikadelis and Tsiatas [2] applied the analog equation method to analyze the non-linear dynamic response of an Euler-Bernoulli beam with variable cross-sectional properties along its axis under large deflection. Nikkhah Bahrami et al. [3] used a wave propagation method to calculate the natural frequencies and related mode shapes of arbitrary non-uniform beams. Huang and Li [4] used Fredholm integral equations to solve natural frequencies of the free vibration of beams with variable flexural rigidity and mass density. Au et al. [5] analyzed vibration and stability of axially loaded non-uniform beams with abrupt changes of cross-sections based on the Euler-Lagrangian approach using a family of  $c^1$  admissible functions as the assumed modes. Naguleswaran [6] investigated the transverse vibration of an Euler-Bernoulli beam of constant depth and linearly varying

\*. Corresponding author. Tel.: +98 21 6601 4828  
E-mail addresses: [emad\\_em@alum.sharif.edu](mailto:emad_em@alum.sharif.edu) (S.E. Motaghian); [mofid@sharif.edu](mailto:mofid@sharif.edu) (M. Mofid); [akin@rice.edu](mailto:akin@rice.edu) (J.E. Akin)

breadth. Firouz-Abadi et al. [7] applied an asymptotic solution to investigate the transverse free vibration of variable cross-sectional beams. Datta and Sil [8] used Bessel's functions to evaluate the natural frequency and mode shape of cantilever beams of varying cross-sections. Banerjee et al. [9] used the dynamic stiffness method to study the free vibration of rotating beams of linearly varying heights or widths. However, some research works have concentrated their analysis on dynamic behavior of beams of some specific variations in cross-sections. This is comprised of such parabolic thickness variation [10], bi-linearly varying thickness [11], linearly variable depth [12], and those with exponentially varying width [13].

In addition to analytical solutions, some numerical and approximation methods, including finite-element method [14-17], variational iteration method [18,19], and approximate fundamental solution [20], have been developed in order to deal with the vibration analysis of beams.

The underlying elastic foundation usually modelled as the Winkler or Pasternak foundation [21,22] has been also a subject of interest in many studies. Some of them have considered the elastic foundations with variable modulus to present a more precise behavior of the foundation. Ding [23] reported a general solution to vibrations of beams on variable Winkler elastic foundation. Eisenberger [24] presented a method based on the dynamic stiffness matrix basically to find the natural frequencies of beams on variable one- and two-parameter elastic foundations. Eisenberger and Clastornik [25] solved eigenvalue problems of vibration and stability pertaining to a beam on a variable Winkler elastic foundation. Soldatos and Selvadurai [26] solved the static problem of the flexure of a beam resting on an elastic foundation with a hyperbolic modulus. Pradhan and Murmu [27] used the modified differential quadrature method to analyze a thermo-mechanical vibration of functionally graded beams. This also includes functionally graded sandwich beams resting on variable Winkler and two-parameter elastic foundations. Payam [28] investigated the effect of the surface contact stiffness on both flexural and torsional sensitivity of a cantilever beam immersed in a fluid.

Another set of mathematical solutions applied by the researcher to study the vibration problems of beams on elastic foundation is series expansion methods. Mutman and Coskun [29] used Homotopy Perturbation Method (HPM) to investigate the problem. This method is an analytical approximate technique deforming a linear or nonlinear problem into a series of linear equations. However, in this procedure, the initial solution approximation should be chosen appropriately to avoid infinite iterations. Ho and Chen [30] employed the Differential Transform Method (DTM) which is derived from Taylor series expansion to

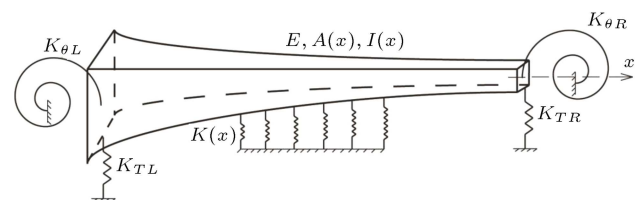
solve the free and forced vibration problems of a general elastically end restrained non-uniform beam. Catal [31] obtained frequency factors of a fixed-simply supported beam with axial load resting on elastic soil, using a differential transform method. Hassanabadi et al. [32] developed a method based on the series expansion of orthonormal polynomial to study the transverse vibration of thin beams with variable thickness. Some researchers have also explored the implementation of the Fourier series for the analysis of beams. Wang and Lin [33] used a Fourier series to study dynamic behavior of uniform beams with arbitrary boundary conditions. However, in their method, the free boundary condition is missing. Li [34] employed polynomial functions as auxiliary functions which are added to a Fourier cosine series merely to solve the free vibration of uniform beams with general boundary conditions.

In the previous papers, the Fourier sine series was applied to investigate the free vibration of beams [35] and plates [36] with a constant cross-section, which are partially supported by uniform elastic foundation. The solution proposed in these two articles has satisfied simply supported and clamped boundary conditions. In this paper, the Fourier sine series was combined with the Fourier cosine series to analyze the free vibration problem of a beam with any arbitrary varying cross-sections and fully or partially supported beams by a variable elastic foundation. The presented approach deals with any boundary conditions of the beam. Fourier analysis takes advantage of the periodic and orthogonal trigonometric functions that can represent any continuous and discontinuous functions to simplify the solution to the original problem. These benefits make the Fourier analysis a simple, extensive and reliable solution in mathematics.

## 2. Mathematical formulation

### 2.1. Governing differential equation

Figure 1 shows an Euler-Bernoulli beam with varying cross-sections resting on an elastic foundation. The beam is supported by a translational spring and a rotational spring at both ends. The differential equation of the transverse vibration of this beam can be expressed as follows [7,23]:



**Figure 1.** An Euler-Bernoulli beam with varying cross-sections resting on a variable elastic foundation, which is supported by a translational and a rotational spring at both ends.

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 F(x,t)}{\partial x^2} \right] + K(x)F(x,t) \\ + \rho A(x) \frac{\partial^2 F(x,t)}{\partial t^2} = P(x,t), \end{aligned} \quad (1)$$

where  $\rho$ ,  $E$ ,  $A(x)$ ,  $I(x)$  are the mass density, modulus of elasticity, cross-sectional area, and the moment of inertia, respectively;  $F(x,t)$  is the dynamic transverse displacement;  $K(x)$  is the Winkler's foundation modulus considered to be inconstant along the beam;  $P(x,t)$  is the dynamic transversely-distributed force;  $x$  represents the position of each point on the beam, and  $t$  represents time. Using separation of variable technique (i.e.,  $F(x,t) = W(x)e^{-i\omega t}$  and  $P(x,t) = p(x)e^{-i\omega t}$ ), one can represent Eq. (1) as follows:

$$\begin{aligned} \frac{d^2}{dx^2} \left[ EI(x) \frac{d^2 W(x)}{dx^2} \right] + K(x)W(x) \\ - \rho A(x)\omega^2 W(x) = p(x), \end{aligned} \quad (2)$$

where  $\omega$  is the natural frequency, and  $i = \sqrt{-1}$ .

