A New Fourier series solution for free vibration of non-uniform beams, resting on variable elastic foundation

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Abstract

In this research, the combination of Fourier sine series and Fourier cosine series is employed to develop an analytical method for free vibration analysis of an Euler-Bernoulli beam of varying cross- section, fully or partially supported by a variable elastic foundation. The foundation stiffness and cross section of the beam are considered as arbitrary functions in the beam length direction. The idea of the proposed method is to superpose Fourier sine and Fourier cosine series to satisfy general elastically end constraints and therefore no auxiliary functions are required to supplement the Fourier series. This method provides a simple, accurate and flexible solution for various beam problems and is also able to be extended to other cases whose governing differential equations are nonlinear. Moreover, this method is applicable for plate problems with different boundary conditions if two-dimensional Fourier sine and cosine series are taken as displacement function.

Numerical examples are carried out illustrating the accuracy and efficiency of the presented approach.

Keywords: *Fourier series solution, free vibration, non-uniform beam, variable elastic foundation, general boundary conditions*

1. Introduction

The application of non-uniform beams resting on elastic foundation in civil and mechanical engineering has resulted in extensive studies. Researchers have proposed various analytical and numerical methods to analyze vibration of Euler-Bernoulli beams of varying cross section properties. Jategaonkar and Chehil [1] calculated the natural frequencies of beams with inertia, area and mass varying in a general manner. Katsikadelis and Tsiatas [2] applied the analog equation method to analyze the non-linear dynamic response of an Euler-Bernoulli beam with variable cross-sectional properties along its axis under large deflection. Nikkhah Bahrami et al. [3] used wave propagation method to calculate the natural frequencies and related mode shapes of arbitrary non-uniform beams. Huang and Li [4] used Fredholm integral equations to solve natural frequencies of free vibration of beams with variable flexural rigidity and mass density. Au et al. [5] analyzed vibration and stability of axially loaded non-uniform beams with abrupt changes of cross-section based on the Euler-Lagrangian approach using a family of c^1 admissible functions as the assumed modes. Naguleswaran [6] investigated the transverse vibration of an Euler-Bernoulli beam of constant depth and linearly varying breadth. Firouz-Abadi et al. [7] applied an asymptotic solution to investigate the transverse free vibration of variable-cross-section beams. Datta and Sil [8] used Bessel's functions to evaluate the natural frequency and mode shape of cantilever beams of varying cross-section. Banerjee et al. [9] used the dynamic stiffness method to study free vibration of rotating beams of linearly varying height or width. However, some work concentrated their analysis on dynamic behavior of beams of some specific variation in cross-section. This is comprised of such as those like parabolic thickness variation [10], bilinearly varying thickness [11], linearly variable depth [12] and those with exponentially varying width [13].
In addition to analytical solutions, some numerical and approximation methods including Finite Element Method [14-17], Variational Iteration Method [18-19] and approximate fundamental solution [20] has been developed in order to treat the vibration analysis of beams.
The underlying elastic foundation modeled usually as the Winkler or Pasternak foundation [21-22] has also been a subject of interest in many studies. Some of them have considered the elastic foundations with variable modulus to present a

more precise behavior of the foundation. Ding [23] reported a general solution to vibration of beams on variable Winkler elastic foundation. Eisenberger [24] presented a method based on the dynamic stiffness matrix basically to find the natural frequencies of beams on variable one and two-parameter elastic foundation. Eisenberger and Clastornik [25] solved eigenvalue problems of vibration and stability pertaining to a beam on a variable Winkler elastic foundation. Soldatos and Selvadurai [26] solved the static problem of the flexure of a beam resting on an elastic foundation with hyperbolic modulus. Pradhan and Murmu [27] used the modified differential quadrature method to analyze thermo-mechanical vibration of functionally graded beams. This also includes functionally graded sandwich beams resting on variable Winkler and two parameter elastic foundations. Payam [28] investigated the effect of the surface contact stiffness on both flexural and torsional sensitivity of a cantilever beam immersed in a fluid.

Another set of mathematical solutions applied by the researcher to study the vibration problems of beams on elastic foundation is series expansion methods. Mutman and Coskun [29] used Homotopy Perturbation Method (HPM) to investigate the problem. This method is an analytical approximate technique deforming a linear or nonlinear problem to a series of linear equations. However, in this procedure the initial approximation of solution should be chosen appropriately in order to avoid infinite iterations. Ho and Chen [30] employed the Differential Transform Method (DTM) which is derived from Taylor series expansion to solve the free and forced vibration problems of a general elastically end restrained non-uniform beam. Catal [31] obtained frequency factors of a fixed-simply supported beam with axial load resting on elastic soil, using differential transform method. Hassanabadi et al. [32] developed a method based on the series expansion of orthonormal polynomial to study the transverse vibration of thin beams with variable thickness. The implementation of Fourier series for the analysis of beams has also been explored by some researchers. Wang and Lin [33] used Fourier series to study dynamic behavior of uniform beams with arbitrary boundary conditions. However, in their method, the free boundary condition is missing. Li [34] employed polynomial functions as auxiliary functions which are added to Fourier cosine series merely to solve free vibration of uniform beams with general boundary conditions.

In the previous papers, the Fourier sine series was applied to investigate the free vibration of beams [35] and plates [36] with constant cross section which are partially supported by uniform elastic foundation. The solution proposed in these two articles has satisfied simply supported and clamped boundary conditions. In this paper, the Fourier sine series was combined with the Fourier cosine series to analyze the free vibration problem of a beam of any arbitrary varying cross-section, as well as, fully or partially supported by variable elastic foundation. The presented approach deals with any boundary conditions of the beam. Fourier analysis takes advantage of the periodic and orthogonal trigonometric functions which can represent any continuous and discontinuous function to simplify the solution to the original problem. These benefits make the Fourier analysis as a simple, extensive and reliable solution in mathematics.

2. Mathematical Formulation

2.1. Governing differential equation

Fig. 1 shows an Euler-Bernoulli beam with varying cross-section resting on elastic foundation. The beam is supported by a translational spring and a rotational spring at both ends. The differential equation of the transverse vibration of this beam can be expressed as follows [7, 23]:

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 F(x,t)}{\partial x^2}]+K(x)F(x,t)+\rho A(x)\frac{\partial^2 F(x,t)}{\partial t^2}=P(x,t) \tag{1}$$

where ρ, E, $A(x)$, $I(x)$ are the mass density, modulus of elasticity, cross-sectional area and the moment of inertia, respectively. $F(x,t)$ is the dynamic transverse displacement and $K(x)$ is the Winkler's foundation modulus having been regarded inconstant along the beam. $P(x,t)$ is the dynamic transversely distributed force. x represents the position of each point on the beam and t represents time. Using separation of variable technique (i.e. $F(x,t)=W(x)e^{-i\omega t}$ and $P(x,t)=p(x)e^{-i\omega t}$), one can represent Eq. (1) as:

$$\frac{d^2}{dx^2}[EI(x)\frac{d^2 W(x)}{dx^2}]+K(x)W(x)-\rho A(x)\omega^2 W(x)=p(x) \tag{2}$$

in which ω is the natural frequency and $i=\sqrt{-1}$.

Fig. 1. An Euler-Bernoulli beam of varying cross-section resting on variable elastic foundation which is supported by a translational and a rotational spring at both ends.

