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Two-warehouse inventory model for deteriorating items with imperfect quality under the conditions of permissible delay in payments

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Two-warehouse;
Inspection;
Trade credit.

Abstract. Regularly, manufacturing systems produce perfect and imperfect quality items. The perfect items start deteriorating as soon as they enter inventory. On the other hand, the suppliers make a delay in payment in order to motivate their buyers to purchase more products. This paper develops a two-warehouse inventory model that jointly considers the imperfect quality items, deterioration, and one level of trade credit. The proposed inventory model optimizes the order quantity to maximize the total profit per unit time. Finally, the proposed inventory model and its solution procedure are validated with numerical examples and a sensitivity analysis is done to show how inventory model reacts to changes in parameters.

1. Introduction

Nowadays, production managers apply and implement efficacious manufacturing planning and complex production and control systems in order to have 100% quality items. However, the manufacturing process may still produce defective items. The defective items reduce the profit for the retailer and their inadvertent supply to customer may cost the retailer to lose goodwill. Thus, to sustain the supply of good quality items, it is vital for the whole lot to be screened as soon as it comes into the inventory and the defective items identified must be removed from the lot. With this in mind, a great amount of research has been made in the direction of the development EOQ/EPQ models for defective items. Porteus [1] found a significant relationship between the fraction of defective items and the lot. Rosenblatt and Lee [2] and Lee and Rosenblatt [3] showed through their papers that the presence of defective items forces the lot size to be smaller. Salameh and Jaber [4] expanded the research made for imperfect quality items considering random yield. They developed an EOQ model that refuted the results of Rosenblatt and Lee [2]. Salameh and Jaber [4] concluded that the batch quantity increased as the average percentage of imperfect quality items rose. Cárdenas-Barrón [5] corrected an error in the model of Salameh and Jaber [4] without affecting its main idea. Goyal and Cárdenas-Barrón [6] proposed an easy to apply method to determine the lot size in the model of Salameh and Jaber [4]. Papachristos and Konstantaras [7] examined the issue of non-shortages in the model of Salameh and Jaber [4] with proportional imperfect quality when the proportion of the imperfect was a random variable. Moussawi-Haidar et al. [8]

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studied the effect of deterioration on the instantaneous replenishment of the lot with imperfect quality items.

The items in inventory are under numerous risks like breakage, obsolescence, and evaporation. Another phenomenon that significantly affects items in inventory, e.g. gasoline, chemicals, and food items, which deteriorate fast through time, is deterioration. Thus, the loss by deterioration cannot be disregarded. Researchers have been progressively modifying the existing inventory models for deteriorating items in order to make them more realistic. Goyal and Giri [9] presented a complete review of research on deteriorating items published up to 2001. Bakker et al. [10] provided an extensive and comprehensive review on advancements in the field of inventory control of deteriorating items.

In order to avoid the losses related to deterioration, a retailer may rent another warehouse with better preserving facilities than those of his/her own warehouse. Moreover, the trade credit period provided by supplier motivates the retailer to purchase a quantity that exceeds the owned warehouse capacity and, consequently, the retailer needs to rent another warehouse for storage. The two-warehouse inventory system was first proposed by Hartley [11]. Das et al. [12] studied joint performance of a supply chain with two warehouse facilities. Hsieh et al. [13] proposed a deterministic inventory model for deteriorating items with two warehouses that minimized the net present value of the total cost. Several research papers in this interesting area were published by researchers in the last few decades [14-18].

The EOQ assumes that when the retailers receive the order, they immediately pay their suppliers. However, this is not always true. To facilitate the business for retailers, the suppliers sometimes make a delay in payment to retailers for a fixed time period to settle the payment of the order without charging any interest on the retailers during this credit period. But, an interest is charged at pre-determined rate if payment is not made by the end of the established credit period. Both the retailer and the supplier benefit from credit period. During the period before the account has to be paid, the buyers can vend products and accumulate revenue, and earn interest by placing revenue within an interest bearing account. The delay in payment period motivates the retailers to order more products and, thus, turns out to be beneficial for the supplier. It is important to remark that large orders increase holding cost and losses due to deterioration. Naturally, the retailer must consider all pros and cons while ordering into bulk in order to earn a maximum profit.

This paper proposes a two-warehouse inventory model that considers imperfect quality items under deteriorating conditions and permissible delay in payments. The lot is screened as soon as it enters the inventory system. The screening rate is assumed to be greater than the demand rate so that the demand can be satisfied along with the screening running in parallel, out of the items which are perfect in quality. Shortages are not allowed. The inventory model optimizes retailer's order quantity by maximizing his/her total profit.