The bending moment and shear force are given respectively by:

$$M(x) = EI(x) \frac{d^2 W(x)}{dx^2}, \quad (3)$$

$$V(x) = \frac{dM(x)}{dx} = \frac{d}{dx} \left[ EI(x) \frac{d^2 W(x)}{dx^2} \right]. \quad (4)$$

Let us introduce dimensionless variables as found in the following:

$$\begin{aligned} \xi = \frac{x}{L}, \quad w(\xi) = \frac{W(x)}{L}, \quad \bar{A}(\xi) = \frac{A(x)}{A(0)}, \\ k(\xi) = \frac{K(x)L^4}{EI(0)}, \quad \bar{\omega}^2 = \frac{\rho A(0)L^4\omega^2}{EI(0)}, \quad \bar{I}(\xi) = \frac{I(x)}{I(0)}, \end{aligned} \quad (5)$$

where  $A(0)$  and  $I(0)$  are respectively the area and moment of inertia for the cross-section of the beam at its left end where  $x = 0$  and  $L$  is the length of the beam. In addition  $\xi$ ,  $w(\xi)$ ,  $\bar{A}(\xi)$ ,  $\bar{I}(\xi)$ ,  $k(\xi)$ , and  $\bar{\omega}$  denote the non-dimensional forms of  $x$ , the beam displacement function, the beam's cross-sectional area function, the moment of inertia function, the stiffness function of the foundation, and frequency parameter, respectively.

Regarding the above variables and Eq. (3), one can rewrite Eq. (2) as follows:

$$\frac{d^2 \bar{M}(\xi)}{d\xi^2} + k(\xi)w(\xi) - \bar{A}(\xi)\bar{\omega}^2 w(\xi) = \bar{p}(\xi), \quad (6)$$

where:

$$\bar{M}(\xi) = \bar{I}(\xi) \frac{d^2 w(\xi)}{d\xi^2}, \quad (7)$$

$$\bar{p}(\xi) = \frac{p(x)L^3}{EI(0)}. \quad (8)$$

## 2.2. The proposed solution based on the Fourier cosine and sine series

The newly proposed method is stated in this section utilizing the Fourier sine and cosine series simultaneously. At first, two beams are defined vibrating transversely with different boundary conditions. Figure 2 shows the first beam supported by a slider at both ends. This kind of support is fixed against  $x$ -direction rotation while allowing the beam ends to translate vertically. Equilibrium equation (6) of this beam can be written as follows:

$$\begin{aligned} \frac{d^2 \bar{M}_c(\xi)}{d\xi^2} + k(\xi)w_c(\xi) - \bar{A}(\xi)\bar{\omega}^2 w_c(\xi) \\ = \bar{p}_L \delta(\xi - 0) + \bar{p}_R \delta(\xi - 1), \end{aligned} \quad (9)$$

where  $w_c(\xi)$  and  $\bar{M}_c(\xi)$  are referred to as the displacement and bending moment of the beam modelled by the cosine series. At the same time,  $\bar{p}_L$  and  $\bar{p}_R$  are dimensionless concentrated loads on the left and right ends of the beam, respectively, and  $\delta$  is the Dirac delta function.

One can assign a cosine series to the bending moment function and similarly to the displacement function as follows:

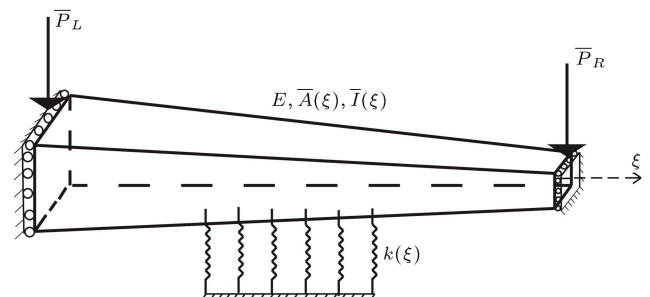
$$\bar{M}_c(\xi) = \sum_{n=0}^{\infty} a_n \cos n\pi\xi, \quad (10)$$

$$w_c(\xi) = \sum_{n=0}^{\infty} b_n \cos n\pi\xi, \quad (11)$$

where  $a_n$  and  $b_n$  are Fourier coefficients considered as unknown parameters. The proposed series satisfies boundary conditions as follows:

$$\xi = 0, 1 \Rightarrow \begin{cases} \frac{dw_c(\xi)}{d\xi} = 0 \\ \frac{d\bar{M}_c(\xi)}{d\xi} = 0 \end{cases} \quad (12)$$

Fourier cosine series defined by Eqs. (10) and (11) can



**Figure 2.** A slider supported beam subjected to a concentration load on each side.

be substituted for the bending moment and displacement functions found in Eqs. (7) and (9). The Dirac delta function is also required to be transformed to Fourier cosine series. Hence, Eqs. (7) and (9) become:

$$\sum_{n=0}^{\infty} a_n \cos n\pi\xi = \sum_{n=0}^{\infty} -b_n n^2 \pi^2 \bar{I}(\xi) \cos n\pi\xi, \quad (13)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} -a_n n^2 \pi^2 \cos n\pi\xi + \sum_{n=0}^{\infty} b_n k(\xi) \cos n\pi\xi \\ & - \sum_{n=0}^{\infty} b_n \bar{\omega}^2 \bar{A}(\xi) \cos n\pi\xi = \bar{p}_L \\ & + \sum_{n=1}^{\infty} 2\bar{p}_L \cos n\pi\xi + \bar{p}_R \\ & + \sum_{n=1}^{\infty} 2\bar{p}_R \cos(n\pi) \cos(n\pi\xi). \end{aligned} \quad (14)$$

Multiplying both sides of Eqs. (13) and (14) by  $\cos m\pi\xi$  ( $m = 0, 1, \dots$ ) and integrating  $\xi = 0$  to 1, one can obtain the following equations:

$$a_0 + \sum_{n=0}^{\infty} C_{n,m}^I b_n n^2 \pi^2 = 0 \quad m = 0, \quad (15.1)$$

$$\frac{a_m}{2} + \sum_{n=0}^{\infty} C_{n,m}^I b_n n^2 \pi^2 = 0 \quad m = 1, 2, \dots \quad (15.2)$$

and:

$$\begin{aligned} & \frac{m^2 \pi^2 a_m}{2} - \sum_{n=0}^{\infty} C_{n,m}^K b_n + \sum_{n=0}^{\infty} C_{n,m}^A b_n \bar{\omega}^2 + \bar{p}_L \\ & + \bar{p}_R \cos m\pi = 0 \quad m = 0, 1, 2, \dots \end{aligned} \quad (16)$$

where  $C_{nm}^I$ ,  $C_{nm}^K$ ,  $C_{nm}^A$  are defined by the following integrations:

$$C_{n,m}^I = \int_0^1 \bar{I}(\xi) \cos(n\pi\xi) \cos(m\pi\xi) d\xi, \quad (17.1)$$

$$C_{n,m}^K = \int_0^1 k(\xi) \cos(n\pi\xi) \cos(m\pi\xi) d\xi, \quad (17.2)$$

$$C_{n,m}^A = \int_0^1 \bar{A}(\xi) \cos(n\pi\xi) \cos(m\pi\xi) d\xi. \quad (17.3)$$

In order to make a system of equations, one can truncate the series to the first  $N+1$  terms and consider  $n$  and  $m$  from 0 to  $N$  and expand Eq. (15) as follows:

$$\begin{cases} a_0 + \pi^2 C_{1,0}^I b_1 + 4\pi^2 C_{2,0}^I b_2 + \dots + N^2 \pi^2 C_{N,0}^I b_N = 0 \\ \frac{a_1}{2} + \pi^2 C_{1,1}^I b_1 + 4\pi^2 C_{2,1}^I b_2 + \dots + N^2 \pi^2 C_{N,1}^I b_N = 0 \\ \vdots \\ \frac{a_N}{2} + \pi^2 C_{1,N}^I b_1 + 4\pi^2 C_{2,N}^I b_2 + \dots + N^2 \pi^2 C_{N,N}^I b_N = 0 \end{cases} \quad (18)$$

