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# Fuzzy multi-objective optimization of linear functions subject to max-arithmetic mean relational inequality constraints

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**Abstract.** The main goal of this work is to find a better solution to a kind of multi-objective optimization problem subject to a system of fuzzy relational inequalities with max-arithmetic mean composition. First, this problem is solved and, then, in the case that the decision maker is not satisfied with any of the solutions, by assigning linear membership functions to the inequalities in the constraints and objective functions and using Bellman-Zadeh decision, a new solution is found. This new solution does not belong to the feasible domain but is considered acceptable based on the decision maker's view. In order to find this solution easier, some simplification processes are given. Afterwards, an algorithm is presented to generate the new solution. Finally, an example is given to illustrate the steps of the algorithm.



## 1. Introduction

Since the time the notion of Fuzzy Relational Equation (FRE) was introduced by Sanchez, many works have been done in the domain of FREs, Fuzzy Relational Inequalities (FRIs), and the problems relevant to them; for instance, see Khorram and Zarei [1], Yang [2], and Zhou et al. [3]. The usage of FREs and FRIs can be observed in many fields such as fuzzy control, fuzzy decision making, knowledge engineering, image processing, image and video compression and decompression, image reconstruction, fuzzy modeling, fuzzy diagnosis, and especially fuzzy medical diagnosis [4].

Max-min composition is the most frequently used composition in FREs and FRIs. Nevertheless, it is shown that the min operator is not always the best selection for the intersection operation [5]. Thus, some researchers have studied FREs and FREIs in the presence of other compositions. For example, Molai [6] and Hassanzadeh et al. [7] considered max-product composition, whereas Khorram et al. [8] and Guu et al. [9] employed max-t-norm composition in their problems. For the first time, Zimmermann used the arithmetic mean operator, which was not a t-norms as an "and" operator. Following Zimmermann's idea [10], Khorram et al. [1,5] and Wu [10] considered FREs and FRIs under max-arithmetic mean composition. In [4], it was shown that with regards to sensitivity, the arithmetic mean was one of the best aggregation operators. Thereafter, a fuzzy optimization problem

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subject to a system of max-arithmetic mean relational inequality was studied.

In addition to problems in which an objective function is optimized over a system of FREs or FRIs, multi- objective optimization problems have been considered by some researchers [8,9]. In [1], a multi-objective optimization problem in the presence of a system of FREs with max-arithmetic mean composition has been considered. In [11], the authors have studied the same problem with FRIs.

As it is mentioned in [12], when the Decision Maker (DM) is not satisfied with the solution of an optimization problem, it is possible to soften the rigid requirements of the DM in order to consider the imprecision of his/her judgment so that a better solution can be obtained. To pursue this idea, in [13], the authors have considered the fuzzy linear optimization problem in the presence of fuzzy relational inequality constraints with max-min composition. Also, in [4], this problem has been studied with max-arithmetic mean composition. To the authors' best knowledge, no work has been done to investigate the multi-objective model of the problem which has been studied in [4]. Here, we study this kind of problems.

As an application of this model, we can consider the given example in [13]. Consider a schoolmaster who decides to cover three educational zones by enhancing the educational quality of his school (A). He considers some criteria to convince the parents to select school A. Also, he has some plans for each potentially poor criterion. Thus, he wants to resolve the problem of parents as desirably as possible in three zones by enhancing the quality such that they prefer to select school A while the cost expended for this purpose becomes less than or equal to the budget. In the case that the schoolmaster has more aims such as maximizing the spent budget for one of the criteria (consider athletic-recreational facilities that can be considered as a potential investment for the school) in comparison to other expenses, we have a multi-objective problem subject to a system of fuzzy relational inequalities. This problem can be considered with any max-aggregation function composition. As we have shown in [4], one of the best choices for an aggregation function to use in FRIs is arithmetic mean, which is the chosen aggregation function of this paper.

The rest of the paper is outlined in the following. In Section 2, a multi-objective optimization of linear functions with ordinary inequalities in the presence of a max-arithmetic mean composition problem is studied [11]. Then, using a selected solution and linear membership functions, the multi-objective optimization problem in the presence of the fuzzy inequalities is converted into another problem with one objective function. In Section 3, the main goal is to reduce the dimension of the feasible domain as much as possible. Section 4 introduces an algorithm to give the solution using the steps of Section 3 and provides one numerical example to illustrate the algorithm. Concluding remarks are given in Section 5.

## 2. Problem formulation

Consider the following linear multi-objective optimization problem:

$$\begin{aligned} \min \quad & \{Z_1(x), \cdots, Z_p(x)\}, \\ \text{s.t.} \quad & A \circ x \leq b, \\ & x \in [0,1]^n. \end{aligned} \tag{1}$$

where, "$\circ$" stands for the max-arithmetic mean composition. Assume that DM is not satisfied with (any of) the solution(s) of Relation (1). In this case, we try to find a better solution, which is called fuzzy solution here. Fuzzy solution, which violates at least one constraint and is still acceptable to be a solution based on the DM's view, is achieved by softening the constraints of Relation (1) [13]. The amount of perturbations imposed on the constraints is determined by having interaction with the DM. To this end, the focus is on solving the following problem:

$$\begin{aligned} \widetilde{\min} \quad & \{Z_1(x), \cdots, Z_p(x)\}, \\ \text{s.t.} \quad & A \circ x \precsim b, \\ & x \in [0,1]^n, \end{aligned} \tag{2}$$

where, $A = (a_{ij})_{m\times n}$ is a matrix, and $b = (b_i)_{m\times 1}$ and $x = (x_j)_{n\times 1}$ are the right-hand-side and unknown vectors, respectively, such that $a_{ij}$, $b_i \in [0,1]$; $i \in I = \{1, 2, \cdots, m\}$ and $j \in J = \{1, 2, \ldots, n\}$. Also, for all $l \in L = \{1, 2, \cdots, p\}$, $Z_l(x) = c_l^t x$ are linear objective functions where, $c_l = (c_{lj})_{n\times 1}$, $c_{lj} \in \mathcal{R}$ and $\mathcal{R}$ is the set of real numbers. Here, "$\widetilde{\min}$" and "$\precsim$" represent moderate or fuzzy types of "min" and "$\leqslant$" meaning that "objective functions should be minimized as much as possible" and "the constraints should be well satisfied", respectively [12].