2. Assumptions and notation

The mathematical model for the two-warehouse inventory problem is based on the following assumptions:

1. The Owned Warehouse (OW) has a fixed capacity of w units; the Rented Warehouse (RW) has unlimited capacity;
2. The initial inventory level and lead time are zero;
3. The deterioration rate of RW (β) is less than the deterioration rate of OW (α);
4. The screening process and demand occur simultaneously, but the screening rate (x) is greater than the demand rate (D), $x > D$;
5. The supplier provides a fixed credit period to the retailer to settle the account;
6. Shortages are not allowed;
7. The defective items exist in lot size y. The percentage of defective items is a random variable (p) with:

$$E(p) = \int_{l_1}^{l_2} pf(p)dp; \quad 0 < l_1 < l_2 < 1.$$

The following notations are used:

y	Order size per cycle (units)-decision variable
w	Storage capacity of OW (units)
D	Demand rate per unit time (unit/time unit)
p	Percentage of defective items in y (%)
$f(p)$	Probability function of p
t_s	Screening time of RW (time unit)
t_w	Screening time of OW (time unit)
t_r	Time point when the stock level of RW reaches zero (time unit)
T	The replenishment cycle (time unit)
M	The retailer's trade credit period provided by the supplier (time unit)
$I_o(t)$	Inventory level of OW at time t (units)
$I_r(t)$	Inventory level of RW at time t (units)
X	Screening rate (unit/time unit)
α	Deterioration rate of OW
β	Deterioration rate of RW
c	Unit purchasing cost per item ($/unit)

k	Fixed cost of placing an order (\$/order)
s	Unit selling price per item of good items (\$/unit)
v	Unit selling price per item of defective items (\$/unit)
d	Unit screening cost per item (\$/unit)
I_e	Interest earned (%/unit time)
I_p	Interest paid (%/unit time)
h_r	Holding cost per unit item per unit time in RW, excluding interest charges (\$/unit/time unit)
h_o	Holding cost per unit item per unit time in OW, excluding interest charges (\$/unit/time unit)
HC_r	Inventory holding cost of RW (\$/time unit)
HC_o	Inventory holding cost of OW (\$/time unit)
$\mathrm{TPU}(y)$	The total profit per unit time (\$/time unit)

3. Model formulation

We consider that a lot of size y enters the inventory system, out of which w units are placed in OW and $(y-w)$ units are stored in RW. The RW is considered to have better preserving facilities than OW and, hence, deterioration rate of RW (β) is less than the deterioration rate of OW (α). The holding cost at RW is greater than the holding cost at OW $(h_r > h_o)$. In any production process, due to certain reasons such as improper transport, low labor skills, low quality of raw materials, among others, the production process may shift to an imperfect production process in which not all the items manufactured are of good quality. Due to this, a screening process must be conducted at screening rate of (x) units per unit time when the whole lot enters the inventory. It is assumed that each received lot y contains p percent of defective items, where p is a random variable with a known probability density function, $f(p)$, and mean, $E(p) = p$. Thus, lot y has py defective items and $(1-p)y$ non-defective items. The defective items found are kept in stock and sold at the end of the screening period at a salvage value of (v) per unit, $v < c$. The screening process takes place in OW and RW, simultaneously, and gets completed at $t_w = w/x$ and $t_s = (y-w)/x$, respectively. Depending on the values of t_w, t_s, and t_r, the following cases are discussed.

Case I: When $t_w < t_r$

This case considers that the screening period of OW (t_w) is less than the time point (t_r) when the stock level of RW reaches zero. The inventory in RW depletes due to demand and deterioration from $[0, t_r]$ and reaches zero. The inventory in OW depletes only due to deterioration during $[0, t_r]$ and, then, by both demand and deterioration during $[t_r, T]$. The behavior of the inventory model through the whole cycle $[0, T]$ is shown graphically in Figures 1 and 2.