Eq. (16) can be also expanded as follows:

$$\begin{cases} (C_{0,0}^A \bar{\omega}^2 - C_{0,0}^K) b_0 + (C_{1,0}^A \bar{\omega}^2 - C_{1,0}^K) b_1 + \dots \\ \quad + (C_{N,0}^A \bar{\omega}^2 - C_{N,0}^K) b_N + \bar{p}_L + \bar{p}_R = 0 \\ \frac{\pi^2 a_1}{2} + (C_{0,1}^A \bar{\omega}^2 - C_{0,1}^K) b_0 + (C_{1,1}^A \bar{\omega}^2 - C_{1,1}^K) b_1 + \dots \\ \quad + (C_{N,1}^A \bar{\omega}^2 - C_{N,1}^K) b_N + \bar{p}_L + \bar{p}_R \cos \pi = 0 \\ \vdots \\ \frac{N^2 \pi^2 a_N}{2} + (C_{0,N}^A \bar{\omega}^2 - C_{0,N}^K) b_0 + (C_{1,N}^A \bar{\omega}^2 - C_{1,N}^K) b_1 \\ \quad + \dots + (C_{N,N}^A \bar{\omega}^2 - C_{N,N}^K) b_N + \bar{p}_L \\ \quad + \bar{p}_R \cos N\pi = 0 \end{cases} \quad (19)$$

The second beam shown in Figure 3 is a simply supported one subjected to concentration moments at both ends. The governing differential equation of this beam is expressed as follows:

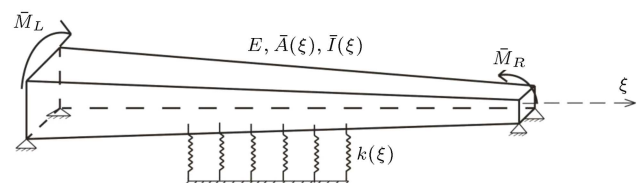
$$\begin{aligned} & \frac{d^2 \bar{M}_s(\xi)}{d\xi^2} + k(\xi) w_s(\xi) - \bar{A}(\xi) \bar{\omega}^2 w_s(\xi) = \bar{p}_1 \delta(\xi - \varepsilon) \\ & - \bar{p}_1 \delta(\xi - 0) + \bar{p}_2 \delta(\xi - (1 - \varepsilon)) - \bar{p}_2 \delta(\xi - 1), \end{aligned} \quad (20)$$

where  $w_s(\xi)$  and  $\bar{M}_s(\xi)$  are respectively the displacement and bending moment functions pertaining to the beam modelled by the Fourier sine series. Thus,  $\bar{p}_1$  and  $\bar{p}_2$  are two force couples creating concentration moments at both ends. In addition,  $\varepsilon$  denotes the distance between two forces of the couple, sufficiently close to zero, while the magnitude of forces approaches infinity. The product converges to the respective moment [35]:

$$\varepsilon \rightarrow 0, \bar{p}_1, \bar{p}_2 \rightarrow \infty \Rightarrow \begin{cases} \bar{p}_1 \varepsilon \rightarrow \bar{M}_L \\ \bar{p}_2 \varepsilon \rightarrow \bar{M}_R \end{cases} \quad (21)$$

where  $\bar{M}_L$  and  $\bar{M}_R$  are the concentrated moments at the left and right ends of the beam, respectively.

For a simply supported beam, the transverse displacement and bending moment at  $x = 0$  and  $x = L$



**Figure 3.** A simply supported beam subjected to a concentration moment on each side.

are equal to zero. Therefore, for their corresponding functions, one can use the Fourier sine series as follows:

$$\bar{M}_s(\xi) = \sum_{n=1}^{\infty} d_n \sin n\pi\xi, \quad (22)$$

$$w_s(\xi) = \sum_{n=1}^{\infty} e_n \sin n\pi\xi, \quad (23)$$

where  $d_n$  and  $e_n$  are unknown coefficients.

By putting Eqs. (22) and (23) into Eqs. (20) and (7) and transforming the right-hand side of Eq. (20) into the Fourier sine series [35], one can get:

$$\sum_{n=1}^{\infty} d_n \sin n\pi\xi = \sum_{n=1}^{\infty} -e_n n^2 \pi^2 \bar{I}(\xi) \sin n\pi\xi, \quad (24)$$

$$\begin{aligned} \sum_{n=1}^{\infty} -d_n n^2 \pi^2 \sin n\pi\xi + \sum_{n=1}^{\infty} e_n k(\xi) \sin n\pi\xi \\ - \sum_{n=0}^{\infty} d_n \bar{\omega}^2 \bar{A}(\xi) \sin n\pi\xi = \sum_{n=1}^{\infty} 2n\pi \bar{M}_L \sin n\pi\xi \\ - \sum_{n=1}^{\infty} 2n\pi \bar{M}_R \cos(n\pi) \sin(n\pi\xi). \end{aligned} \quad (25)$$

Similar to the beam problem solved by the cosine series, one can multiply both sides of Eqs. (24) and (25) by  $\sin n\pi\xi$  and, then, take an integral from  $\xi = 0$  to 1 as in the following:

$$\frac{d_m}{2} + \sum_{n=1}^{\infty} S_{n,m}^I e_n n^2 \pi^2 = 0 \quad m = 1, 2, \dots \quad (26)$$

and:

$$\begin{aligned} \frac{m^2 \pi^2 d_m}{2} - \sum_{n=1}^{\infty} S_{n,m}^K e_n + \sum_{n=0}^{\infty} S_{n,m}^A e_n \bar{\omega}^2 - 2m\pi \bar{M}_L \\ + 2m\pi \cos m\pi \bar{M}_R = 0 \quad m = 1, 2, \dots, \end{aligned} \quad (27)$$

where  $S_{n,m}^I$ ,  $S_{n,m}^K$ , and  $S_{n,m}^A$  are:

$$S_{n,m}^I = \int_0^1 \bar{I}(\xi) \sin(n\pi\xi) \sin(m\pi\xi) d\xi, \quad (28.1)$$

$$S_{n,m}^K = \int_0^1 k(\xi) \sin(n\pi\xi) \sin(m\pi\xi) d\xi, \quad (28.2)$$

$$S_{n,m}^A = \int_0^1 \bar{A}(\xi) \sin(n\pi\xi) \sin(m\pi\xi) d\xi. \quad (28.3)$$

The first  $N$  terms of the above series are selected in order to develop the system of equations observed earlier. Hence, Eq. (26) can be expanded as follows:

$$\begin{cases} \frac{d_1}{2} + \pi^2 S_{1,1}^I e_1 + 4\pi^2 S_{2,1}^I e_2 + \dots \\ \quad + N^2 \pi^2 S_{N,1}^I e_N = 0 \\ \vdots \\ \frac{d_N}{2} + \pi^2 S_{1,N}^I e_1 + 4\pi^2 S_{2,N}^I e_2 + \dots \\ \quad + N^2 \pi^2 S_{N,N}^I e_N = 0 \end{cases} \quad (29)$$

Likewise, Eq. (27) can be rewritten as follows:

$$\begin{cases} \frac{\pi^2 d_1}{2} + (S_{1,1}^A \bar{\omega}^2 - S_{1,1}^K) e_1 + \dots \\ \quad + (S_{N,1}^A \bar{\omega}^2 - S_{N,1}^K) e_N - 2\pi \bar{M}_L \\ \quad + 2\pi \cos(\pi) \bar{M}_R = 0 \\ \vdots \\ \frac{N^2 \pi^2 d_N}{2} + (S_{1,N}^A \bar{\omega}^2 - S_{1,N}^K) e_1 + \dots \\ \quad + (S_{N,N}^A \bar{\omega}^2 - S_{N,N}^K) e_N - 2N\pi \bar{M}_L \\ \quad + 2N\pi \cos(N\pi) \bar{M}_R = 0 \end{cases} \quad (30)$$

Eqs. (18), (19), (29), and (30) constitute a collection of  $2N + 1$  equations with  $2N + 5$  unknowns. This includes Fourier sine and cosine coefficients joined with four unknown end forces and moments (i.e.  $\bar{p}_L$ ,  $\bar{p}_R$ ,  $\bar{M}_L$ , and  $\bar{M}_R$ ). In order to have a system of equations with the same set of unknowns, one needs four more equations which are developed through the exertion of boundary conditions. Moreover, boundary condition equations do establish an association between two sets of equations developed separately with the cosine and sine series.