The bending moment and shear force are respectively given by:

$$M(x) = EI(x)\frac{d^2W(x)}{dx^2} \tag{3}$$

$$V(x) = \frac{dM(x)}{dx} = \frac{d}{dx}[EI(x)\frac{d^2W(x)}{dx^2}] \tag{4}$$

Let us introduce dimensionless variables as found in the following:

$$\begin{aligned}
&\xi = \frac{x}{L} \\
&w(\xi) = \frac{W(x)}{L} \\
&\overline{A}(\xi) = \frac{A(x)}{A(0)} \\
&k(\xi) = \frac{K(x)L^4}{EI(0)} \\
&\overline{\omega}^2 = \frac{\rho A(0)L^4\omega^2}{EI(0)} \\
&\overline{I}(\xi) = \frac{I(x)}{I(0)}
\end{aligned} \tag{5}$$

where $A(0)$ and $I(0)$ are respectively the area and moment of inertia for the cross-section of the beam at its left end where $x=0$ and L is the length of the beam. ξ, $w(\xi)$, $\overline{A}(\xi)$, $\overline{I}(\xi)$, $k(\xi)$ and $\overline{\omega}$ denote the non-dimensional forms of x, the beam displacement function, the beam cross-section area function, the moment of inertia function, the stiffness function of the foundation and frequency parameter, respectively.

Regarding above variables and relation (3), one can rewrite Eq. (2) as:

$$\frac{d^2\overline{M}(\xi)}{d\xi^2} + k(\xi)w(\xi) - \overline{A}(\xi)\overline{\omega}^2 w(\xi) = \overline{p}(\xi) \tag{6}$$

in which

$$\overline{M}(\xi) = \overline{I}(\xi)\frac{d^2w(\xi)}{d\xi^2} \tag{7}$$

$$\bar{p}(\xi) = \frac{p(x)L^3}{EI(0)} \tag{8}$$

2.2. Proposed solution based on the Fourier cosine and sine series

The newly proposed method is stated in this section utilizing the Fourier sine and cosine series simultaneously. At first, two beams are defined vibrating transversely with different boundary conditions. Fig. 2 shows the first beam supported by slider at both ends. This kind of support is fixed against *x*-direction rotation while allowing the beam ends to translate vertically. Equilibrium equation (6) of this beam can be written as:

$$\frac{d^2\bar{M}_c(\xi)}{d\xi^2} + k(\xi)w_c(\xi) - \bar{A}(\xi)\bar{\omega}^2 w_c(\xi) = \bar{p}_L\delta(\xi - 0) + \bar{p}_R\delta(\xi - 1) \tag{9}$$

where $w_c(\xi)$ and $\bar{M}_c(\xi)$ are referred as the displacement and bending moment of the beam modeled by the cosine series. At the same time, $\bar{p}_L$ and $\bar{p}_R$ are dimensionless concentrated loads on the left and right end of the beam respectively and δ is the Dirac delta function.

Fig. 2. A slider supported beam subjected to a concentration load at each side.

One can assign a cosine series to the bending moment function and similarly to the displacement function as:

$$\bar{M}_c(\xi) = \sum_{n=0}^{\infty} a_n \cos n\pi\xi \tag{10}$$

$$w_c(\xi) = \sum_{n=0}^{\infty} b_n \cos n\pi\xi \tag{11}$$

where a_n and b_n are Fourier coefficients considered as unknown parameters.

The proposed series satisfies boundary conditions as:

$$\xi = 0,1 \Rightarrow \begin{cases} \dfrac{dw_c(\xi)}{d\xi} = 0 \\ \dfrac{d\overline{M}_c(\xi)}{d\xi} = 0 \end{cases} \tag{12}$$

Fourier cosine series defined by Eqs. (10) and (11) can be substituted for the bending moment and displacement functions found in Eqs. (7) and (9). The Dirac delta function is also required to be transformed to Fourier cosine series. Hence, Eqs. (7) and (9) become:

$$\sum_{n=0}^{\infty} a_n \cos n\pi\xi = \sum_{n=0}^{\infty} -b_n n^2\pi^2 \overline{I}(\xi)\cos n\pi\xi \tag{13}$$

$$\begin{gathered}\sum_{n=0}^{\infty} -a_n n^2\pi^2 \cos n\pi\xi + \sum_{n=0}^{\infty} b_n k(\xi)\cos n\pi\xi - \sum_{n=0}^{\infty} b_n \overline{\omega}^2 \overline{A}(\xi)\cos n\pi\xi = \overline{p}_L + \sum_{n=1}^{\infty} 2\overline{p}_L \cos n\pi\xi \\ + \overline{p}_R + \sum_{n=1}^{\infty} 2\overline{p}_R \cos(n\pi)\cos(n\pi\xi)\end{gathered} \tag{14}$$

Multiplying both sides of Eqs. (13) and (14) by $\cos m\pi\xi$ (m=0,1,…) and integration from $\xi = 0$ to 1, one can obtain the following equations:

$$a_0 + \sum_{n=0}^{\infty} C^I{}_{n,m} b_n n^2\pi^2 = 0 \qquad m=0 \tag{15.1}$$

$$\frac{a_m}{2} + \sum_{n=0}^{\infty} C^I{}_{n,m} b_n n^2\pi^2 = 0 \qquad m=1,2,\ldots \tag{15.2}$$

and

$$\frac{m^2\pi^2 a_m}{2} - \sum_{n=0}^{\infty} C^K{}_{n,m} b_n + \sum_{n=0}^{\infty} C^A{}_{n,m} b_n \overline{\omega}^2 + \overline{p}_L + \overline{p}_R \cos m\pi = 0 \qquad m=0,1,2,\ldots \tag{16}$$

in which $C^I{}_{nm}$, $C^K{}_{nm}$, $C^A{}_{nm}$ are defined by the following integrations:

$$C^I{}_{n,m} = \int_0^1 \overline{I}(\xi)\cos(n\pi\xi)\cos(m\pi\xi)d\xi \tag{17.1}$$

$$C^K{}_{n,m} = \int_0^1 k(\xi)\cos(n\pi\xi)\cos(m\pi\xi)d\xi \tag{17.2}$$

$$C^A{}_{n,m} = \int_0^1 \overline{A}(\xi)\cos(n\pi\xi)\cos(m\pi\xi)d\xi \tag{17.3}$$

In order to make a system of equations, one can truncate the series to the first N+1 terms and consider n and m from 0 to N and expand the Eq. (15) as follows:

$$\begin{cases} a_0+\pi^2 C^I{}_{1,0}b_1+4\pi^2 C^I{}_{2,0}b_2+\ldots+N^2\pi^2 C^I{}_{N,0}b_N=0 \\ \dfrac{a_1}{2}+\pi^2 C^I{}_{1,1}b_1+4\pi^2 C^I{}_{2,1}b_2+\ldots+N^2\pi^2 C^I{}_{N,1}b_N=0 \\ . \\ . \\ . \\ \dfrac{a_N}{2}+\pi^2 C^I{}_{1,N}b_1+4\pi^2 C^I{}_{2,N}b_2+\ldots+N^2\pi^2 C^I{}_{N,N}b_N=0 \end{cases} \tag{18}$$