The differential equations that model the inventory level in both warehouses RW and OW at any time, t, over the period $(0, T)$ are:

$$\frac{dI_r(t)}{d(t)} + \beta I_r(t) = -D \qquad 0 \le t \le t_r, \tag{1}$$

$$\frac{dI_o(t)}{dt} = -\alpha I_o(t) \qquad 0 \le t \le t_r, \tag{2}$$

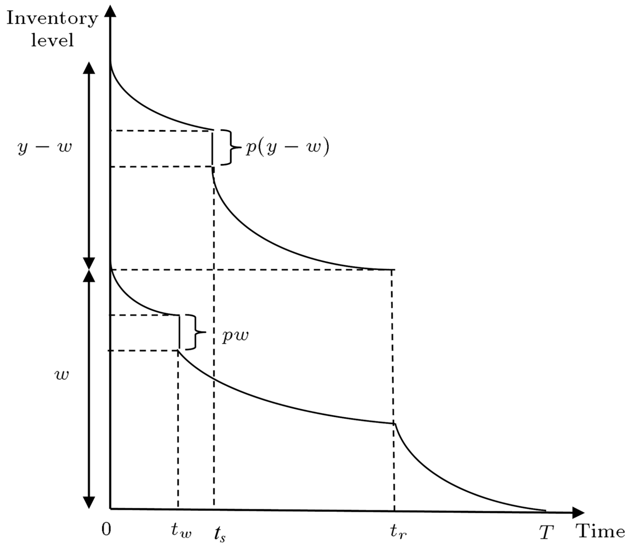

Figure 1. Inventory level for the two-warehouse system when $t_w < t_s < t_r$.

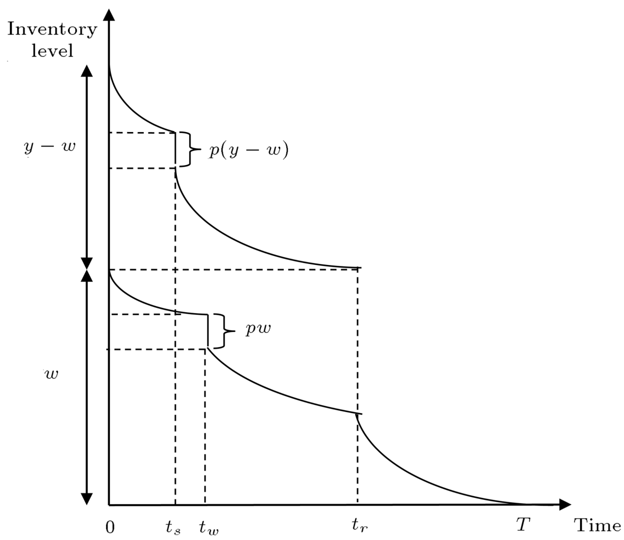

Figure 2. Inventory level for the two-warehouse system when $t_s < t_w < t_r$.

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -D \qquad t_r \leq t \leq T. \tag{3}$$

Solving the above differential equations with the boundary conditions $I_r(0) = y - w$, $I_r(t_s^+) = I_r(t_s) - p(y - w)$, $I_o(0) = w$, $I_o(t_w^+) = I_o(t_w) - pw$, and $I_o(T) = 0$, the solutions are:

$$I_r(t) = -\frac{D}{\beta} + \left(y - w + \frac{D}{\beta}\right) e^{-\beta t} \qquad 0 \leq t \leq t_s, \tag{4}$$

$$I_r(t) = -\frac{D}{\beta} + \left((y-w) + D/\beta - p(y-w)e^{\beta t_s}\right) e^{-\beta t}$$

$$t_s < t \leq t_r, \tag{5}$$

$$I_o(t) = we^{-\alpha t} \qquad 0 \leq t \leq t_w, \tag{6}$$

$$I_o(t) = \left\{w - pwe^{\alpha t_w}\right\} e^{-\alpha t} \qquad t_w < t \leq t_r, \tag{7}$$

$$I_o(t) = \frac{D}{\alpha}\left(e^{\alpha(T-t)} - 1\right) \qquad t_r \leq t \leq T. \tag{8}$$

Applying the boundary condition $I_r(t_r) = 0$, the value of t_r is:

$$t_r = \frac{1}{\beta}\left\{\ln\left(1 + \frac{\beta}{D}(y - w)\left(1 - pe^{\beta t_s}\right)\right)\right\}. \tag{9}$$

Considering the continuity of $I_o(t)$ at $t = t_r$, the value for T is:

$$\left(w - wpe^{\alpha t_w}\right) e^{-\alpha t_r} = \frac{D}{\alpha}\left\{e^{\alpha(T-t_r)} - 1\right\},$$

$$T = t_r + \frac{1}{\alpha}\ln\left[\frac{\alpha}{D}\left(w - pwe^{\alpha t_w}\right) e^{-\alpha t_r} + 1\right]. \tag{10}$$

Case II: When $t_w > t_r$

Figure 3 shows the behavior of the inventory model over the time interval $[0, T]$. It is assumed that the screening period of OW (t_w) is greater than the time point (t_r) when the stock level of RW reaches zero. The inventory in RW diminishes due to demand and deterioration from $[0, t_r]$ and reaches zero. The inventory in OW diminishes only due to deterioration during $[0, t_r]$ and, then, by both demand and deterioration during $[t_r, T]$.