### 3. Boundary conditions

The beam studied in this paper is restrained with a transversal spring and a rotational spring at both ends. Therefore, the sum of concentrated loads applied to both ends of the first beam along with shear forces generated at both edges of the second beam should be proportional to the respective end displacement of the first beam. Thus, one can use Eq. (4) and write the following equations:

$$P_L + \frac{dM_s(x)}{dx} \Big|_{x=0} = K_{TL} W_c(x=0), \quad (31)$$

$$P_R + \frac{dM_s(x)}{dx} \Big|_{x=L} = K_{TR} W_c(x=L), \quad (32)$$

where  $K_{TL}$  and  $K_{TR}$  are translational spring constants at the left and right ends of the beam, respectively. Moreover,  $W_c$  and  $M_s$  are respectively the displacement function of the beam analyzed by the cosine series (the first beam) and bending moment function of the beam analyzed by the sine series (the second beam).

Applying non-dimensional parameters gives:

$$\bar{p}_L + \left. \frac{d\bar{M}_s(\xi)}{d\xi} \right|_{\xi=0} = \bar{K}_{TL} w_c(\xi=0), \quad (33)$$

$$\bar{p}_R + \left. \frac{d\bar{M}_s(\xi)}{d\xi} \right|_{\xi=1} = \bar{K}_{TR} w_c(\xi=1), \quad (34)$$

where:

$$\bar{K}_{TL} = \frac{K_{TL} L^3}{EI(0)}, \quad (35.1)$$

$$\bar{K}_{TR} = \frac{K_{TR} L^3}{EI(0)}. \quad (35.2)$$

Using Eqs. (11) and (22), we can extend Eqs. (33) and (34) as follows:

$$\bar{p}_L + \sum_{n=1}^N d_n(n\pi) - \bar{K}_{TL} \sum_{n=1}^N b_n = 0, \quad (36)$$

$$\bar{p}_R + \sum_{n=1}^N d_n(n\pi) \cos n\pi - \bar{K}_{TR} \sum_{n=1}^N b_n \cos n\pi = 0. \quad (37)$$

Similarly, the sum of bending moments of the first beam and concentrated moments of the second beam at both ends is commensurate with the rotation angle of the respective edge of the second beam as:

$$M_L + M_c(x=0) = K_{\theta L} \left. \frac{dW_s(x)}{dx} \right|_{x=0}, \quad (38)$$

$$M_R + M_c(x=L) = K_{\theta R} \left. \frac{dW_s(x)}{dx} \right|_{x=L}, \quad (39)$$

where  $K_{\theta L}$  and  $K_{\theta R}$  are rotational spring constants at the left and right ends of the beam, respectively. Moreover,  $W_s$  and  $M_c$  are respectively the displacement function of the beam modelled by the sine series (the second beam) and bending moment function of the beam modelled by the cosine series (the first beam).

In the non-dimensional form, Eqs. (38) and (39) become:

$$\bar{M}_L + \bar{M}_c(\xi=0) = \bar{K}_{\theta L} \left. \frac{dw_s(\xi)}{d\xi} \right|_{\xi=0}, \quad (40)$$

$$\bar{M}_R + \bar{M}_c(\xi=1) = \bar{K}_{\theta R} \left. \frac{dw_s(\xi)}{d\xi} \right|_{\xi=1}, \quad (41)$$

in which:

$$\bar{K}_{\theta L} = \frac{K_{\theta L} L}{EI(0)}, \quad (42.1)$$

$$\bar{K}_{\theta R} = \frac{K_{\theta R} L}{EI(0)}. \quad (42.2)$$

Substituting Eqs. (10) and (23) into Eqs. (40) and (41) gives:

$$\bar{M}_L + \sum_{n=0}^N a_n - \bar{K}_{\theta L} \sum_{n=0}^N e_n(n\pi) = 0, \quad (43)$$

$$\bar{M}_R + \sum_{n=0}^N a_n \cos n\pi - \bar{K}_{\theta R} \sum_{n=0}^N e_n(n\pi) \cos n\pi = 0. \quad (44)$$

Eqs. (36), (37), (43), and (44) create a link between two groups of equations representing respectively the behavior of the first and second beams.

Three classical boundary conditions, including simply supported, clamped and free ends, are the specialized cases of the general elastically constraint as listed below:

- Simply supported end:

$$\begin{cases} w_c(\xi=0 \text{ or } 1) = 0 \\ \bar{M}_L \text{ or } \bar{M}_R + \bar{M}_c(\xi=0 \text{ or } 1) = 0 \end{cases} \quad (45)$$

- Fixed end:

$$\begin{cases} w_c(\xi=0 \text{ or } 1) = 0 \\ \left. \frac{dw_s(\xi)}{d\xi} \right|_{\xi=0 \text{ or } 1} = 0 \end{cases} \quad (46)$$

- Free end:

$$\begin{cases} \bar{M}_L \text{ or } \bar{M}_R + \bar{M}_c(\xi=0 \text{ or } 1) = 0 \\ \bar{p}_L \text{ or } \bar{p}_R + \left. \frac{d\bar{M}_s(\xi)}{d\xi} \right|_{\xi=0 \text{ or } 1} = 0 \end{cases} \quad (47)$$

A system of  $2N + 5$  equations is made by writing the four remaining equations provided by boundary conditions. This system can be written in a matrix form as follows:

$$[A]_{2N+5, 2N+5} [U]_{2N+5, 1} = 0, \quad (48)$$

where  $[U]$  is the unknown coefficients vector, and  $[A]$  is the coefficient matrix containing the natural circular frequency term. The non-trivial solution to this problem requires matrix  $A$  to have determination zero

$$|A|_{2N+5, 2N+5} = 0. \quad (49)$$

Solving Eq. (49) leads to obtaining the natural frequencies of the beam and their related mode shapes.

#### 4. Numerical examples and verification

In this section, a number of case studies are conducted to demonstrate the introduced method. Initially, two examples are modelled while choosing different terms of truncated series,  $N$ , and are compared with those of

the previous work to verify the proposed solution. The first example is a cantilever beam with a rectangular cross-section. The width of the beam is kept constant, while the height varies linearly as follows:

$$\bar{h}(\xi) = \frac{h(x)}{h_0} = 1 - c\xi, \quad (50)$$

where  $h(x)$  is the height of the beam,  $h_0$  is the height of the beam at  $x = 0$ , and  $c$  is the taper ratio constant. The beam is clamped at  $\xi = 0$  and is free at  $\xi = 1$ .

The first three dimensionless natural frequencies  $\bar{\omega}$  of this beam with different values of the tapered ratio constant are calculated using a developed solution and are shown in Table 1. The results are compared with those obtained by Banerjee et al. [9]. The comparison indicates excellent agreement between two results.