Eq. (16) can also be expanded as:

$$\begin{cases} (C^A{}_{0,0}\overline{\omega}^2-C^K{}_{0,0})b_0+(C^A{}_{1,0}\overline{\omega}^2-C^K{}_{1,0})b_1+\ldots+(C^A{}_{N,0}\overline{\omega}^2-C^K{}_{N,0})b_N+\overline{p}_L+\overline{p}_R=0 \\ \dfrac{\pi^2 a_1}{2}+(C^A{}_{0,1}\overline{\omega}^2-C^K{}_{0,1})b_0+(C^A{}_{1,1}\overline{\omega}^2-C^K{}_{1,1})b_1+\ldots+(C^A{}_{N,1}\overline{\omega}^2-C^K{}_{N,1})b_N+\overline{p}_L+\overline{p}_R\cos\pi=0 \\ . \\ . \\ . \\ \dfrac{N^2\pi^2 a_N}{2}+(C^A{}_{0,N}\overline{\omega}^2-C^K{}_{0,N})b_0+(C^A{}_{1,N}\overline{\omega}^2-C^K{}_{1,N})b_1+\ldots+(C^A{}_{N,N}\overline{\omega}^2-C^K{}_{N,N})b_N+\overline{p}_L+\overline{p}_R\cos N\pi=0 \end{cases} \tag{19}$$

The second beam shown in Fig. 3 is a simply supported beam subjected to concentration moments at both ends. The governing differential equation of this beam is expressed as:

$$\begin{aligned} &\frac{d^2\overline{M}_s(\xi)}{d\xi^2}+k(\xi)w_s(\xi)-\overline{A}(\xi)\overline{\omega}^2 w_s(\xi)=\overline{p}_1\delta(\xi-\varepsilon)-\overline{p}_1\delta(\xi-0) \\ &+\overline{p}_2\delta(\xi-(1-\varepsilon))-\overline{p}_2\delta(\xi-1) \end{aligned} \tag{20}$$

where $w_s(\xi)$ and $\overline{M}_s(\xi)$ are respectively the displacement and bending moment functions pertaining to the beam modeled by the Fourier sine series. Thus $\overline{p}_1$ and $\overline{p}_2$ are two force couples creating concentration moments at both ends. In addition, ε denotes the distance between two forces of a couple which is sufficiently close to zero while the magnitude of forces approaches infinity. The product from them converges to the respective moment [35]:

$$\varepsilon \to 0, \overline{p}_1, \overline{p}_2 \to \infty \Rightarrow \begin{cases} \overline{p}_1 \varepsilon \to \overline{M}_L \\ \overline{p}_2 \varepsilon \to \overline{M}_R \end{cases} \tag{21}$$

in which $\overline{M}_L$ and $\overline{M}_R$ are the concentrated moments at the left and right end of the beam respectively.

Fig. 3. A simply supported beam subjected to a concentration moment at each side.

For a simply supported beam, the transverse displacement and bending moment at *x*=0 and *x*=*L* are equal to zero. Therefore, for their corresponding functions one can use the Fourier sine series as:

$$\overline{M}_s(\xi) = \sum_{n=1}^{\infty} d_n \sin n\pi\xi \tag{22}$$

$$w_s(\xi) = \sum_{n=1}^{\infty} e_n \sin n\pi\xi \tag{23}$$

where d_n and e_n are unknown coefficients.

Putting Eqs. (22) and (23) into Eqs. (20) and (7) and transforming the right hand of Eq. (20) to the Fourier sine series [35], one can get:

$$\sum_{n=1}^{\infty} d_n \sin \pi\xi = \sum_{n=1}^{\infty} -e_n n^2 \pi^2 \overline{I}(\xi) \sin n\pi\xi \tag{24}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} -d_n n^2 \pi^2 \sin n\pi\xi + \sum_{n=1}^{\infty} e_n k(\xi) \sin n\pi\xi - \sum_{n=0}^{\infty} d_n \overline{\omega}^2 \overline{A}(\xi) \sin n\pi\xi = \sum_{n=1}^{\infty} 2n\pi \overline{M}_L \sin n\pi\xi \\ &- \sum_{n=1}^{\infty} 2n\pi \overline{M}_R \cos(n\pi) \sin(n\pi\xi) \end{aligned} \tag{25}$$

Similar to the beam solved by the cosine series, one can multiply both sides of Eqs. (24) and (25) by $\sin n\pi\xi$ and then take an integral from $\xi = 0$ to 1 as in the following:

$$\frac{d_m}{2}+\sum_{n=1}^{\infty}S^I{}_{n,m}e_n n^2\pi^2=0 \qquad m=1,2,\ldots \qquad (26)$$

and

$$\frac{m^2\pi^2 d_m}{2}-\sum_{n=1}^{\infty}S^K{}_{n,m}e_n+\sum_{n=0}^{\infty}S^A{}_{n,m}e_n\overline{\omega}^2-2m\pi\overline{M}_L+2m\pi\cos m\pi\overline{M}_R=0 \quad m=1,2,\ldots \quad (27)$$

where $S^I{}_{n,m}$, $S^K{}_{n,m}$ and $S^A{}_{n,m}$ are

$$S^I{}_{n,m}=\int_0^1\overline{I}(\xi)\sin(n\pi\xi)\sin(m\pi\xi)d\xi \qquad (28.1)$$

$$S^K{}_{n,m}=\int_0^1 k(\xi)\sin(n\pi\xi)\sin(m\pi\xi)d\xi \qquad (28.2)$$

$$S^A{}_{n,m}=\int_0^1\overline{A}(\xi)\sin(n\pi\xi)\sin(m\pi\xi)d\xi \qquad (28.3)$$

The first N terms of the above series are selected in order to develop the system of equations that was made earlier. Hence, Eq. (26) can be expanded as:

$$\begin{cases}\frac{d_1}{2}+\pi^2S^I{}_{1,1}e_1+4\pi^2S^I{}_{2,1}e_2+\ldots+N^2\pi^2S^I{}_{N,1}e_N=0\\ .\\ .\\ .\\ \frac{d_N}{2}+\pi^2S^I{}_{1,N}e_1+4\pi^2S^I{}_{2,N}e_2+\ldots+N^2\pi^2S^I{}_{N,N}e_N=0\end{cases} \qquad (29)$$

Likewise, Eq. (27) can be rewritten as:

$$\begin{cases}\frac{\pi^2 d_1}{2}+(S^A{}_{1,1}\overline{\omega}^2-S^K{}_{1,1})e_1+\ldots+(S^A{}_{N,1}\overline{\omega}^2-S^K{}_{N,1})e_N-2\pi\overline{M}_L+2\pi\cos(\pi)\overline{M}_R=0\\ .\\ .\\ .\\ \frac{N^2\pi^2 d_N}{2}+(S^A{}_{1,N}\overline{\omega}^2-S^K{}_{1,N})e_1+\ldots+(S^A{}_{N,N}\overline{\omega}^2-S^K{}_{N,N})e_N-2N\pi\overline{M}_L+2N\pi\cos(N\pi)\overline{M}_R=0\end{cases} \qquad (30)$$

Eqs. (18), (19), (29) and (30) make up a collection of 2N+1 equations with 2N+5 unknowns. This includes Fourier sine and cosine coefficients joining with four unknown end forces and moments (i.e. $\overline{p}_L$, $\overline{p}_R$, $\overline{M}_L$ and $\overline{M}_R$). In order to have a system of equations involving the same set of unknowns, one needs four more equations which are contributed by exerting boundary conditions. Moreover, boundary condition equations do establish an association between two sets of equations developed separately with the cosine and sine series.