Let $a_i$ denote the $i$th row of matrix $A$; then, Relation (2) can be demonstrated as follows:

$$\begin{aligned} \widetilde{\min} \quad & \{Z_1(x), \cdots, Z_p(x)\}, \\ \text{s.t.} \quad & a_i \circ x \precsim b_i; \qquad i \in I, \\ & x \in [0,1]^n, \end{aligned}$$

where, $a_i \circ x \precsim b_i$ means $\max_{j\in J}(\frac{a_{ij}+x_j}{2}) \precsim b_i$ for all $i \in I$.

In order to solve Relation (2), it is necessary to find solutions of Relation (1); then, define membership functions for $\precsim$ and objective functions; and use

Bellman-Zadeh decision [14]. Accordingly, Relation (1) and its solutions have a significant role in solving Relation (2). Correspondingly, in the following, some previously obtained results are stated [4,5,11,12] that are in the direction of solving Relation (1).

**Notation 1 [4].** Set:

$$S^i(A,b) = \{x \in [0,1]^n : a_i \circ x \le b_i\} \quad \text{for all} \quad i \in I,$$

$$S(A,b) = \bigcap_{i\in I} S^i(A,b) = \{x \in [0,1]^n : A \circ x \le b\}.$$

**Definition 1 [12].** $\widehat{x} \in S(A,b)$ is a complete optimal solution to Relation (1) if and only if $Z_l(\widehat{x}) \le Z_l(x)$, for all $l \in L$ and all $x \in S(A,b)$.

Nevertheless, generally, when the objective functions conflict with each other, such a complete optimal solution which concurrently minimizes all of the objective functions does not always exist. Therefore, Pareto optimal solution is used as a substitute [12].

**Definition 2 [12].** $x' \in S(A,b)$ is said to be a Pareto optimal solution to Relation (1) if and only if there does not exist another $x \in S(A,b)$ such that $Z_l(x) \le Z_l(x')$ for all $l \in L$ and $Z_l(x) \ne Z_l(x')$ for at least one $l$.

Throughout this work, any kind of (complete/Pareto) solution is called an "optimal solution". However, it is clear that a complete (Pareto) optimal solution yields a fuzzy complete (Pareto) optimal solution.

The following theorems state some properties of $S(A,b)$. For more details and proof of theorems see [4,5].

**Theorem 1 [4].**

a) $S(A,b) \ne \emptyset$, if and only if for all $i \in I$ and all $j \in J$, $2b_i - a_{ij} \geqslant 0$;

b) If $S(A,b) \ne \emptyset$, then $\overline{0} = [0,0,\cdots,0]^t_{1\times n}$ is the unique minimal element of $S(A,b)$.

**Theorem 2 [4].** If $S(A,b) \ne \emptyset$, then $\overline{x} = (\overline{x}_j)_{n\times 1}$ is the unique maximal element of $S(A,b)$ where, $\overline{x}_j = \min\{1, \min_{i\in I}\{2b_i - a_{ij}\}\}$.

Here, the feasible domain of Relation (1) can be given.

**Corollary 1.** *If $S(A,b) \ne \emptyset$, then $S(A,b) = [\overline{0}, \overline{x}]$.*

Now, one simplified form of Relation (1) can be presented.

**Theorem 3.** *Relation (1) is equivalent to the following problem:*

$$\min \quad \{Z_1(x), \cdots, Z_p(x)\},$$
$$\text{s.t.} \quad x \in [\overline{0}, \overline{x}]. \tag{3}$$

***Proof.*** Due to Corollary 1, the proof follows immediately.□

**Notation 2.** Set:

$$J' = \{j \in J | c_{lj} < 0 \quad \text{for all} \quad l \in L\},$$

$$J'' = \{j \in J | c_{lj} > 0 \quad \text{for all} \quad l \in L\},$$

$$\overline{J} = J \setminus (J' \cup J'') \quad \text{and}$$

$$J^{\cdot} = J \setminus J''.$$

**Theorem 4.** *If $x_{os}$ is an optimal solution to Relation (3), then $(x_{os})_j = \overline{x}_j$ and $j \in J'$ and $(x_{os})_j = 0$ for all $j \in J''$.*

***Proof.*** Let $x \in S(A,b)$ be an optimal solution where, for some $j' \in J'$, $x_{j'} \ne \overline{x}_{j'}$. Set $x'_j = x_j$ for all $j \in J \setminus \{j'\}$ and $x'_{j'} = \overline{x}_{j'}$. Thus, $Z_l(x') < Z_l(x)$ for all $l \in L$, which is a contradiction. Similarly, the other part could be proven.□

According to Theorem 4, it is enough to compute $(x_{os})_j$ for all $j \in \overline{J}$. Thus, in order to solve Relation (3) and, as a result of Theorem 3 to solve Relation (1), we just consider the following problem:

$$\min \quad \sum_{j\in\overline{J}} c_{lj}x_j + \sum_{j\in J'} c_{lj}\overline{x}_j \quad \text{for all} \quad l \in L,$$

$$\text{s.t.} \quad x_j \in [0, \overline{x}_j] \quad \text{for all} \quad j \in \overline{J}. \tag{4}$$

Since the feasible domain of Relation (4) is no longer a system of FRIs, it can be solved by any existing method for these kinds of problems that are explained in [12]. Also, it can be solved by heuristic methods such as the Genetic algorithm [15]. If the problem has several solutions and the DM is satisfied with none of them, then the DM shall choose one of these solutions based on his/her point of view in order to obtain one fuzzy solution [11].