The second example is a simply supported beam resting on the variable Winkler elastic foundation with linear and parabolic moduli, investigated by Ding [23]. In this example, the elastic modulus for the linear foundation is defined as follows:

$$k(\xi) = k_0(1 - \alpha\xi). \quad (51)$$

For parabolic foundation, we have:

$$k(\xi) = k_0(1 - \beta\xi^2), \quad (52)$$

where  $\alpha$  and  $\beta$  are variation parameters, and  $k_0$  is the foundation constant at  $\xi = 0$ .

Tables 2 and 3 report the first three non-dimensional natural frequency parameters  $\sqrt{\bar{\omega}}$  which pertain respectively to linear and parabolic foundations regarding different amounts of  $\alpha$ ,  $\beta$ , and  $k_0$ ; they are compared with the results presented in [23]. In this case, the results, which entirely comply with those obtained by Ding [23], are independent of  $N$  and do not change when  $N$  increases.

The results presented in Tables 2 and 3 indicate that the natural frequencies of the beam resting on a foundation with a parabolic modulus are relatively greater than those of the linear foundation with the same  $k_0$  and a variation parameter. Thus, a parabolically varying foundation provides stiffer support than an equivalent foundation does, with a linear modulus. However, this increase is more considerable in the first mode than in the other two is and also becomes more discernible when the variation parameter gets larger.

Some other case studies with various characteristics are included in this section to illustrate the applications of the presented method. In all the following cases,  $h(x)$  stands for the height of the beam, and  $h_0$  is the height of the left side of the beam. For obtaining the results, the first 80 terms of truncated series have been considered (i.e.,  $N = 80$ ).

Figure 4 displays a beam whose one end is fixed and the other is supported by a spring with stiffness

**Table 1.** The first three natural frequencies,  $\bar{\omega}$ , of the linearly height tapered beam versus taper ratio,  $c$ , and the first  $N$  terms of truncated series.

$c$	$N$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0.1	10	3.576	21.554	59.392
	30	3.560	21.395	59.438
	50	3.559	21.286	59.330
	80	3.559	21.430	59.241
	<b>Ref. [9]</b>	<b>3.559</b>	<b>21.338</b>	<b>58.980</b>
0.3	10	3.687	19.741	54.811
	30	3.670	20.011	54.007
	50	3.668	19.983	53.784
	80	3.667	19.951	53.636
	<b>Ref. [9]</b>	<b>3.667</b>	<b>19.881</b>	<b>53.322</b>
0.5	10	3.845	18.442	48.804
	30	3.827	18.432	47.907
	50	3.825	18.397	47.678
	80	3.824	18.371	47.535
	<b>Ref. [9]</b>	<b>3.824</b>	<b>18.317</b>	<b>47.265</b>
0.6	10	3.954	17.609	45.475
	30	3.937	17.583	44.602
	50	3.935	17.553	44.389
	80	3.934	17.531	44.259
	<b>Ref. [9]</b>	<b>3.934</b>	<b>17.488</b>	<b>44.025</b>
0.8	10	4.308	15.787	38.081
	30	4.294	15.787	37.267
	50	4.292	15.774	37.116
	80	4.292	15.764	37.029
	<b>Ref. [9]</b>	<b>4.292</b>	<b>15.743</b>	<b>36.885</b>
0.9	10	4.645	14.946	34.114
	30	4.631	14.941	33.082
	50	4.630	14.941	32.976
	80	4.630	14.939	32.919
	<b>Ref. [9]</b>	<b>4.631</b>	<b>14.931</b>	<b>32.833</b>
0.99	10	5.232	14.967	31.702
	30	5.215	14.948	29.920
	50	5.214	14.958	29.786
	80	5.214	14.963	29.747
	<b>Ref. [9]</b>	<b>5.214</b>	<b>14.967</b>	<b>29.727</b>

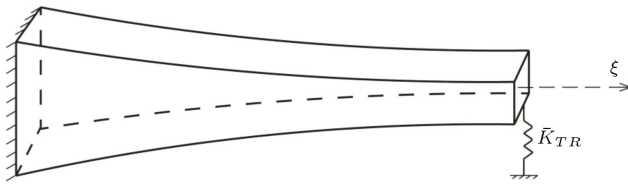
**Table 2.** The first three natural frequency parameters,  $\bar{\omega}$ , of a uniform simply supported beam resting on Winkler foundation with linear modulus versus  $k_0$  and variation parameter  $\alpha$ .

$k_0$	$\alpha$	$\sqrt{\bar{\omega}_1}$		$\sqrt{\bar{\omega}_2}$		$\sqrt{\bar{\omega}_3}$	
		Presented method	Ref. [23]	Presented method	Ref. [23]	Presented method	Ref. [23]
500	0.2	4.836	4.837	6.694	6.695	9.556	9.577
	0.4	4.720	4.721	6.653	6.652	9.542	9.541
	0.6	4.594	4.595	6.610	6.610	9.527	9.527
	0.8	4.455	4.456	6.568	6.568	9.513	9.513
1000	0.2	5.618	5.618	7.041	7.041	9.682	9.682
	0.4	5.468	5.468	6.971	6.971	9.655	9.656
	0.6	5.300	5.301	6.898	6.898	9.627	9.627
	0.8	5.113	5.113	6.823	6.824	9.599	9.599
1500	0.2	6.165	6.165	7.345	7.345	9.864	9.864
	0.4	5.991	5.991	7.251	7.251	9.765	9.765
	0.6	5.797	5.798	7.154	7.155	9.723	9.724
	0.8	5.574	5.574	7.057	7.057	9.684	9.684
2000	0.2	6.596	6.597	7.614	7.614	9.922	9.921
	0.4	6.405	6.405	7.503	7.503	9.870	9.870
	0.6	6.186	6.187	7.390	7.390	9.819	9.819
	0.8	5.937	5.937	7.271	7.272	9.767	9.767

**Table 3.** The first three natural frequency parameters,  $\bar{\omega}$ , of a uniform simply supported beam resting on Winkler foundation with parabolic modulus versus  $k_0$  and variation parameter  $\beta$ .

$k_0$	$\beta$	$\sqrt{\bar{\omega}_1}$		$\sqrt{\bar{\omega}_2}$		$\sqrt{\bar{\omega}_3}$	
		Presented method	Ref. [23]	Presented method	Ref. [23]	Presented method	Ref. [23]
500	0.2	4.884	4.884	6.710	6.710	9.562	9.562
	0.4	4.820	4.821	6.683	6.683	9.552	9.552
	0.6	4.753	4.753	6.656	6.657	9.542	9.542
	0.8	4.682	4.682	6.630	6.630	9.534	9.534
1000	0.2	5.679	5.679	7.067	7.067	9.692	9.692
	0.4	5.597	5.597	7.022	7.022	9.675	9.675
	0.6	5.507	5.507	6.980	6.980	9.657	9.657
	0.8	5.409	5.409	6.935	6.935	9.638	9.638
1500	0.2	6.232	6.233	7.378	7.378	9.817	9.817
	0.4	6.137	6.138	7.321	7.321	9.792	9.792
	0.6	6.032	6.032	7.263	7.263	9.766	9.767
	0.8	5.917	5.917	7.205	7.206	9.741	9.741
2000	0.2	6.671	6.671	7.653	7.653	9.939	9.939
	0.4	6.564	6.564	7.587	7.587	9.905	9.905
	0.6	6.444	6.444	7.521	7.521	9.873	9.874
	0.8	6.312	6.312	7.454	7.454	9.841	9.841





**Figure 4.** A cantilever beam of exponentially varying thickness, which is supported by a translational spring at its free end.

coefficient,  $\bar{K}_{TR}$ . The beam's cross-section is assumed to be rectangular with constant width and exponentially varying thickness. It is worth mentioning that exponentially varying beam's cross-sections may provide an appropriate distribution of mass and strength to resist dynamic forces such as wind, earthquake, fluid pressure, etc. Therefore, they are used in mechanical components (such as turbine blades, helicopter blades, etc.) and also in robotics, aeronautics, architecture, and other innovative engineering applications.