3. Boundary conditions

The beam studied at this paper is restrained with a transversal spring and a rotational spring at both ends. Therefore, the sum of concentrated loads applied to both ends of the first beam along with shear forces generated at both edges of the second beam should be proportional to the respective end displacement of the first beam. Thus, one can use relation (4) and write the following equations:

$$P_L + \left.\frac{dM_s(x)}{dx}\right|_{x=0} = K_{TL} W_c(x=0) \tag{31}$$

$$P_R + \left.\frac{dM_s(x)}{dx}\right|_{x=L} = K_{TR} W_c(x=L) \tag{32}$$

where K_{TL} and K_{TR} are translational spring constants at the left and right end of the beam, respectively. Moreover, W_c and M_s are respectively the displacement function of the beam analyzed by the cosine series (the first beam) and bending moment function of the beam analyzed by the sine series (the second beam).

Applying non-dimensional parameters gives:

$$\overline{p}_L + \left.\frac{d\,\overline{M}_s(\xi)}{d\,\xi}\right|_{\xi=0} = \overline{K}_{TL} w_c(\xi=0) \tag{33}$$

$$\overline{p}_R + \left.\frac{d\,\overline{M}_s(\xi)}{d\,\xi}\right|_{\xi=1} = \overline{K}_{TR} w_c(\xi=1) \tag{34}$$

where

$$\overline{K}_{TL} = \frac{K_{TL}L^3}{EI(0)} \tag{35.1}$$

$$\overline{K}_{TR} = \frac{K_{TR}L^3}{EI(0)} \tag{35.2}$$

Using Eqs. (11) and (22), we can extend Eqs. (33) and (34) as follows:

$$\overline{p}_L + \sum_{n=1}^{N} d_n (n\pi) - \overline{K}_{TL} \sum_{n=1}^{N} b_n = 0 \tag{36}$$

$$\overline{p}_R + \sum_{n=1}^{N} d_n (n\pi) \cos n\pi - \overline{K}_{TR} \sum_{n=1}^{N} b_n \cos n\pi = 0 \tag{37}$$

Analogously, the sum of bending moments of the first beam and concentrated moments of the second beam at both ends is commensurate with the rotation angle of the respective edge of the second beam as:

$$M_L + M_c(x = 0) = K_{\theta L} \left. \frac{dW_s(x)}{dx} \right|_{x=0} \tag{38}$$

$$M_R + M_c(x = L) = K_{\theta R} \left. \frac{dW_s(x)}{dx} \right|_{x=L} \tag{39}$$

where $K_{\theta L}$ and $K_{\theta R}$ are rotational spring constants at the left and right end of the beam, respectively. Moreover, W_s and M_c are respectively the displacement function of the beam modeled by the sine series (the second beam) and bending moment function of the beam modeled by the cosine series (the first beam).

In non-dimensional form, Eqs. (38) and (39) become:

$$\overline{M}_L + \overline{M}_c(\xi = 0) = \overline{K}_{\theta L} \left. \frac{dw_s(\xi)}{d\xi} \right|_{\xi=0} \tag{40}$$

$$\overline{M}_R + \overline{M}_c(\xi = 1) = \overline{K}_{\theta R} \left. \frac{dw_s(\xi)}{d\xi} \right|_{\xi=1} \tag{41}$$

in which

$$\overline{K}_{\theta L} = \frac{K_{\theta L} L}{EI(0)} \tag{42.1}$$

$$\overline{K}_{\theta R} = \frac{K_{\theta R} L}{EI(0)} \tag{42.2}$$

Substituting Eqs. (10) and (23) in Eqs. (40) and (41) gives:

$$\overline{M}_L + \sum_{n=0}^{N} a_n - \overline{K}_{\theta L} \sum_{n=0}^{N} e_n (n\pi) = 0 \tag{43}$$

$$\overline{M}_R + \sum_{n=0}^{N} a_n \cos n\pi - \overline{K}_{\theta L} \sum_{n=0}^{N} e_n (n\pi) \cos n\pi = 0 \tag{44}$$

Eqs. (36), (37), (43) and (44) create a link between two groups of equations representing respectively the behavior of the first and second beam.

Three classical boundary conditions including simply supported, clamped and free end are the specialized cases of the general elastically constraint as listed below:

- Simply supported

$$\begin{cases} w_c(\xi = 0 or 1) = 0 \\ \overline{M}_{LorR} + \overline{M}_c(\xi = 0 or 1) = 0 \end{cases} \tag{45}$$

- Fixed end

$$\begin{cases} w_c(\xi = 0 or 1) = 0 \\ \left. \dfrac{dw_s(\xi)}{d\xi} \right|_{\xi = 0 or 1} = 0 \end{cases} \tag{46}$$

- Free end

$$\begin{cases} \overline{M}_{LorR} + \overline{M}_c(\xi = 0 or 1) = 0 \\ \left. \overline{p}_{LorR} + \dfrac{d\overline{M}_s(\xi)}{d\xi} \right|_{\xi = 0 or 1} = 0 \end{cases} \tag{47}$$

By writing the four remaining equations provided by boundary conditions, a system of $2N$+5 equations is made. This system can be written in matrix form as:

$$[A]_{2N+5,2N+5}[U]_{2N+5,1}=0 \quad (48)$$

Where *[U]* is unknown coefficients vector and *[A]* is coefficient matrix containing the natural circular frequency term. The non-trivial solution to this problem requires matrix *A* to have determination zero

$$|A|_{2N+5,2N+5}=0 \quad (49)$$

Solving Eq. (49) leads to obtaining the natural frequencies of the beam and their related mode shapes.

4. Numerical examples and verification

In this section, a number of case studies are conducted to demonstrate the introduced method. Initially, two examples are modeled choosing different terms of truncated series, *N*, and compared with the previous work to verify the proposed solution. The first example is a cantilever beam with rectangular cross-section. The width of the beam is kept constant while the height varies linearly as:

$$\bar{h}(\xi)=\frac{h(x)}{h_0}=1-c\xi \quad (50)$$

where *h(x)* is the height of the beam, h_0 is the height of the beam at x=0 and *c* is the taper ratio constant. The beam is clamped at $\xi=0$ and is free at $\xi=1$.
The first three dimensionless natural frequencies $\bar{\omega}$ of this beam with different values of the tapered ratio constant are calculated using developed solution and are shown in Table 1. The results are compared with those obtained by Banerjee et al. [9]. The comparison indicates an excellent agreement between two results.

Table 1. The first three natural frequencies $\bar{\omega}$ of the linearly height tapered beam versus taper ratio *c* and the first *N* terms of truncated series

The second example is a simply supported beam resting on variable Winkler elastic foundation with linear and parabolic moduli which is investigated by

Ding [23]. In this example, the elastic modulus for linear foundation is defined as:

$$k(\xi) = k_0(1-\alpha\xi) \quad (51)$$

For parabolic foundation, we have:

$$k(\xi) = k_0(1-\beta\xi^2) \quad (52)$$

where α and β are variation parameters and k_0 is the foundation constant at $\xi = 0$.