Notice that the equations for the inventory level of RW are identical to those in Case I. Thus, the differential equations and their solutions for OW are:

$$\frac{dI_o(t)}{dt} = -\alpha I_o(t) \qquad 0 \leq t \leq t_r, \tag{11}$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -D \qquad t_r \leq t \leq T. \tag{12}$$

Solving the above differential equations with the boundary conditions, $I_o(0) = w$, $I_o(t_{w^+}) = I_o(t_w) - p_w$,

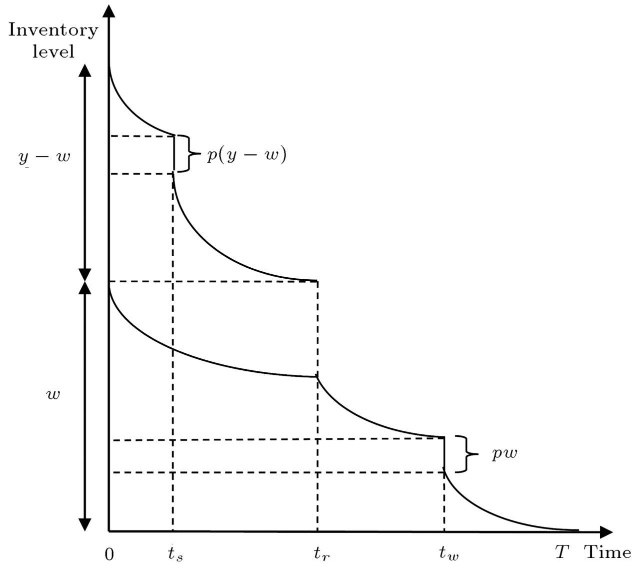

Figure 3. Inventory level for the two-warehouse system when $t_s < t_r < t_w$.

and continuity of $I_o(t)$ at $t = t_r$, the solutions are:

$$I_o(t) = we^{-\alpha t} \qquad 0 \leq t \leq t_r, \tag{13}$$

$$I_o(t) = \frac{-D}{\alpha} + \left(we^{-\alpha t_r} + \frac{D}{\alpha}\right) e^{-\alpha(t-t_r)}$$

$$t_r \leq t \leq t_w, \tag{14}$$

$$I_o(t) = \frac{-D}{\alpha} + \left(we^{-\alpha t_r} + \frac{D}{\alpha} - pwe^{\alpha(t_w-t_r)}\right) e^{-\alpha(t-t_r)}$$

$$t_w < t \leq T. \tag{15}$$

$I_o(T) = 0$ implies:

$$T = t_r + \frac{1}{\alpha}\ln\left[\frac{\alpha}{D}\left(w - pwe^{\alpha t_w}\right) e^{-\alpha t_r} + 1\right]. \tag{16}$$

It is easy to see that the total time cycle T is identical to that in Case I.

Now, let $N_r(y, p)$ and $N_o(y, p)$ be the totals of good items in each lot at time t with respect to RW and OW, respectively. These items are obtained by removing the defective and deteriorated items from the inventory. Let w_{1r} and w_{1o} be the total numbers of deteriorated items during time intervals $[0, t_r]$ and $[0, T]$ in RW and OW, respectively:

$$N_r(y, p) = (y - w)(1 - p) - w_{1r}, \tag{17}$$

$$N_o(y, p) = w(1 - p) - w_{1o}. \tag{18}$$

Let $I_{01r}(t)$ be the inventory level of RW at time t when both effects of lot quality and deterioration are ignored; thus, $I_{01r}(t) = -Dt + y - w$. Let $I_{02r}(t)$ be the inventory level of RW when only the effect of deterioration is

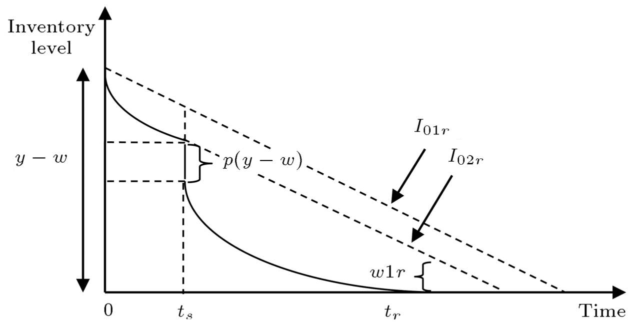

Figure 4. Inventory level for rented warehouse showing the deteriorated items at time t_r.