Now, using the selected solution of Relation (1), we try to solve Relation (2). In fact, we are going to investigate if it is possible to minimize all objective functions considering aspiration levels $z_l$ of the DM by imposing certain flexibility on the constraints. That means we consider the following problem:

$$c_l^t x \preccurlyeq z_l; \quad \text{for all} \quad l \in L,$$

$$A \circ x \preccurlyeq b,$$

$$x \geqslant 0. \tag{5}$$

Assume that $z_l$ for all $l \in L$ and the amount of acceptable flexibility on the constraints are specified having interaction with the DM. Then, the following linear membership functions for fuzzy inequalities in Relation (5) can be employed as in [4,13]:

$$\mu_i(a_i \circ x) = \begin{cases} 1 & a_i \circ x \leqslant b_i \\ 1 - \frac{a_i \circ x - b_i}{d_i} & b_i \leqslant a_i \circ x \leqslant b_i + d_i \\ 0 & a_i \circ x \geqslant b_i + d_i, \end{cases} \tag{6}$$

$$\mu_l(c_l^t x) = \begin{cases} 1 & c_l^t x \leqslant z_l \\ 1 - \frac{c_l^t x - z_l}{d_0^l} & z_l \leqslant c_l^t x \leqslant z_l + d_0^l \\ 0 & c_l^t x \geqslant z_l + d_0^l. \end{cases} \tag{7}$$

where, $z_l = Z_l(x_{os}) - v d_0^l$ for some fixed $v \in (0,1)$. Each $d_i$ and $d_0^l$ is a chosen constant expressing the limit of the permissible violation of the $i$th inequality for all $i \in I$ and all $l \in L$. Further, $\mu_l(z_l) = 1$, $\mu_l(Z_l(x_{os})) = 1 - v$, and $\mu_l(Z_l(x_{os}) + (1-v)d_0^l) = 0$ for all $l \in L$. The parameters $v$, $d_0^l$, and $d_i$ for all $i \in I$ and $l \in L$ can usually be found based on the empirical-technical views of the DM. Note that Eqs. (6) and (7) allocate a higher degree to those points that are closer to the feasible solution set. Assignment of these membership functions is crucial to find the best fuzzy solution as near as possible to the feasible solution set. On the occasion that the flexibility of the constraints is not sufficient, the optimal solution will not change and, afterwards, more flexibility is enforced on the constraints to find a better fuzzy solution [13].

**Remark 1 [12].** Some other membership functions can be used, such as piecewise, exponential, hyperbolic, or hyperbolic inverse ones, besides the linear membership function.

**Notation 3 [13].** Set $S_\Lambda = \{x \in [0,1]^n : x \notin S(A,b)\}$.

In fact, only the vectors of $x \in [0,1]^n$ can be better solutions than $x_{os}$ for Relation (5) that violate at least one inequality $a_i \circ x \leqslant b_i$. That is, $x$ is an infeasible solution or, by Notation 3, $x \in S_\Lambda$ equivalently [13].

The next theorem represents the most important problem of Section 2.

**Theorem 5.** *Relation (5) is equivalent to the following problem*:

$$\begin{aligned} &\max \quad \lambda, \\ &\text{s.t.} \quad D_i\left(\max_{j\in J}(a_{ij} + x_j)\right) + \lambda \leq B_i; \qquad i \in I, \\ &\qquad D_0^l(c_l^t x) + \lambda \leq B_0^l; \qquad l \in L \\ &\qquad x \in [0,1]^n. \end{aligned} \tag{8}$$

***Proof.*** Similar to [4,12,13], following the decision of Bellman and Zadeh, Relation (5) has the following form:

$$\Lambda = \max_{x\in[0,1]^n} \left\{ \min_{l\in L} \left\{ \mu_l(c_l^t x), \min_{i\in I} \mu_i(a_i \circ x) \right\} \right\}. \tag{9}$$

Considering Eqs. (6) and (7) and substituting $D_i = \frac{1}{2d_i}$ and $B_i = 1 + \frac{b_i}{d_i}$ for all $i \in I$, $D_0^l = \frac{1}{d_0^l}$ and $B_0^l = 1 + \frac{z_l}{d_0^l}$, Eq. (9) is rewritten as:

$$\begin{aligned} \Lambda = \max_{x\in[0,1]^n} \Bigg\{ \min_{l\in L} \Bigg\{ & B_0^l - D_0^l(c_l^t x), \\ & \min_{i\in I} \left\{ B_i - D_i\left(\max_{j\in J}(a_{ij} + x_j)\right) \right\} \Bigg\} \Bigg\}. \end{aligned} \tag{10}$$