Tables 2 and 3 report the first three non-dimensional natural frequency parameters $\sqrt{\omega}$ which pertains respectively to linear and parabolic foundation regarding different amounts of α, β and k_0, as well as, compare them with the results presented in [23]. In this case, the results which entirely comply with those obtained by Ding [23] are independent from *N* and do not change when *N* increases.

Table 2. The first three natural frequency parameters $\bar{\omega}$ of a uniform simply supported beam resting on Winkler foundation with linear modulus vs. k_0 and variation parameter α

Table 3. The first three natural frequency parameters $\bar{\omega}$ of a uniform simply supported beam resting on Winkler foundation with parabolic modulus vs. k_0 and variation parameter β

The results presented in Tables 2 and 3 indicate that the natural frequencies of the beam resting on a foundation with parabolic modulus are relatively greater than those of linear foundation with the same k_0 and variation parameter. Thus, a parabolically varying foundation provides a stiffer support than an equivalent foundation with linear modulus. However, this increase is more considerable in the first mode than the other two and also becomes more discernible when the variation parameter gets larger.
Some other case studies with various characteristics are included in this section to illustrate the applications of the presented method. In all following cases, *h(x)*

stands for the height of the beam and h_0 is the height of the left side of the beam. For obtaining the results, the first 80 terms of truncated series have been considered (i.e. *N*=80).

Fig. 4 displays a beam whose one end is fixed and the other is supported by a spring with stiffness coefficient $\overline{K}_{TR}$ **.** The beam cross-section is assumed to be rectangular with constant width and exponentially varying thickness. It is worth mentioning that exponentially varying cross-section beams may provide an appropriate distribution of mass and strength to resist dynamic forces such as wind, earthquake, fluid pressure, etc. Therefore, they are being used in mechanical components (such as turbine blades, helicopter blades, etc.) and also in robotics, aeronautics, architecture and other innovative engineering applications.

The non-dimensional height function can be written as:

$$\overline{h}(\xi)=\frac{h(x)}{h_0}=e^{-\eta\xi} \tag{53}$$

where η is the variation parameter of the beam height.

Fig. 4. A cantilever beam of exponentially varying thickness which is supported by a translational spring at its free end.

Fig. 5 shows the first, second and third natural frequency parameter of this beam, $\sqrt{\omega}$, versus spring parameter $\overline{K}_{TR}$ for $\eta = -0.4,-1,0.4,1$. According to Fig. 5, since the mass distribution in the exponentially decreasing height beam is farther to the spring in comparison with the exponentially heightening beam, the natural frequencies of the former converge more rapidly to a constant value than those of the latter when $\overline{K}_{TR}$ increases. However, for smaller amounts of $\overline{K}_{TR}$, the first and second natural frequencies of the exponentially decreasing height beam are more sensitive to the stiffness parameter of the spring.

Fig. 5. Natural frequency parameters $\sqrt{\omega}$ versus spring parameter $\overline{K}_{TR}$ (a) The first mode (b) The second mode (c) The third mode.

In the next case, a beam of rectangular cross-section with parabolic thickness variation and constant width which is supported partially by Winkler elastic foundation is studied. The elastic foundation is distributed under the beam from *x*=*L*/4 to *x*=2*L*/3 and the foundation stiffness varies linearly. The non-dimensional thickness and foundation stiffness functions can be expressed as follows:

$$\bar{h}(\xi) = \frac{h(x)}{h_0} = 1 - c\xi^2 \qquad (54.1)$$

$$k(\xi) = k_0(1 - \alpha\xi) \qquad (54.2)$$

Where c and α are variation parameter of the thickness and foundation stiffness respectively.

The first four natural frequency parameters $\sqrt{\varpi}$ of this beam with different boundary conditions, i.e. Simply Supported (SS) beam, Clamped-Simply Supported (CS) beam, Clamped-Clamped (CC) beam and Clamped-Free (CF) beam, are tabulated in Table 4 for c=0.8 and variable amounts for α and k_0.

Table 4. The first four natural frequency parameters $\sqrt{\varpi}$ of a beam with parabolic varying thickness partially supported by linear elastic foundation for *c=0.8* regarding different boundary conditions

The subject of our next case is the movement of a partial elastic foundation spread under one-fourth of the beam. Fig. 6 shows a cantilever beam with rectangular cross-section in which the characteristic height is constant while the characteristic width varies exponentially along the length of the beam. Therefore, one can define the non-dimensional width function as follows:

$$\bar{b}(\xi) = \frac{b(x)}{b_0} = e^{-\lambda\xi} \qquad (55)$$

in which *b(x)* is the function of the width , b_0 is the width of the beam at the left side and λ is the variation parameter of the width function.

The partial elastic foundation occupying one-fourth of the beam length moves under the beam. The parameter δ used in the Fig. 6 is the distance between the center of the elastic foundation and the left side of the beam. In addition, k_0 is the dimensionless stiffness parameter of the elastic foundation which is assumed to be constant.

Fig. 6. A cantilever beam with exponentially varying width which is supported by a moving elastic foundation whose length is one-fourth of the beam length.

The first three natural frequencies $\bar{\omega}$ against δ are drawn and shown in Fig. 7 for $k_0 = 1000$ and $\lambda = +1, -1$. As it can be seen in Fig. 7, both the exponentially narrowing beam (i.e. $\lambda = +1$) and the exponentially widening beam (i.e. $\lambda = -1$) roughly show similar behavior in all three modes against the movement of the elastic foundation. Moreover, it can be inferred from Fig.7 that in higher mode frequencies in which the modal shape functions have more turning points, the frequency curves versus the movement of the foundation show more periodic and complicated behavior.

Fig. 7. Natural frequencies $\bar{\omega}$ versus δ (a) The first mode (b) The second mode (c) The third mode.

Conclusion

The sine and cosine Fourier series are employed in this paper in order to obtain free natural frequencies of beams with variable cross-sections, fully or partially supported by non-uniform elastic foundation. It is shown that this new mathematical method can deal with any function describing the variability of the beam's cross-section, as well as, the variability of the foundation stiffness. The proposed solution combines the properties of sine and cosine series to satisfy any boundary condition. Presented results do agree with those reported in the literature showing the accuracy of it. Some other numerical examples are

incorporated to this research in order to signify the applications of this new analysis.
The advantages of the Fourier series make this analysis easier and more extensive compared to other existing solutions. This analysis can be extended to the forced vibration and buckling problems of beams. If two-dimensional Fourier series is considered, the presented method will be also applicable for vibration of plates with varying cross-section resting on variable elastic foundation which is a subject for our future studies.