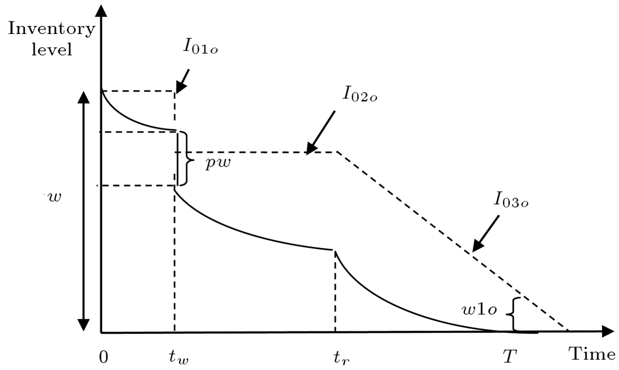

Figure 5. Inventory level for the owned warehouse showing the deteriorated items at time T.

ignored; hence, $I_{02r}(t) = -Dt + (y-w)(1-p)$. According to Figure 4, w_{1r} is:

$$w_{1r} = I_{02r}(t_r). \tag{19}$$

Let $I_{01o}(t)$ be the inventory level of OW at time t when both effects of lot quality and deterioration are ignored; thus, $I_{01o}(t) = w$. Let $I_{02o}(t)$ and $I_{03o}(t)$ be the inventory levels of OW when only the effect of deterioration is ignored; hence, $I_{02o}(t) = w - pw$ and $I_{03o}(t) = -D(t-t_r) + w - pw$. According to Figure 5, w_{1o} is:

$$w_{1o} = I_{03o}(T). \tag{20}$$

To avoid shortages, it is assumed that the totals of good items $N_r(y,p)$ and $N_o(y,p)$ are at least equal to the demands during screening times, i.e. t_s and t_w:

$$N_r(y,p) \geq Dt_s, \tag{21}$$

$$N_o(y,p) \geq Dt_w. \tag{22}$$

From Eqs. (21) and (22), the percent of defective items (p) must satisfy:

$$p \leq \min\left(\frac{-D}{\beta(y-w)} + \left(1 + \frac{D}{\beta(y-w)}\right) e^{-\beta t_s}, \left(1 - \frac{D}{w\alpha}\left(e^{\alpha t_w} - 1\right) a^{\alpha t_r}\right) e^{-\alpha t_w}\right). \tag{23}$$

The retailer's total profit during a cycle $TP_{i.j}(y)$, (where i = Case 1 - Case 3 and j = Sub-case 1 to Sub-case 5) is comprised of the following terms:

$$TP_{i.j}(y) = \text{Sales revenue} + \text{Interest earned} - \text{Ordering cost} - \text{Purchasing cost} - \text{Screening cost} - \text{Holding cost} - \text{Interest paid}.$$

Thus:

$$TP_{i.j}(y) = \text{Sales revenue} + \text{Interest earned} - k - cy - dy - \text{Holding cost} - \text{Interest paid}. \tag{24}$$

Sales revenue, TR, is the sum of the total sale volumes of good quality and imperfect quality items from RW and OW:

$$TR = s((y-w)(1-p) - w_{1r} + w(1-p) - w_{1o}) + v(y-w)p + vwp.$$

By replacing w_{1r} and w_{1o}, TR becomes:

$$TR = sDT + vyp. \tag{25}$$

The inventory holding cost per cycle in RW is:

$$\mathrm{HC}_r = h_r\left\{\int_0^{t_s} I_r(t)d(t) + \int_{t_s}^{t_r} I_r(t)d(t)\right\} = h_r\left\{\frac{y-w}{\beta}(1-p) - \frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\}. \tag{26}$$

The inventory holding cost per cycle in OW for the case with $t_w < t_r$ is:

$$\mathrm{HC}_o = h_o\left\{\int_0^{t_w} I_o(t)dt + \int_{t_w}^{t_r} I_o(t)dt + \int_{t_r}^{T} I_o(t)dt\right\} = h_o\left\{\frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w - wpe^{\alpha t_w}\right)e^{-\alpha t_r} + 1\right\}\right\}. \tag{27}$$

The inventory holding cost per cycle in OW for the case with $t_r < t_w$ is:

Table 1. Different sub-cases in each main case.