Now, by introducing $\lambda$ as the auxiliary variable:

$$\lambda = \min_{l\in L} \left\{ B_0^l - D_0^l(c_l^t x), \min_{i\in I} \left\{ B_i - D_i\left(\max_{j\in J}(a_{ij} + x_j)\right) \right\} \right\},$$

we have:

$$\lambda \leq B_i - D_i\left(\max_{j\in J}(a_{ij} + x_j)\right), \tag{11}$$

for all $i \in I$ and

$$\lambda \leq B_0^l - D_0^l(c_l^t x), \tag{12}$$

for all $l \in L$. Using Relations (11) and (12), Problem (8) is equivalent to the following problem:

$$\begin{aligned} &\max \quad \lambda, \\ &\text{s.t.} \quad \lambda \leq B_i - D_i\left(\max_{j\in J}(a_{ij} + x_j)\right); \qquad i \in I \\ &\qquad \lambda \leq B_0^l - D_0^l(c_l^t x); \qquad l \in L \\ &\qquad x \in [0,1]^n. \end{aligned} \tag{13}$$

Now, from Problem (13), Problem (8) is derived immediately. □

Therefore, according to Theorem 5, in order to find the fuzzy solution to Relation (2), it is adequate to consider Relation (8). In the next section, the dimension of Relation (8) is reduced as much as possible.

## 3. Simplification process

In this section, some theorems are given in order to convert Relation (8) into the equivalent problems that are more simplified and more easily solvable as well. Similar to [4], we use the following notations:

**Notation 4.** Set

$$\lambda_0^l(x) = B_0^l - D_0^l(c_l^t x) \quad \text{for all} \quad l \in L,$$

$$\lambda_i(x) = B_i - D_i\left(\max_{j\in J}(a_{ij}+x_j)\right) \quad \text{for all} \quad i \in I,$$

$$\lambda_{ij}(x_j) = B_i - D_i(a_{ij}+x_j) \quad \text{for all}$$

$$i \in I \quad \text{and} \quad j \in J \quad \text{and}$$

$$\Lambda(x) = \min\left\{\min_{i\in I}\{\lambda_i(x)\}, \min_{l\in L}\{\lambda_0^l(x)\}\right\}.$$

**Theorem 6.** *The functions $\lambda_i$ for all $i \in I$ and $\lambda_{ij}$ for all $i \in I$ and $j \in J$ are non-increasing continuous functions. Especially, $\lambda_{ij}$ decreases for each component $x_j \in [0,1]$. Also, $\lambda_0^l$ for all $l \in L$ decreases with respect to component $x_j$ if $c_j > 0$, and increases if $c_j < 0$.*

***Proof.*** Straightforward. □

The next theorem provides a simplification process to solve Problem (8) by finding some components of its solution.

**Theorem 7.** *If $x^*$ is the optimal solution of Problem (8), then $x_j^* = 0$ for all $j \in J''$.*

***Proof.*** The proof is similar to that of Theorem 2 in [13]. □

**Remark 2.** According to Theorem 7, in the solving procedure of Problem (8), some columns of matrix $A$ can be removed. Thus, to solve Problem (8), it is adequate to consider only the columns of matrix $A$ that belong to $J^{\cdot}$.

By Remark 2, Problem (8) is converted into the following simplified form:

$$\begin{aligned}
\max \quad & \lambda,\\
\text{s.t.} \quad & D_i\left(\max_{j\in J^{\cdot}}(a_{ij}+x_j)\right) + \lambda \le B_i; \quad i \in I\\
& D_0^l\left(\sum_{j\in J^{\cdot}} c_{lj}x_j\right) + \lambda \le B_0^l; \quad l \in L\\
& x \in [0,1]^{|J^{\cdot}|},
\end{aligned} \tag{14}$$

where $|J^{\cdot}|$ is cardinality of $J^{\cdot}$.

**Theorem 8 [4].** *For all $i \in I$ and $j \in J^{\cdot}$,*

a) *If $2b_i - a_{ij} \geqslant 1$ then, $\lambda_{ij}(x_j) \geqslant 1$ for all $x_j \in [0,1]$;*
b) *If $2b_i - a_{ij} < 1$ then, $\lambda_{ij}(x_j) \geqslant 1$ for all $x_j \in [0, 2b_i - a_{ij}]$.*

***Proof.*** See the proof of Theorem 7 in [4]. □

Theorem 8 has two interesting conclusions.

**Corollary 2 [4].** *Let $i \in I$ and $j \in J^{\cdot}$. Then, $\lambda_{ij}(x_j) \geqslant 1$ for all $x_j \in [0,1]$ if and only if $x_j$ does not violate the inequality $a_{ij} \circ x \leqslant b_i$.*

***Proof.*** See the proof of Corollary 3 in [4]. □

**Corollary 3 [4].** *Under the simplification burdened by Remark 2, $x \in S_\Lambda$ if and only if there exist $i \in I$ such that $\lambda_i(x) < 1$.*

***Proof.*** See the proof of Corollary 4 in [4]. □

The next theorem introduces another simplification to convert Relation (14) to the more simplified form.

**Theorem 9 [4].** *Suppose that the simplification by Remark 2 is done and $i \in I$. Then, $\lambda_i(x) = \min_{j\in J_i^{\cdot}}\{\lambda_{ij}(x_j)\}$, where $J_i^{\cdot} = \{j \in J^{\cdot} : 2b_i - a_{ij} < 1\}$.*

***Proof.*** See the proof of Theorem 8 in [4]. □

**Corollary 4.** *Relation (14) is equivalent to the following problem:*

$$\begin{aligned}
\max \quad & \lambda,\\
\text{s.t.} \quad & D_i\left(\max_{j\in J_i^{\cdot}}(a_{ij}+x_j)\right) + \lambda \le B_i; \quad i \in I\\
& D_0^l\left(\sum_{j\in J^{\cdot}} c_{lj}x_j\right) + \lambda \le B_0^l; \quad l \in L\\
& x \in [0,1]^{|J^{\cdot}|}.
\end{aligned} \tag{15}$$