The non-dimensional height function can be written as follows:

$$\bar{h}(\xi) = \frac{h(x)}{h_0} = e^{-\eta\xi}, \quad (53)$$

where  $\eta$  is the variation parameter of the beam's height.

Figure 5 shows the first, second, and third natural frequency parameters of this beam,  $\sqrt{\bar{\omega}}$ , versus spring parameter  $\bar{K}_{TR}$  for  $\eta = -0.4, -1.0, 0.4, 1$ . According to Figure 5, since the mass distribution in the exponentially decreasing height of the beam to the spring is further than that in the exponentially heightening height of the beam, the natural frequencies of the former converge more rapidly to a constant value than those of the latter, when  $\bar{K}_{TR}$  increases. However, for lower amounts of  $\bar{K}_{TR}$ , the first and second natural frequencies of the exponentially decreasing height of the beam are more sensitive to the stiffness parameter of the spring.

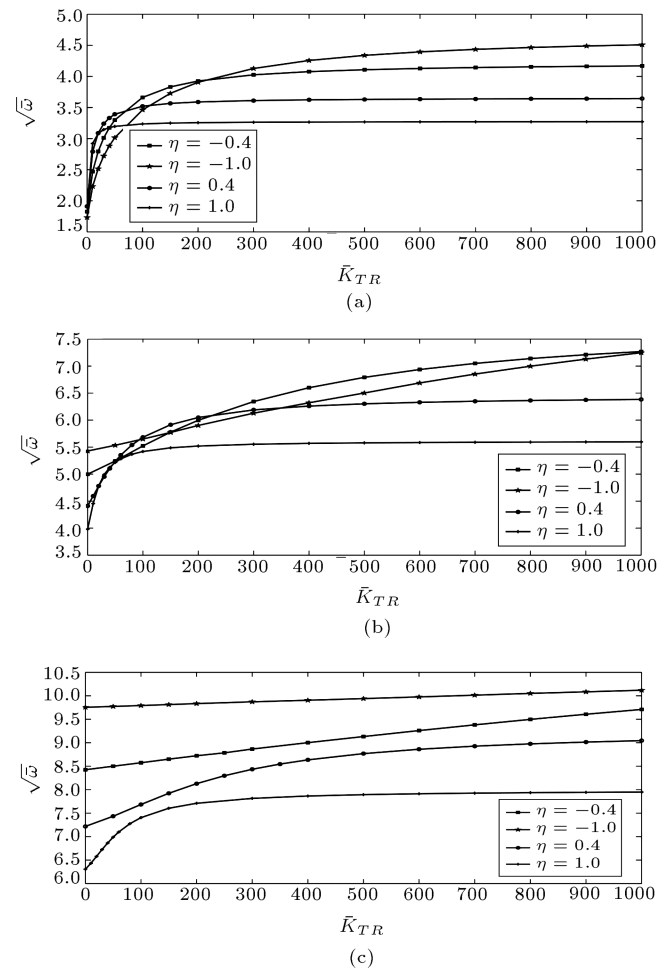
In the next case, a beam with a rectangular cross-section, a parabolic thickness variation, and constant width, which is supported partially by the Winkler elastic foundation, is studied. The elastic foundation is distributed under the beam from  $x = L/4$  to  $x = 2L/3$ , and the foundation stiffness varies linearly. The non-dimensional thickness and foundation stiffness functions can be expressed as follows:

$$\bar{h}(\xi) = \frac{h(x)}{h_0} = 1 - c\xi^2, \quad (54.1)$$

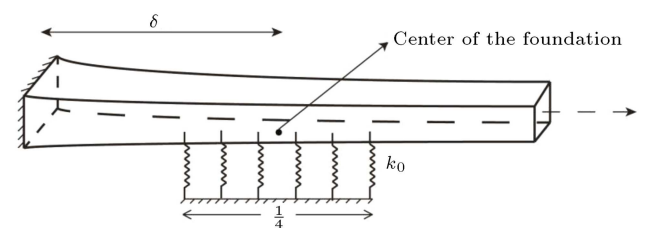
$$k(\xi) = k_0(1 - \alpha\xi), \quad (54.2)$$

where  $c$  and  $\alpha$  are variation parameters of the thickness and foundation stiffness, respectively.

The first four natural frequency parameters  $\sqrt{\bar{\omega}}$  of this beam with different boundary conditions, i.e.,



**Figure 5.** Natural frequency parameters  $\sqrt{\bar{\omega}}$  versus spring parameter,  $\bar{K}_{TR}$ : (a) The first mode, (b) the second mode, and (c) the third mode.



**Figure 6.** A cantilever beam with exponentially varying width, which is supported by a moving elastic foundation whose length is one-fourth of the beam's length.

Simply Supported (SS) beam, Clamped-Simply Supported (CS) beam, Clamped-Clamped (CC) beam, and Clamped-Free (CF) beam, are tabulated in Table 4 for  $c = 0.8$  and variable amounts for  $\alpha$  and  $k_0$ .

The subject of our next case is the movement of a partial elastic foundation spread under one-fourth of the beam. Figure 6 shows a cantilever beam with a rectangular cross-section in which the characteristic height is constant, while the characteristic width varies exponentially along the length of the beam. Therefore,

**Table 4.** The first four natural frequency parameters,  $\sqrt{\bar{\omega}}$ , of a beam with parabolic varying thickness partially supported by linear elastic foundation for  $c = 0.8$  regarding different boundary conditions.

$\alpha$	$k_0$	Mode number	$\sqrt{\bar{\omega}}$			
			SS	CS	CC	CF
0.5	200	1	3.558	3.876	4.147	2.768
		2	5.279	5.959	6.328	4.739
		3	7.747	8.412	8.863	6.903
		4	10.256	10.917	11.424	9.311
	500	1	4.220	4.340	4.621	3.091
		2	5.457	6.129	6.461	5.105
		3	7.801	8.463	8.907	6.991
		4	10.275	10.934	11.438	9.344
	800	1	4.597	4.639	4.951	3.250
		2	5.651	6.303	6.597	5.412
		3	7.855	8.514	8.950	7.079
		4	10.294	10.950	11.453	9.378
0.8	200	1	3.382	3.743	4.025	2.624
		2	5.266	5.945	6.318	4.677
		3	7.737	8.404	8.856	6.895
		4	10.253	10.915	11.422	9.306
	500	1	3.965	4.116	4.403	2.916
		2	5.425	6.088	6.433	4.961
		3	7.775	8.444	8.890	6.971
		4	10.268	10.928	11.433	9.331
	800	1	4.307	4.371	4.677	3.071
		2	5.595	6.234	6.550	5.205
		3	7.814	8.484	8.923	7.049
		4	10.282	10.941	11.444	9.357
-0.5	200	1	4.001	4.232	4.487	3.068
		2	5.322	6.010	6.365	4.945
		3	7.782	8.439	8.886	6.931
		4	10.266	10.925	11.431	9.328
	500	1	4.802	4.857	5.163	3.392
		2	5.567	6.273	6.555	5.537
		3	7.892	8.529	8.964	7.055
		4	10.300	10.952	11.457	9.388
	800	1	5.195	5.199	5.579	3.532
		2	5.853	6.553	6.759	5.987
		3	8.005	8.619	9.043	7.177
		4	10.334	10.980	11.482	9.449
-0.8	200	1	4.107	4.319	4.574	3.128
		2	5.334	6.026	6.375	5.005
		3	7.793	8.447	8.893	6.938
		4	10.269	10.927	11.433	9.333
	500	1	4.928	4.969	5.289	3.443
		2	5.601	6.319	6.584	5.652
		3	7.920	8.549	8.982	7.073
		4	10.307	10.958	11.462	9.401
	800	1	5.308	5.308	5.715	3.578
		2	5.920	6.632	6.811	6.130
		3	8.053	8.651	9.072	7.205
		4	10.345	10.989	11.490	9.471