Case 1: $t_w < t_s < t_r$	**Case 2:** $t_s < t_w < t_r$	**Case 3:** $t_s < t_r < t_w$
Sub-case 1.1: $0 < M \le t_w < t_s < t_r$	Sub-case 2.1: $0 < M \le t_s < t_w < t_r$	Sub-case 3.1: $0 < M \le t_s < t_r < t_w$
Sub-case 1.2: $t_w < M \le t_s < t_r$	Sub-case 2.2: $t_s < M \le t_w < t_r$	Sub-case 3.2: $t_s < M \le t_r < t_w$
Sub-case 1.3: $t_w < t_s < M \le t_r$	Sub-case 2.3: $t_s < t_w < M \le t_r$	Sub-case 3.3: $t_s < t_r < M \le t_w$
Sub-case 1.4: $t_w < t_s < t_r < M \le T$	Sub-case 2.4: $t_s < t_w < t_r < M \le T$	Sub-case 3.4: $t_s < t_r < t_w < M \le T$
Sub-case 1.5: $t_w < t_s < t_r < T < M$	Sub-case 2.5: $t_s < t_w < t_r < T < M$	Sub-case 3.5: $t_s < t_r < t_w < T < M$

$$\mathrm{HC}_o = h_o\left\{\int_0^{t_r} I_o(t)dt + \int_{t_r}^{t_w} I_o(t)dt + \int_{t_w}^{T} I_o(t)dt\right\}$$

$$= h_o\left\{\frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w - wpe^{\alpha t_w}\right)e^{-\alpha t_r} + 1\right\}\right\}. \quad (28)$$

Hence, from Eqs. (27) and (28), we see that the holding costs in two cases with $t_w < t_r$ and $t_r < t_w$ are the same.

The interest earned, interest paid, and profit functions are calculated for different cases, which are shown in Table 1.

The solution procedure is as follows.

Case 1: $t_w < t_s < t_r$

Sub-case 1.1: $0 < M \le t_w < t_s < t_r$

The retailer earns interest on revenue generated from the sale of good quality items up to M. Although, the account must be paid at M and for that, the money has to be arranged at some specified rate of interest in order to get remaining stocks financed for the period M to T. The interest earned per cycle is equal to the area of triangle OAM in Figure 6. The interest earned is:

$$\frac{1}{2}(OM \times MA) = \frac{sI_eDM^2}{2}. \quad (29)$$

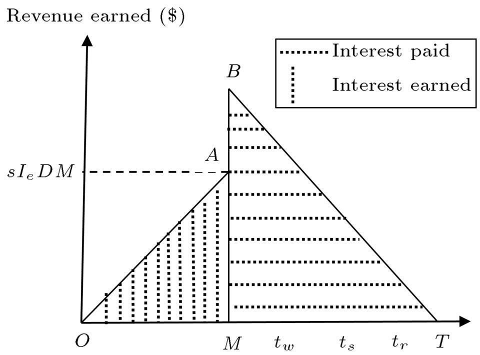

Figure 6. Graphical representation of interest earned and interest charged for $0 < M \le t_w < t_s < t_r$.

The interest payable per cycle for the inventory not sold after the due period M from RW and OW is given by:

Interest paid from RW =

$$cI_p\left\{\int_M^{t_s} I_r(t)d(t) + \int_{t_s}^{t_r} I_r(t)d(t)\right\}$$

$$= cI_p\left\{\frac{D}{\beta}(M - t_r) + \frac{1}{\beta}\left(y - w + \frac{D}{\beta}\right)\left(e^{-\beta M} - e^{-\beta t_r}\right) + \frac{p}{\beta}(y-w)e^{\beta t_s}\left(e^{-\beta t_r} - e^{-\beta t_s}\right)\right\}. \quad (30)$$

Interest paid from OW =

$$cI_p\left\{\int_M^{t_w} I_o(t)dt + \int_{t_w}^{t_r} I_o(t)dt + \int_{t_r}^{T} I_o(t)dt\right\}$$

$$= cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M} - p\right) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w - wpe^{\alpha t_w}\right)e^{-\alpha t_r} + 1\right\}\right\}. \quad (31)$$

Substituting the values of Eqs. (25)-(27) and (29)-(31) into Eq. (24), the total profit for Sub-case 1.1 is obtained by Eq. (32) as shown in Box I.

Sub-case 1.2: $t_w < M \le t_s < t_r$

The retailer receives interest on revenue created from the sale of good quality items up to M and also from the defective items sold as a single lot from OW for $t_w < t \le M$. The account has to be paid at M and, hence, the retailer must arrange the money at a rate of interest in order to obtain the remaining stock financed from M to T.