***Proof.*** Straightforward. □

**Theorem 10.** *Let $I_j^{\cdot} = \{i \in I : 2b_i - a_{ij} < 1\}$ for all $j \in J^{\cdot}$ and $\widehat{J} = \{j \in J^{\cdot} : I_j^{\cdot} \neq \emptyset\}$. Then:*

a) *We can remove all columns $j \notin \widehat{J}$ from matrix $A$ with no effect on the optimal solution to Problem (8);*
b) *$x_j^* = 1$ for all $j \notin \widehat{J}$, where $x^*$ is the optimal solution to Problem (8).*

***Proof.*** Proof is followed by a modification of Corollary 6 in [13]. □

**Corollary 5.** *Relation (15) is equivalent to the following problem:*

$$\begin{aligned}
&\max \quad \lambda,\\
&\text{s.t.} \quad D_i\left(\max_{j\in\widehat{J}}(a_{ij}+x_j)\right)+\lambda\le B_i; \quad i\in I\\
&\qquad D_0^l\left(\sum_{j\in\widehat{J}} c_{lj}x_j\right)+\lambda\le B_0^l; \quad l\in L\\
&\qquad x\in[0,1]^{|\widehat{J}|}.
\end{aligned} \tag{16}$$

***Proof.*** Straightforward. □

## 4. An algorithm to solve Relation (2)

Until now, Relation (8) has been in the process of being simplified to Relation (16) as its equivalent problem. Now, some definitions are presented to provide an algorithm improving the objective functions of Relation (16) in each step and stopping at the optimal solution. Here, it is presumed that all the simplifications mentioned earlier have been done on Relation (16). Similar to [4], we consider the following notations.

**Notation 5.** Let $\lambda_K\in(0,1)$. For all $j\in\widehat{J}$ set $I_j=\{i\in I_j^{\cdot}:\lambda_{ij}(x_j^i)=\lambda_K$ for some $x_j^i\in(0,1)\}$ and $x_j^{i'}=\min_{i\in I_j}\{x_j^i\}$. Also, see Remark 2 in [13].

**Remark 3 [4].** $\lambda_{ij}(x_j^i)=\lambda_K$ and $x_j^{i'}\le x_j^i$ implies $\lambda_{ij}(x_j^{i'})\ge\lambda_{ij}(x_j^i)=\lambda_K$ by Theorem 6; thus, $\lambda_{ij}(x_j^{i'})\ge\lambda_K$ for all $i\in I_j$. Therefore, it can be assumed that $I_j^{\cdot}=\{i'\}$.

**Remark 4 [4].** If $x_j^{i''}=x_j^{i'''}=\min_{i\in I_j}\{x_j^i\}$ for some $i''\ne i'''$, then set $i'=i''$ in the case $\frac{d_{i''}+a_{i''j}}{b_{i''}}\le\frac{d_{i'''}+a_{i'''j}}{b_{i'''}}$, otherwise, set $i'=i'''$.

**Remark 5.** It is possible that, for some $i\in I$ we have $i\in I_j^{\cdot}$ for more than one $j\in\widehat{J}$. This means there exist $I_{\cdot}\subseteq I$ such that for all $i\in I_{\cdot}$, there exist $J_i'\subseteq\widehat{J}$ such that $i\in I_j^{\cdot}$ for all $j\in J_i'$.

**Theorem 11.**

a) *If for all $j,j'\in\widehat{J}$ such that $j\ne j'$, $I_j^{\cdot}$ and $I_{j'}^{\cdot}$ are disjoint, then Relation (16) is equivalent to:*

$$\begin{aligned}
&\max \quad \lambda,\\
&\text{s.t.} \quad D_i(a_{ij}+x_j)+\lambda\le B_i; \quad \forall i\in I_j^{\cdot}\\
&\qquad\qquad \text{and} \quad \forall\, j\in\widehat{J}\\
&\qquad D_0^l\left(\sum_{j\in\widehat{J}} c_{lj}x_j\right)+\lambda\le B_0^l; \quad l\in L\\
&\qquad x\in[0,1]^{|\widehat{J}|}.
\end{aligned} \tag{17}$$

b) *Let for some $i\in I$, $i\in I_j^{\cdot}$ for more than one $j\in\widehat{J}$. In this case, Relation (16) is equivalent to:*

$$\begin{aligned}
&\max \quad \lambda,\\
&\text{s.t.} \quad D_i(a_{ij}+x_j)+\lambda\le B_i; \quad \forall\, i\in I_j^{\cdot} \quad \forall\, j\in J^*\\
&\qquad D_i(a_{ij}+x_j)+\lambda\le B_i; \quad \forall\, i\in I_j^{\cdot} \quad \forall\, j\in J_i'\\
&\qquad D_0^l\left(\sum_{j\in\widehat{J}} c_{lj}x_j\right)+\lambda\le B_0^l; \quad l\in L\\
&\qquad x\in[0,1]^{|\widehat{J}|},
\end{aligned} \tag{18}$$

where, $J^*=\widehat{J}\setminus\bigcup_{i\in I_{\cdot}}J_i'$.