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List of symbols

L:	beam length
ρ:	beam mass density
E:	modulus of elasticity
ξ:	non-dimensional form of x $(=\frac{x}{L})$
$A(x)$:	beam cross-section area function
$A(0)$:	beam cross-section area at the left end
$\bar{A}(\xi)$:	dimensionless cross-section area $(=\frac{A(x)}{A(0)})$
$I(x)$:	beam moment of inertia function
$I(0)$:	beam moment of inertia at the left end
$\bar{I}(\xi)$:	dimensionless moment of inertia $(=\frac{I(x)}{I(0)})$
$F(x,t)$:	dynamic transverse displacement
$W(x)$:	transverse displacement
$\bar{w}(\xi)$:	dimensionless displacement $(=\frac{W(x)}{L})$
$K(x)$:	Winkler's foundation modulus function
$k(\xi)$:	dimensionless foundation modulus $(=\frac{K(x)L^4}{EI(0)})$
ω:	angular frequency
$\bar{\omega}$:	dimensionless frequency parameter $(=\omega L^2\sqrt{\frac{\rho A(0)}{EI(0)}})$
$P(x,t)$:	dynamic transverse force function
$p(x)$:	transverse force function
$\bar{p}(\xi)$:	dimensionless transverse force $(=\frac{p(x)L^3}{EI(0)})$
$M(x)$:	bending moment function
$V(x)$:	shearing force function
$\bar{M}(\xi)$:	dimensionless bending moment $(=\bar{I}(\xi)\frac{d^2w(\xi)}{d\xi^2})$
$\bar{p}_L, \bar{p}_R$:	dimensionless concentrated loads at, respectively, x=0 and x=L

$\overline{M}_L, \overline{M}_R$: dimensionless concentrated moments at, respectively, x=0 and x=L

n: Fourier series index

$w_c(\xi), \overline{M}_c(\xi)$: respectively, displacement and moment of the slider supported beam

$w_s(\xi), \overline{M}_s(\xi)$: respectively, displacement and moment of the simply supported beam

a_n, b_n: Fourier cosine coefficients

c_n, d_n: Fourier sine coefficients

K_{TL}, K_{TR}: translational stiffnesses at, respectively, x=0 and x=L

$K_{\theta L}, K_{\theta R}$: rotational stiffnesses at, respectively, x=0 and x=L

N: Fourier series truncation number

An Euler-Bernoulli beam of varying cross-section resting on variable elastic foundation which is supported by a translational and a rotational spring at both ends.

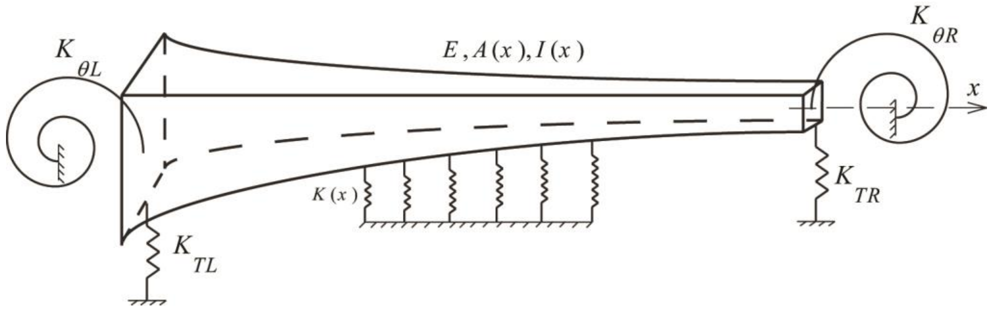

Fig. 1

A slider supported beam subjected to a concentration load at each side.

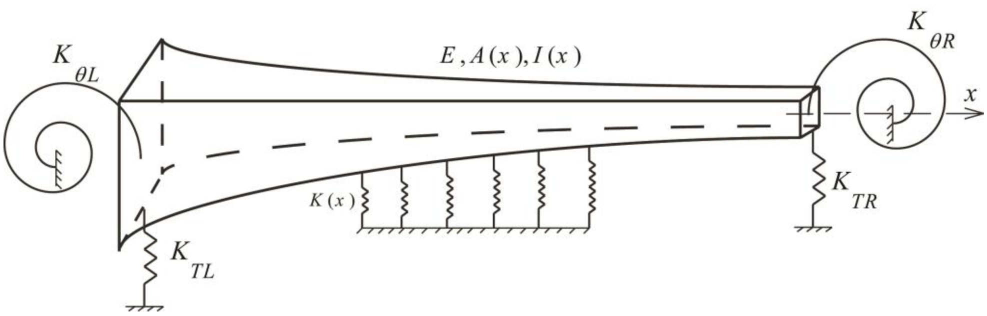

Fig. 2

A simply supported beam subjected to a concentration moment at each side.

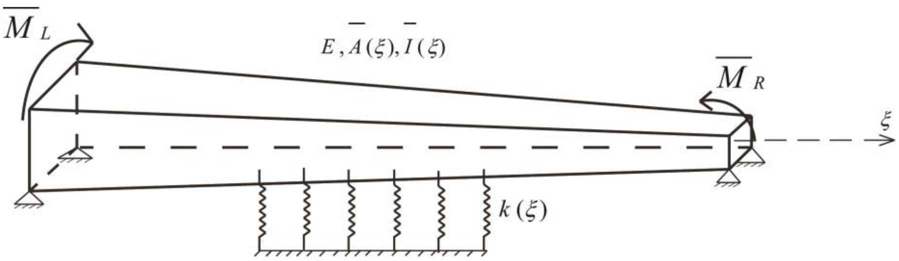

Fig. 3

A cantilever beam of exponentially varying thickness which is supported by a translational spring at its free end.

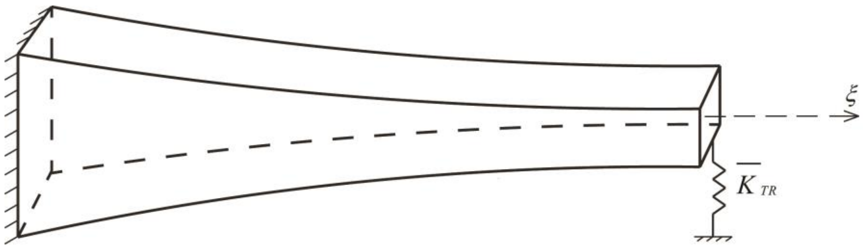

Fig. 4

Natural frequency parameters $\sqrt{\omega}$ versus spring parameter $\overline{K}_{TR}$ (a) The first mode (b) The second mode (c) The third mode.