Therefore, the interest earned on good items per cycle is equal to the area of triangle OAM. Thus, the interest earned on good items is determined as $\frac{sI_eDM^2}{2}$. In addition, the retailer can earn interest on the sale

$$TP_{1.1}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}\right\}-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left(y-w+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right. \\ \left.+\frac{p}{\beta}(y-w)e^{\beta t_s}\left(e^{-\beta t_r}-e^{-\beta t_s}\right)\right\} \\ +cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M}-p\right)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \quad (32)$$

Box I

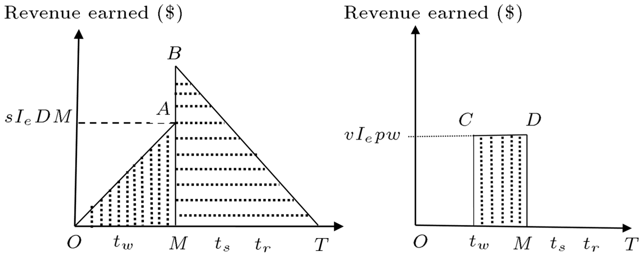

Figure 7. Graphical representation of interest earned and interest charged for $t_w < M \leq t_s < t_r$.

of defective items from OW, which is equal to the area of the rectangle $CDMt_w$, as shown in Figure 7. Therefore, the interest earned from the sale of defective items from OW is given by $vI_epw(M-t_w)$ hence:

The total interest earned =

$$\frac{sI_eDM^2}{2}+vI_epw(M-t_w). \quad (33)$$

The interest payable per cycle for the inventory not sold after the due period M from RW and OW is given by:

Interest paid from RW =

$$cI_p\left\{\int_M^{t_s} I_r(t)d(t)+\int_{t_s}^{t_r} I_r(t)d(t)\right\}$$

$$=cI_p\left\{\frac{D}{\beta}(M-t_r)\right.$$

$$+\frac{1}{\beta}\left(y-w+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)$$

$$\left.+\frac{p}{\beta}(y-w)e^{\beta t_s}\left(e^{-\beta t_r}-e^{-\beta t_s}\right)\right\}. \quad (34)$$

Interest paid from OW =

$$cI_p\left\{\int_M^{t_r} I_o(t)d(t)+\int_{t_r}^{T} I_o(t)d(t)\right\}$$

$$=cI_p\left\{\frac{w}{\alpha}\left(1-pe^{\alpha t_w}\right)e^{-\alpha M}\right.$$

$$\left.-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\}. \quad (35)$$

Substituting the values from Eqs. (25)-(27) and (33)-(35) in Eq. (24), the total profit for Sub-case 1.2 is obtained by Eq. (36) as shown in Box II.

Sub-case 1.3: $t_w < t_s < M \leq t_r$

In this sub-case, the retailer wins interest on revenue produced from the sale of good quality items up to M and, likewise, from the sale of defective items wholesaled as one lot from OW for $t_w < t \leq M$ and from RW for $t_s < t \leq M$. The retailer has to arrange the money at a specified rate of interest to settle the account at M in order to finance the remaining stocks for the period M to T.

Therefore, the interest earned on good items per cycle is determined with the area of triangle OAM. Thus, the interest earned on good items is given by $\frac{sI_eDM^2}{2}$. Furthermore, the retailer can gain interest on the sale of defective items from OW and RW, which is equal to the sum of the areas of rectangles $CDMt_w$ and $EFMt_s$ that are depicted in Figure 8. Consequently, the interest earned from the sale of defective items from OW is $vI_epw(M-t_w)$ and the interest earned from the sale of defective items from RW is $vI_ep(y-w)(M-t_s)$. Thus:

The total interest earned =

$$TP_{1.2}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left(y-w+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)+\frac{p}{\beta}(y-w)e^{\beta t_s}\left(e^{-\beta t_r}-e^{-\beta t_s}\right)\right\} \\ +cI_p\left\{\frac{w}{\alpha}\left(1-pe^{\alpha t_w}\right)e^{-\alpha M}-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \tag{36}$$