***Proof.***

a) By Notation 5 and Remark 3, for each $j\in J$, the key role to find the fuzzy solution is played by $i\in I_j^{\cdot}$. Now, if all $I_j^{\cdot}$'s are mutually disjoint, then Relation (17) is derived from Relation (16) immediately;

b) Now, let for some $i\in I$, $i\in I_j^{\cdot}$. for more than one $j\in\widehat{J}$. Then, by Part (a), Relation (16) is equivalent to the following form:

$$\begin{aligned}
&\max \quad \lambda,\\
&\text{s.t.} \quad D_i(a_{ij}+x_j)+\lambda\le B_i; \quad \forall\, i\in I_j^{\cdot}\\
&\qquad\qquad \text{and} \quad \forall\, j\in J^*\\
&\qquad D_i\left(\max_{j\in J_i'}(a_{ij}+x_j)\right)+\lambda\le B_i; \quad \forall\, i\in I_j^{\cdot}\\
&\qquad D_0^l\left(\sum_{j\in\widehat{J}} c_{lj}x_j\right)+\lambda\le B_0^l; \quad l\in L\\
&\qquad x\in[0,1]^{|\widehat{J}|},
\end{aligned}$$

where, $J^*=\widehat{J}\setminus\bigcup_{i\in I_{\cdot}}J_i'$.

Since $D_i(\max_{j\in J_i'}(a_{ij}+x_j))+\lambda\le B_i$ implies $D_i(a_{ij}+x_j)+\lambda\le B_i$ for all $j\in J_i'$ Relation (16) is concluded directly. □

Now, all the requirements are ready to present the algorithm. The following algorithm obtains a fuzzy solution to Relation (2).

**Algorithm 1.** Suppose Relation (2) is given and $d_i(i \in I)$, $d_0^l(l \in L)$, and $v$ are suggested by the DM; then do the following steps:

- **Step 1.** Consider Relation (1) and compute $\overline{x}$ by Theorem 2;
- **Step 2.** Obtain $J'$, $J''$, $\overline{J}$, and $J^{\cdot}$ by Notation 2;
- **Step 3.** Using Theorem 4, convert Relations (1) to (4) and solve it by any multi-objective linear programming method such as the heuristic one. Completely derive all $x_{os}$'s and $Z_l(x_{os})$'s by Theorem 4. Choose one optimal solution having interaction with the DM;
- **Step 4.** Derive Relation (8) considering the selected solution in Step 3;
- **Step 5.** Set $x_j^* = 0$ for all $j \in J''$ by Theorem 7. Then, convert the problem of Step 4 to the form of Relation (14);
- **Step 6.** Obtain $J_i^{\cdot}$ for all $i \in I$ and convert the problem obtained in Step 5 into Relation (15) using Theorem 9;
- **Step 7.** By Theorem 10, derive $I_j^{\cdot}$ for all $j \in J^{\cdot}$ and obtain $\widehat{J}$. Set $x_j^* = 1$ for all $j \notin \widehat{J}$ and then, convert the problem of Step 6 into Relation (16) by Theorem 10;
- **Step 8.** Get $\epsilon, P$. Let $K = 1$, $\lambda_K = 1 - v$, and $(x^*)_j = (x_{os})_j$ for all $j \in \widehat{J}$;
- **Step 9.** Until $\lambda_K \geq 1 - \epsilon$ or $K = P$ or $\lambda_K = \lambda_{K-1}$, do:

  9-1. Derive $I_j$ for all $j \in \widehat{J}$ using Notation 5;

  9-2. If for some $j \in \widehat{J}$, $I_j = \emptyset$, then if $\lambda_K = 1 - v$, then Relation (2) with this $\lambda_K$ has no better solution than $x^*$ and so, $x^*$ and $Z_l(x^*)$ for all $l \in L$ are optimal. Otherwise, $\lambda^* = \lambda_{K-1}$, $x^* = x$ and go to Step 11;

  9-3. Obtain $I_j^{\cdot}$ for all $j \in \widehat{J}$ using Notation 5, Remark 3, and Remark 4;

  9-4. If for all $j \neq j'$ and $j, j' \in \widehat{J}$; $I_j^{\cdot} \cap I_{j'}^{\cdot} = \emptyset$, then convert the problem obtained in Step 7 to Relation (17); otherwise, convert it to Relation (18) using Theorem 11;

  9-5. Solve the problem obtained in step 9-4 with any linear programming method such as the simplex method and find $x$, $\lambda$. If it has no optimal solution, then set $\lambda^* = \lambda_{K-1}$ and $x^* = x$ is the optimal solution and go to Step 11;

  9-6. $K := K + 1$, $\lambda_K = \lambda$.

  **Step 10.** $x_j^* = x_j$ for all $j \in \widehat{J}$ and $\lambda^* = \lambda$;

  **Step 11.** $Z_l(x^*) = c_l^t x^*$ for all $l \in L$;

  **Step 12.** End.

**Remark 6.** Algorithm 1 is a polynomial time algorithm, and in the following, it is illustrated by one example.

**Example 1.** Consider the following problem:

$$\min \quad \{2x_1 + x_2 - x_3 - 6x_4, -3x_1 + x_2 - 3x_3 + 2x_4\},$$

$$\text{s.t.} \quad \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0.3 \\ 0.4 & 0.8 & 0.1 & 0.2 \\ 0.0 & 0.3 & 0.7 & 0.6 \\ 0.1 & 0.3 & 0.1 & 0.4 \end{bmatrix} \circ x \preceq \begin{bmatrix} 0.4 \\ 0.7 \\ 0.5 \\ 0.6 \end{bmatrix}.$$

Let $v = 0.5$, $d_0^1 = \frac{2}{3}$, $d_0^2 = \frac{1}{2}$, $d_1 = 0.3$, $d_2 = 0.1$, $d_3 = 0.2$, $d_4 = 0.1$.