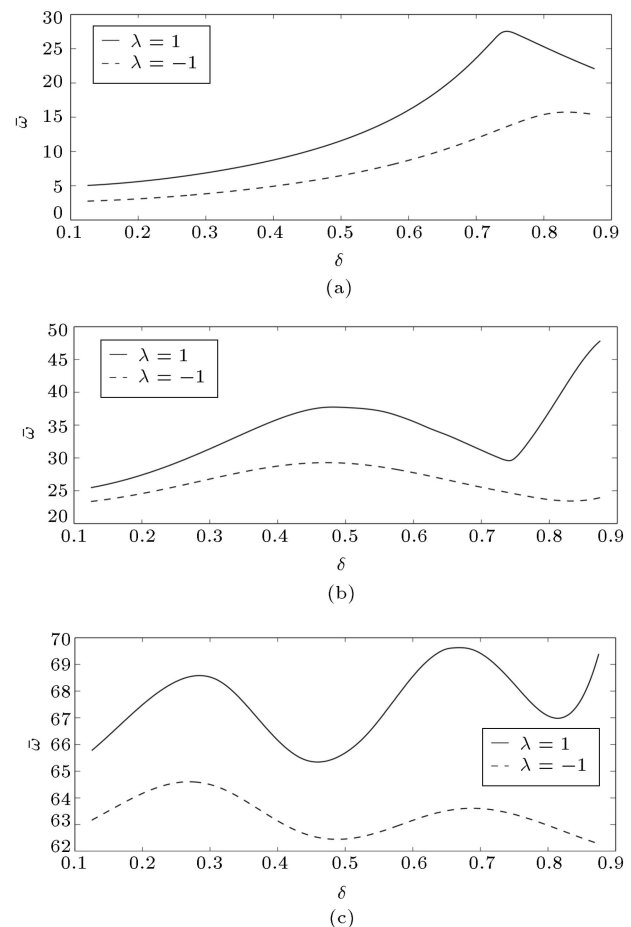
one can define the non-dimensional width function as follows:

$$\bar{b}(\xi) = \frac{b(x)}{b_0} = e^{-\lambda \xi}, \quad (55)$$

where  $b(x)$  is the function of the width,  $b_0$  is the width of the beam on the left side, and  $\lambda$  is the variation parameter of the width function.

The partial elastic foundation occupying one-fourth of the beam's length moves under the beam. Parameter  $\delta$  used in Figure 6 is the distance between the center of the elastic foundation and the left side of the beam. In addition,  $k_0$  is the dimensionless stiffness parameter of the elastic foundation, which is assumed to be constant.

The first three natural frequencies  $\bar{\omega}$  against  $\delta$  are drawn and shown in Figure 7 for  $k_0 = 1000$  and  $\lambda = +1, -1$ . According to Figure 7, both the exponentially narrowing beam (i.e.,  $\lambda = +1$ ) and exponentially widening beam (i.e.,  $\lambda = -1$ ) roughly show similar behavior in all three modes against the movement of the elastic foundation. Moreover, it can be inferred from Figure 7 that in higher mode frequencies in



**Figure 7.** Natural frequencies,  $\bar{\omega}$ , versus  $\delta$ : (a) The first mode, (b) the second mode, and (c) the third mode.

which the modal shape functions have more turning points, the frequency curves versus the movement of the foundation show more C and complicated behavior.

## 5. Conclusion

The sine and cosine Fourier series were employed in this paper in order to obtain free natural frequencies of beams with variable cross-sections, fully or partially supported by the non-uniform elastic foundation. It was shown that this new mathematical method could deal with any function describing the variability of the beam's cross-section and the variability of the foundation stiffness. The proposed solution combined the properties of sine and cosine series to satisfy any boundary condition. The presented results do agree with those reported in the literature, showing the related accuracy. Some other numerical examples were incorporated into this research in order to signify the applications of this new analysis.

The advantages of the Fourier series make this analysis easier and more extensive than any other existing solutions. This analysis can be extended to the forced vibration and buckling problems of beams. If a two-dimensional Fourier series is considered, the presented method will be also applicable to the vibration of plates with varying cross-sections resting on the variable elastic foundation, which is the subject of our future studies.

## Nomenclature

$L$	Beam's length
$\rho$	Beam's mass density
$E$	Modulus of elasticity
$\xi$	Non-dimensional form of $x(= \frac{x}{L})$
$A(x)$	Beam's cross-sectional area function
$A(0)$	Beam's cross-sectional area at the left end
$\bar{A}(\xi)$	Dimensionless cross-sectional area ( $= \frac{A(x)}{A(0)}$ )
$I(x)$	Beam's moment of inertia function
$I(0)$	Beam's moment of inertia at the left end
$\bar{I}(\xi)$	Dimensionless moment of inertia ( $= \frac{I(x)}{I(0)}$ )
$F(x, t)$	Dynamic transverse displacement
$W(x)$	Transverse displacement
$\bar{w}(\xi)$	Dimensionless displacement ( $= \frac{W(x)}{L}$ )
$K(x)$	Winkler's foundation modulus function

$k(\xi)$	Dimensionless foundation modulus ( $= \frac{K(x)L^4}{EI(0)}$ )
$\omega$	Angular frequency
$\bar{\omega}$	Dimensionless frequency parameter ( $= \omega L^2 \sqrt{\frac{\rho A(0)}{EI(0)}}$ )
$P(x, t)$	Dynamic transverse force function
$p(x)$	Transverse force function
$\bar{p}(\xi)$	Dimensionless transverse force ( $= \frac{p(x)L^3}{EI(0)}$ )
$M(x)$	Bending's moment function
$V(x)$	Shearing force function
$\bar{M}(\xi)$	Dimensionless bending moment ( $= \bar{I}(\xi) \frac{d^2 w(\xi)}{d\xi^2}$ )
$\bar{p}_L, \bar{p}_R$	Dimensionless concentrated loads at, respectively, $x = 0$ and $x = L$
$\bar{M}_L, \bar{M}_R$	Dimensionless concentrated moments at, respectively, $x = 0$ and $x = L$
$n$	Fourier series index
$w_c(\xi), \bar{M}_c(\xi)$	Displacement and moment of the slider supported beam, respectively
$w_s(\xi), \bar{M}_s(\xi)$	Displacement and moment of the simply supported beam, respectively
$a_n, b_n$	Fourier cosine coefficients
$c_n, d_n$	Fourier sine coefficients
$K_{TL}, K_{TR}$	Translational stiffnesses at, respectively, $x = 0$ and $x = L$
$K_{\theta L}, K_{\theta R}$	Rotational stiffnesses at, respectively, $x = 0$ and $x = L$
$N$	Fourier series truncation number

## References

1. Jategaonkar, R. and Chehil, D.S. "Natural frequencies of a beam with varying section properties", *J. Sound Vib.*, **133**(2), pp. 303-322 (1989).
2. Katsikadelis, J.T. and Tsiatas, G.C. "Non-linear dynamic analysis of beams with variable stiffness", *J. Sound Vib.*, **270**(4-5), pp. 847-863 (2004).
3. Nikkiah Bahrami, M., Khoshbayani Arani, M., and Rasekh Saleh, N. "Modified wave approach for calculation of natural frequencies and mode shapes in arbitrary non-uniform beams", *Sci. Iran., B*, **18**(5), pp. 1088-1094 (2011).
4. Huang, Y. and Li, X-F. "A new approach for free vibration of axially functionally graded beams with non-uniform cross-section", *J. Sound Vib.*, **329**(11), pp. 2291-2303 (2010).
5. Au, F.T.K., Zheng, D.Y., and Cheung, Y.K. "Vibration and stability of non-uniform beams with abrupt