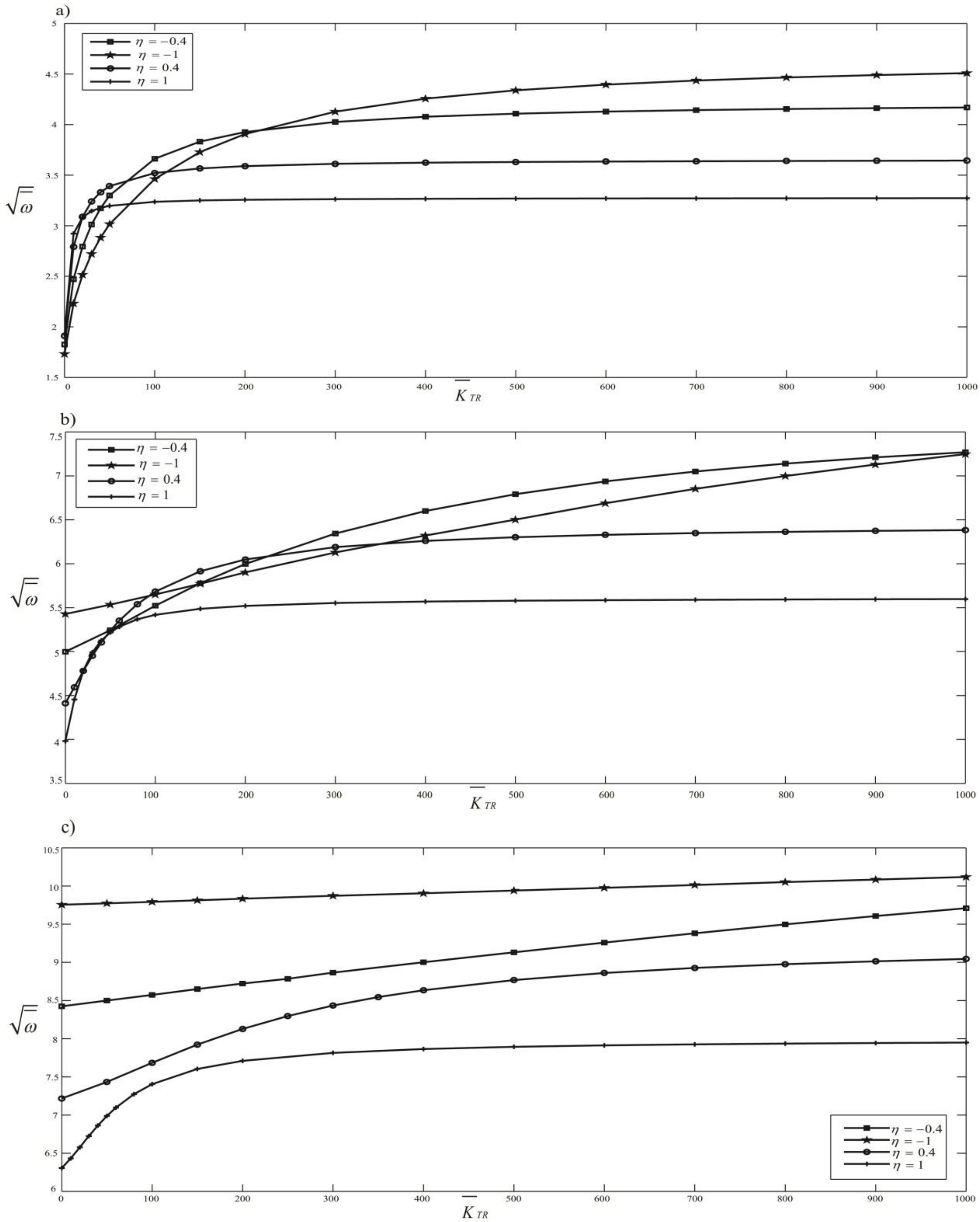

Fig. 5

A cantilever beam with exponentially varying width which is supported by a moving elastic foundation whose length is one-fourth of the beam length.

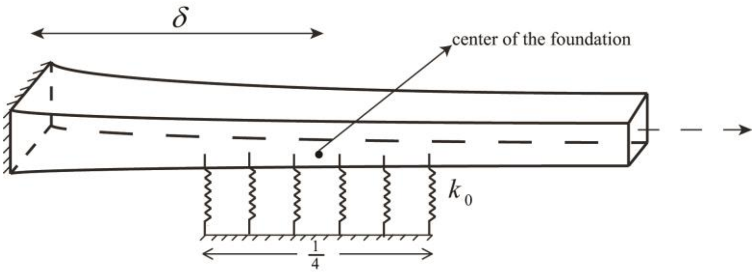

Fig. 6

Natural frequencies $\bar{\omega}$ versus δ (a) The first mode (b) The second mode (c) The third mode.

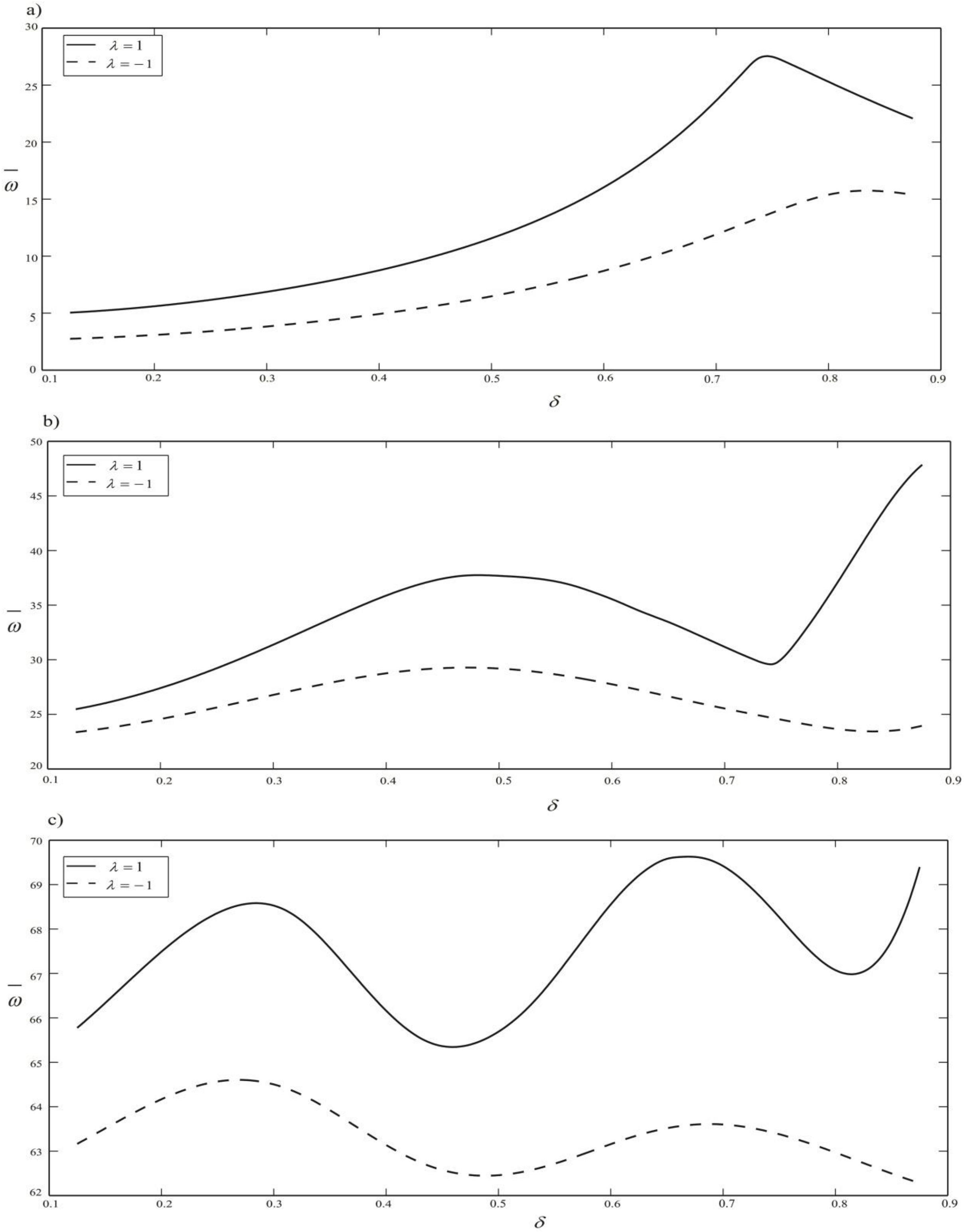

Fig. 7

The first three natural frequencies $\bar{\omega}$ of the linearly height tapered beam versus taper ratio c and the first N terms of truncated series