Using Algorithm 1, we have the following steps:

- **Step 1.** We consider the following problem:

$$\min \quad \{2x_1 + x_2 - x_3 - 6x_4, -3x_1 + x_2 - 3x_3 + 2x_4\}$$

$$\text{s.t.} \quad \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0.3 \\ 0.4 & 0.8 & 0.1 & 0.2 \\ 0.0 & 0.3 & 0.7 & 0.6 \\ 0.1 & 0.3 & 0.1 & 0.4 \end{bmatrix} \circ x \leq \begin{bmatrix} 0.4 \\ 0.7 \\ 0.5 \\ 0.6 \end{bmatrix},$$

  and we have $\overline{x} = (0.3, 0.6, 0.3, 0.4)$.

- **Step 2.** We have $J' = \{3\}$, $J'' = \{2\}$, $\overline{J} = \{1, 4\}$, and $J^{\cdot} = \{1, 3, 4\}$.
- **Step 3.** In this step, we have the following problem:

$$\min \quad \{2x_1 - 6x_4 - 0.3, -3x_1 + 2x_4 - 0.9\}$$

$$\text{s.t.} \quad x_1 \in [0, 0.3]$$

$$x_4 \in [0, 0.4].$$

  By the optimization toolbox of Matlab software and "gamultiobj" solver, which uses the Genetic algorithm for solving multi-objective problems, 11 Pareto optimal solutions have been computed for the problem of this step and they are presented in Table 1, where $(x_{os})_2 = 0$ and $(x_{os})_3 = 0.3$ for all Pareto optimal solutions. Assume the DM has chosen $x_{os} = (0.239, 0, 0.3, 0.307)$. Therefore, $Z_1(x_{os}) = -1.664$ and $Z_2(x_{os}) = -1.003$.

- **Step 4.** We have $D_1 = \frac{5}{3}$, $D_2 = 10$, $D_3 = 5$, $D_4 = 10$, $D_0^1 = \frac{3}{2}$, $D_0^2 = 2$, $B_1 = \frac{7}{3}$, $B_2 = 8$, $B_3 = \frac{7}{2}$, $B_4 = 7$, $z_1 = -1.997$, $z_2 = -1.253$, $B_0^1 = -1.995$, and $B_0^2 = -1.506$. Thus, the problem is converted to the following problem:

**Table 1.** Pareto optimal solutions for Example 1.

| | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{Z_1(x_{os})}$ | -2.691 | -0.301 | -2.691 | -2.377 | -1.664 | -2.313 | -2.007 | -0.913 | -0.448 | -2.118 | 0.257 |
| $\boldsymbol{Z_2(x_{os})}$ | -0.112 | -1.507 | -0.112 | -0.386 | -1.003 | -0.539 | -0.922 | -1.361 | -1.404 | -0.694 | -1.783 |
| $\boldsymbol{(x_{os})_1}$ | 0.004 | 0.26 | 0.004 | 0.077 | 0.239 | 0.133 | 0.253 | 0.285 | 0.237 | 0.171 | 0.299 |
| $\boldsymbol{(x_{os})_4}$ | 0.4 | 0.087 | 0.4 | 0.372 | 0.307 | 0.38 | 0.369 | 0.197 | 0.104 | 0.36 | 0.007 |

$$\begin{aligned}
\max \quad & \lambda, \\
\text{s.t.} \quad & \frac{5}{3}\left(\max_{j\in J}(a_{1j}+x_j)\right)+\lambda \leqslant \frac{7}{3}, \\
& 10\left(\max_{j\in J}(a_{2j}+x_j)\right)+\lambda \leqslant 8 \\
& 5\left(\max_{j\in J}(a_{3j}+x_j)\right)+\lambda \leqslant \frac{7}{2} \\
& 10\left(\max_{j\in J}(a_{4j}+x_j)\right)+\lambda \leqslant 7 \\
& \frac{3}{2}(2x_1+x_2-x_3-6x_4)+\lambda \leqslant -1.995 \\
& 2(-3x_1+x_2-3x_3+2x_4)+\lambda \leqslant -1.506 \\
& x\in[0,1]^4.
\end{aligned}$$

- **Step 5.** Since $2\in J''$, we have $x_2^*=0$ and then:

$$\begin{aligned}
\max \quad & \lambda, \\
\text{s.t.} \quad & \frac{5}{3}\left(\max_{j\in J^\cdot}(a_{1j}+x_j)\right)+\lambda \leqslant \frac{7}{3} \\
& 10\left(\max_{j\in J^\cdot}(a_{2j}+x_j)\right)+\lambda \leqslant 8 \\
& 5\left(\max_{j\in J^\cdot}(a_{3j}+x_j)\right)+\lambda \leqslant \frac{7}{2} \\
& 10\left(\max_{j\in J^\cdot}(a_{4j}+x_j)\right)+\lambda \leqslant 7 \\
& \frac{3}{2}(2x_1-x_3-6x_4)+\lambda \leqslant -1.995 \\
& 2(-3x_1-3x_3+2x_4)+\lambda \leqslant -1.506 \\
& x\in[0,1]^3.
\end{aligned}$$

- **Step 6.** We have $J_1^\cdot=\{1,3,4\}$, $J_2^\cdot=\emptyset$, $J_3^\cdot=\{3,4\}$, $J_4^\cdot=\{4\}$ and then:

$$\begin{aligned}
\max \quad & \lambda \\
\text{s.t.} \quad & \frac{5}{3}(\max((0.5+x_1),(0.3+x_3),(0.3+x_4))) \\
& \quad +\lambda \leqslant \frac{7}{3} \\
& 5(\max((0.7+x_3),(0.6+x_4)))+\lambda \leqslant \frac{7}{2} \\
& 10(\max(0.4+x_4))+\lambda \leqslant 7 \\
& \frac{3}{2}(2x_1-x_3-6x_4)+\lambda \leqslant -1.995 \\
& 2(-3x_1-3x_3+2x_4)+\lambda \leqslant -1.506 \\
& x\in[0,1]^3.
\end{aligned}$$