- changes of cross-section by using modified beam vibration functions", *Appl. Math. Model.*, **29**(1), pp. 19-34 (1999).
6. Naguleswaran, S. "vibration of an Euler-Bernoulli beam of constant depth and with linearly varying breadth", *J. Sound Vib.*, **153**(3), pp. 509-522 (1992).
  7. Firouz-Abadi, R.D., Haddadpoura, H., and Novinzadehb, A.B. "An asymptotic solution to transverse free vibrations of variable-section beams", *J. Sound Vib.*, **304**(3-5), pp. 530-54 (2007).
  8. Datta, A.K. and Sil, S.N. "An analytical of free undamped vibration of beams of varying cross-section", *Comput. Struct.*, **59**(3), pp. 479-483 (1996).
  9. Banerjee, J.R., Su, H., and Jackson, D.R. "Free vibration of rotating tapered beams using the dynamic stiffness method", *J. Sound Vib.*, **298**(4-5), pp. 1034-1054 (2006).
  10. Caruntu, D.I. "Dynamic modal characteristics of transverse vibrations of cantilevers of parabolic thickness", *Mech. Res. Commun.*, **36**(3), pp. 391-404 (2009).
  11. Laura, P.A.A., Gutierrez, R.H., and Rossi, R.E. "Free vibrations of beams of bilinearly varying thickness", *Ocean. Eng.*, **23**(1), pp. 1-6 (1996).
  12. Chaudhari, T.D. and Maiti, S.K. "Modelling of transverse vibration of beam of linearly variable depth with edge crack", *Eng. Fract. Mech.*, **63**(4), pp. 425-445 (1999).
  13. Ece, M.C., Aydogdu, M., and Taskin, V. "Vibration of a variable cross-section beam", *Mech. Res. Commun.*, **34**(1), pp. 78-84 (2007).
  14. Tanaka, M. and Bercin, A.N. "Finite element modelling of the coupled bending and torsional free vibration of uniform beams with an arbitrary cross-section", *Appl. Math. Model.*, **21**(6), pp. 339-344 (1997).
  15. Malekzadeh, P. and Karami, G. "A mixed differential quadrature and finite element free vibration and buckling analysis of thick beams on two-parameter elastic foundations", *Appl. Math. Model.*, **32**(7), pp. 1381-1394 (2008).
  16. Laura, P.A.A. and Valerga De Greco, B. "Numerical experiments on free and forced vibrations of beams of non-uniform cross-section", *J. Sound Vib.*, **120**(3), pp. 587-596 (1998).
  17. Zohoor, H. and Kakavand, F. "Vibration of Euler-Bernoulli and Timoshenko beams in large overall motion on flying support using finite element method", *Sci. Iran., B*, **19**(4), pp. 1105-1116 (2012).
  18. Daeichi, M. and Ahmadian, M.T. "Application of variational iteration method to large vibration analysis of slenderness beams considering mid-plane stretching", *Sci. Iran., B*, **22**(5), pp. 1911-1917 (2015).
  19. Baghani, M., Asgarshamsi, A., and Goharkhaha, M. "Analytical solution for large amplitude vibrations of microbeams actuated by an electro-static force", *Sci. Iran., B*, **20**(5), pp. 1499-1507 (2013).
  20. Lee, S.Y., Ke, H.Y., and Kuo, Y.H. "Analysis of non-uniform beam vibration", *J. Sound Vib.*, **142**(1), pp. 15-29 (1990).
  21. Chen, W.Q., Lu, C.F., and Bian Z.G. "A mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation", *Appl. Math. Model.*, **28**(10), pp. 877-890 (2004).
  22. Thambiratnam, D. and Zuge, Y. "Free vibration analysis of beams on elastic foundation", *Comput. Struct.*, **60**(6), pp. 971-980 (1996).
  23. Ding, Z. "A general solution to vibrations of beams on variable Winkler elastic foundation", *Comput. Struct.*, **47**(1), pp. 83-90 (1993).
  24. Eisenberger, M. "Vibration frequencies for beams on variable one and two-parameter elastic foundations", *J. Sound Vib.*, **176**(5), pp. 577-584 (1994).
  25. Eisenberger, M. and Clastornik, J. "Vibrations and buckling of a beam on a variable Winkler elastic foundation", *J. Sound Vib.*, **115**(2), pp. 233-241 (1987).
  26. Soldatos, K.P. and Selvadurai, A.P.S. "Flexure of beams resting on hyperbolic elastic foundations", *Int. J. Solids. Struct.*, **21**(4), pp. 373-388 (1985).
  27. Pradhan, S.C. and Murmu, T. "Thermo-mechanical vibration of FGM sandwich beam under variable elastic foundations using differential quadrature method", *J. Sound Vib.*, **321**(1-2), pp. 342-362 (2009).
  28. Payam, A.F. "Sensitivity analysis of vibration modes of rectangular cantilever beams immersed in fluid to surface stiffness variations", *Sci. Iran., B*, **20**(4), pp. 1221-1227 (2013).
  29. Mutman, U. and Coskun, S.B. "Free vibration analysis of non-uniform Euler beams on elastic foundation via homotopy perturbation method", *Int. J. Mech. Aerosp. Ind. Mechatron. Eng.*, **7**(7), pp. 432-437 (2013).
  30. Ho, S.H. and Chen, C. "Analysis of general elastically end restrained non-uniform beams using differential transform", *Appl. Math. Model.*, **22**(4-5), pp. 219-234 (1998).
  31. Catal, S. "Solution of free vibration equations of beam on elastic soil by using differential transform method", *Appl. Math. Model.*, **32**(9), pp. 1744-1757 (2008).
  32. Ebrahimzadeh Hassanabadi, M., Nikkhoo, A., Vaseghi Amiri, J., and Mehri, B. "A new orthonormal polynomial series expansion method in vibration analysis of thin beams with non-uniform thickness", *Appl. Math. Model.*, **37**(18-19), pp. 8543-8556 (2013).
  33. Wang, J.T.-S. and Lin, C.-C. "Dynamic analysis of generally supported beams using Fourier series", *J. Sound Vib.*, **196**(3), pp. 285-293 (1996).
  34. Li, W.L. "Free vibrations of beams with general boundary conditions", *J. Sound Vib.*, **237**(4), pp. 709-725 (2000).

35. Motaghian, S.E., Mofid, M., and Alanjari, P. “Exact solution to free vibration of beams partially supported by an elastic foundation”, *Sci. Iran., A*, **18**(4), pp. 861-866 (2011).
36. Motaghian, S.E., Mofid, M., and Akin J.E. “On the free vibration response of rectangular plates, partially supported on elastic foundation”, *Appl. Math. Model.*, **36**(9), pp. 4473-4482 (2012).

### Biographies

**Seyedemad Motaghian** is currently a PhD Candi-

date in Structural Engineering at University of Tehran, Iran. He received his MSc degree in Structural Engineering from Sharif University of Technology and his BSc degree from University of Tehran. His research interests include mechanical vibration, nonlinear elasticity, and mathematical science.

**Massood Mofid** is Professor of Civil and Structural Engineering at Sharif University of Technology.

**John E. Akin** is Professor of Mechanical Engineering at Rice University, Houston, Texas, USA.