c	N	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0.1	10	3.576	21.554	59.392
	30	3.560	21.395	59.438
	50	3.559	21.286	59.330
	80	3.559	21.430	59.241
	Ref. [9]	**3.559**	**21.338**	**58.980**
0.3	10	3.687	19.741	54.811
	30	3.670	20.011	54.007
	50	3.668	19.983	53.784
	80	3.667	19.951	53.636
	Ref. [9]	**3.667**	**19.881**	**53.322**
0.5	10	3.845	18.442	48.804
	30	3.827	18.432	47.907
	50	3.825	18.397	47.678
	80	3.824	18.371	47.535
	Ref. [9]	**3.824**	**18.317**	**47.265**
0.6	10	3.954	17.609	45.475
	30	3.937	17.583	44.602
	50	3.935	17.553	44.389
	80	3.934	17.531	44.259
	Ref. [9]	**3.934**	**17.488**	**44.025**
0.8	10	4.308	15.787	38.081
	30	4.294	15.787	37.267
	50	4.292	15.774	37.116
	80	4.292	15.764	37.029
	Ref. [9]	**4.292**	**15.743**	**36.885**
0.9	10	4.645	14.946	34.114
	30	4.631	14.941	33.082
	50	4.630	14.941	32.976
	80	4.630	14.939	32.919
	Ref. [9]	**4.631**	**14.931**	**32.833**
0.99	10	5.232	14.967	31.702
	30	5.215	14.948	29.920
	50	5.214	14.958	29.786
	80	5.214	14.963	29.747
	Ref. [9]	**5.214**	**14.967**	**29.727**

Table 1

The first three natural frequency parameters $\bar{\omega}$ of a uniform simply supported beam resting on Winkler foundation with linear modulus vs. k_0 and variation parameter α

k_0	α	$\sqrt{\bar{\omega}_1}$		$\sqrt{\bar{\omega}_2}$		$\sqrt{\bar{\omega}_3}$	
		Presented Method	Ref. [23]	Presented Method	Ref. [23]	Presented Method	Ref. [23]
500	0.2	4.836	4.837	6.694	6.695	9.556	9.577
	0.4	4.720	4.721	6.653	6.652	9.542	9.541
	0.6	4.594	4.595	6.610	6.610	9.527	9.527
	0.8	4.455	4.456	6.568	6.568	9.513	9.513
1000	0.2	5.618	5.618	7.041	7.041	9.682	9.682
	0.4	5.468	5.468	6.971	6.971	9.655	9.656
	0.6	5.300	5.301	6.898	6.898	9.627	9.627
	0.8	5.113	5.113	6.823	6.824	9.599	9.599
1500	0.2	6.165	6.165	7.345	7.345	9.864	9.864
	0.4	5.991	5.991	7.251	7.251	9.765	9.765
	0.6	5.797	5.798	7.154	7.155	9.723	9.724
	0.8	5.574	5.574	7.057	7.057	9.684	9.684
2000	0.2	6.596	6.597	7.614	7.614	9.922	9.921
	0.4	6.405	6.405	7.503	7.503	9.870	9.870
	0.6	6.186	6.187	7.390	7.390	9.819	9.819
	0.8	5.937	5.937	7.271	7.272	9.767	9.767

Table 2

The first three natural frequency parameters $\overline{\omega}$ of a uniform simply supported beam resting on Winkler foundation with parabolic modulus vs. k_0 and variation parameter β

k_0	β	$\sqrt{\overline{\omega}_1}$		$\sqrt{\overline{\omega}_2}$		$\sqrt{\overline{\omega}_3}$	
		Presented Method	Ref. [23]	Presented Method	Ref. [23]	Presented Method	Ref. [23]
500	0.2	4.884	4.884	6.710	6.710	9.562	9.562
	0.4	4.820	4.821	6.683	6.683	9.552	9.552
	0.6	4.753	4.753	6.656	6.657	9.542	9.542
	0.8	4.682	4.682	6.630	6.630	9.534	9.534
1000	0.2	5.679	5.679	7.067	7.067	9.692	9.692
	0.4	5.597	5.597	7.022	7.022	9.675	9.675
	0.6	5.507	5.507	6.980	6.980	9.657	9.657
	0.8	5.409	5.409	6.935	6.935	9.638	9.638
1500	0.2	6.232	6.233	7.378	7.378	9.817	9.817
	0.4	6.137	6.138	7.321	7.321	9.792	9.792
	0.6	6.032	6.032	7.263	7.263	9.766	9.767
	0.8	5.917	5.917	7.205	7.206	9.741	9.741
2000	0.2	6.671	6.671	7.653	7.653	9.939	9.939
	0.4	6.564	6.564	7.587	7.587	9.905	9.905
	0.6	6.444	6.444	7.521	7.521	9.873	9.874
	0.8	6.312	6.312	7.454	7.454	9.841	9.841

Table 3

The first four natural frequency parameters $\sqrt{\omega}$ of a beam with parabolic varying thickness partially supported by linear elastic foundation for *c=0.8* regarding different boundary conditions

α	k_0	Mode Number	$\sqrt{\omega}$			
			SS	**CS**	**CC**	**CF**
0.5	200	1	3.558	3.876	4.147	2.768
		2	5.279	5.959	6.328	4.739
		3	7.747	8.412	8.863	6.903
		4	10.256	10.917	11.424	9.311
	500	1	4.220	4.340	4.621	3.091
		2	5.457	6.129	6.461	5.105
		3	7.801	8.463	8.907	6.991
		4	10.275	10.934	11.438	9.344
	800	1	4.597	4.639	4.951	3.250
		2	5.651	6.303	6.597	5.412
		3	7.855	8.514	8.950	7.079
		4	10.294	10.950	11.453	9.378
0.8	200	1	3.382	3.743	4.025	2.624
		2	5.266	5.945	6.318	4.677
		3	7.737	8.404	8.856	6.895
		4	10.253	10.915	11.422	9.306
	500	1	3.965	4.116	4.403	2.916
		2	5.425	6.088	6.433	4.961
		3	7.775	8.444	8.890	6.971
		4	10.268	10.928	11.433	9.331

	800	1	4.307	4.371	4.677	3.071
		2	5.595	6.234	6.550	5.205
		3	7.814	8.484	8.923	7.049
		4	10.282	10.941	11.444	9.357
-0.5	200	1	4.001	4.232	4.487	3.068
		2	5.322	6.010	6.365	4.945
		3	7.782	8.439	8.886	6.931
		4	10.266	10.925	11.431	9.328
	500	1	4.802	4.857	5.163	3.392
		2	5.567	6.273	6.555	5.537
		3	7.892	8.529	8.964	7.055
		4	10.300	10.952	11.457	9.388
	800	1	5.195	5.199	5.579	3.532
		2	5.853	6.553	6.759	5.987
		3	8.005	8.619	9.043	7.177
		4	10.334	10.980	11.482	9.449
-0.8	200	1	4.107	4.319	4.574	3.128
		2	5.334	6.026	6.375	5.005
		3	7.793	8.447	8.893	6.938
		4	10.269	10.927	11.433	9.333
	500	1	4.928	4.969	5.289	3.443
		2	5.601	6.319	6.584	5.652
		3	7.920	8.549	8.982	7.073
		4	10.307	10.958	11.462	9.401
	800	1	5.308	5.308	5.715	3.578

		2	5.920	6.632	6.811	6.130
		3	8.053	8.651	9.072	7.205
		4	10.345	10.989	11.490	9.471

Table 4

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