- **Step 7.** We have $I_1^\cdot=\{1\}$, $I_3^\cdot=\{1,3\}$, $I_4^\cdot=\{1,3,4\}$, and $\widehat{J}=\{1,3,4\}$. Thus:

$$\begin{aligned}
\max \quad & \lambda, \\
\text{s.t.} \quad & \frac{5}{3}\,(\max((0.5+x_1),(0.3+x_3),(0.3+x_4))) \\
& \quad +\lambda \leqslant \frac{7}{3} \\
& 5\,(\max((0.7+x_3),(0.6+x_4)))+\lambda \leqslant \frac{7}{2} \\
& 10\,(\max(0.4+x_4))+\lambda \leqslant 7 \\
& \frac{3}{2}(2x_1-x_3-6x_4)+\lambda \leqslant -1.995 \\
& 2(-3x_1-3x_3+2x_4)+\lambda \leqslant -1.506 \\
& x\in[0,1]^3.
\end{aligned}$$

- **Step 8.** We have $\epsilon=10^{-2}$, $P=10$, $K=1$, $\lambda_K=0.5$, $x_1^*=0.239$, $x_3^*=0.3$, and $x_4^*=0.307$.
- **Step 9.** This step includes six parts:
  - 9-1. We have $I_1=I_3=\{1\}$ and $I_4=\{1,4\}$;
  - 9-2. Since $I_j\neq\emptyset$ for all $j\in\widehat{J}$, we go to the next part;
  - 9-3. In this part, $I_j^\cdot$'s are derived as $I_1^\cdot=I_3^\cdot=\{1\}$, and $I_4^\cdot=\{4\}$;

9-4 Since $I_1^{\cdot} \bigcap I_3^{\cdot} \neq \emptyset$, the problem of Step 7 should be converted to the form Relation (18). We have $I_{\cdot} = \{1\}$, $J_1' = \{1,3\}$, $J^* = \{4\}$ and thus:

$$\begin{aligned} \max \quad & \lambda, \\ \text{s.t.} \quad & \frac{5}{3}(0.5 + x_1) + \lambda \leqslant \frac{7}{3} \\ & \frac{5}{3}(0.3 + x_3) + \lambda \leqslant \frac{7}{3} \\ & 10(0.4 + x_4) + \lambda \leqslant 7 \\ & \frac{3}{2}(2x_1 - x_3 - 6x_4) + \lambda \leqslant -1.995 \\ & 2(-3x_1 - 3x_3 + 2x_4) + \lambda \leqslant -1.506 \\ & x \in [0,1]^3. \end{aligned}$$

9-5. Solving the problem in step 9-4, we have $x_1 = 0.0$, $x_3 = 0.595$, $x_4 = 0.215$, and $\lambda = 0.841$;

9-6. Since $\lambda < 1 - \epsilon$ by considering $K = 2$ and $\lambda_K = 0.841$, we repeat Step 9.

In the second repetition of Step 9, we have:

9-1. $I_1 = I_3 = \{1\}$ and $I_4 = \{1,4\}$;

9-2. $I_j \neq \emptyset$ for all $j \in \widehat{J}$ and thus, we go to the next part;

9-3. We have $I_1^{\cdot} = I_3^{\cdot} = \{1\}$ and $I_4^{\cdot} = \{4\}$;

9-4. Since $I_{\cdot} = \{1\}$, $J_1' = \{1,3\}$ and $J^* = \{4\}$, the problem in this part is similar to the problem in step 9-4 in the previous repetition;

9-5. We have $x_1 = 0.0$, $x_3 = 0.595$, $x_4 = 0.215$, and $\lambda = 0.841$.

As it is seen that $\lambda_K = \lambda_{K-1}$ and hence, we should break this step.

- **Step 10.** We have $x_1^* = 0.0$, $x_3^* = 0.595$, $x_4^* = 0.215$, and $\lambda^* = 0.841$.
- **Step 11.** $Z_1(x^*) = -1.885$ and $Z_2(x^*) = -1.355$, which are the optimal values of the problem.
- **Step 12.** End.

Hence, in this example, the fuzzy solution is $(0, 0, 0.595, 0.215)$ and its values in objective functions are -1.885 and -1.355, respectively. If the DM is not satisfied with this fuzzy solution, he/she should accept more perturbation in constraints or choose another solution in Step 3 of Algorithm 1.

## 5. Conclusion

We have used Max-Arithmetic mean composition in a multi-objective optimization problem subject to a system of fuzzy relational inequalities in which ordinary inequalities have been replaced by fuzzy inequalities to benefit from the advantages of this composition and obtain more realistic solutions. Assigning linear membership functions to the inequalities and objective functions using one selected solution of the same multi-objective optimization problem with ordinary inequalities and employing Bellman-Zadeh decision, we have converted the multi-objective optimization problem in the presence of fuzzy inequalities in its constraints into a new simpler one in order to use infeasible points to obtain better solutions. Afterwards, we have diminished the dimension of the problem and proposed an algorithm to generate the optimal solution. If the algorithm yields a solution similar to the one which is obtained using only the feasible points, then the decision maker should accept more perturbation on the constraints. Also, in the case that the decision maker is not satisfied with the obtained solution by the algorithm, he/she should accept more perturbation on the constraints as well or choose another solution to the ordinary multi-objective optimization problem. This process should be continued until the desired solution of the decision maker is achieved. For future studies, it seems useful to employ other kinds of membership functions as they have been mentioned in Remark 1.

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