Box II

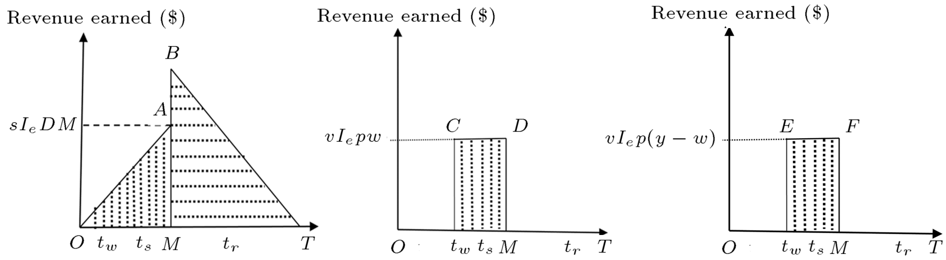

Figure 8. Graphical representation of interest earned and interest charged for $t_w < t_s < M \leq t_r$.

$$\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w). \tag{37}$$

The interest payable per cycle for the inventory not sold after the due period M from RW and OW is given by:

Interest paid from RW =

$$cI_p\left\{\int_M^{t_r} I_r(t)d(t)\right\}=cI_p\left\{\frac{D}{\beta}(M-t_r)\right.$$
$$\left.+\frac{1}{\beta}\left((y-w)(1-p)e^{\beta t_s}+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right\}. \tag{38}$$

Interest paid from OW =

$$cI_p\left\{\int_M^{t_r} I_o(t)d(t)+\int_{t_r}^{T} I_o(t)d(t)\right\}$$
$$=cI_p\left\{\frac{w}{\alpha}\left(1-pe^{\alpha t_w}\right)e^{-\alpha M}\right.$$
$$\left.-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\}. \tag{39}$$

Substituting the values given in Eqs. (25)-(27) and (37)-(39) into Eq. (24), the total profit for Sub-case 1.3 is obtained by Eq. (40) as shown in Box III.

Sub-case 1.4: $t_w < t_s < t_r < M \leq T$

The interest earned in this sub-case is calculated in the similar way to that in Sub-case 1.3. The interest earned on good items per cycle is obtained with the area of triangle OAM. Thus, the interest earned on good items is calculated with $\frac{sI_eDM^2}{2}$.

Additionally, the retailer obtains interest on the sale of defective items from both OW and RW, which is computed with the sum of the areas of rectangles $CDMt_w$ and $EFMt_s$ that are displayed in Figure 9. Hence, the interest earned from the sale of defective items from OW is determined as $vI_epw(M-t_w)$ and the interest earned from the sale of defective items from RW is obtained by $vI_ep(y-w)(M-t_s)$. Then:

The total interest earned =

$$\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w). \tag{41}$$

The interest paid in this case will result only from OW, since RW is exhausted, and it is given by:

Interest paid from OW =

$$TP_{1.3}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left((y-w)(1-p)e^{\beta t_s}+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right\} \\ +cI_p\left\{\frac{w}{\alpha}\left(1-pe^{\alpha t_w}\right)e^{-\alpha M}-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \quad (40)$$

Box III

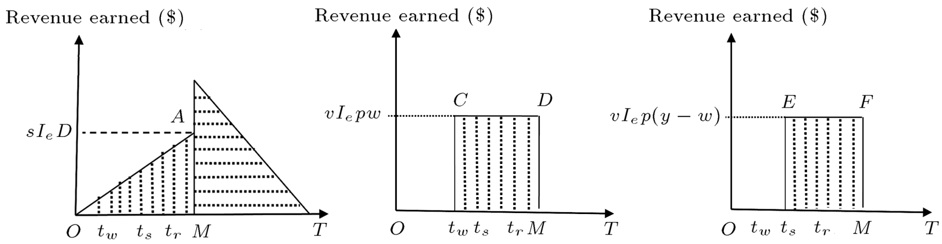

Figure 9. Graphical representation of interest earned and interest charged for $t_w < t_s < t_r < M \le T$.

$$cI_p\left\{\int_M^T I_{3o}(t)d(t)\right\}$$
$$=cI_p\left\{\frac{D}{\alpha^2}\left(e^{\alpha(T-M)}-1\right)-\frac{D}{\alpha}(T-M)\right\}. \quad (42)$$

Substituting Eqs. (25)-(27) and (40)-(42) into Eq. (24), the total profit for Sub-case 1.4 is obtained by Eq. (43) as shown in Box IV.

Sub-case 1.5: $t_w < t_s < t_r < T < M$

In this sub-case, the retailer obtains interest on revenue generated from the sale of good quality items up to T for time $T < t \le M$. Moreover, the retailer earns interest from the defective items vended as a sole lot from OW for $t_w < t \le M$ and from RW for $t_s < t \le M$.

The interest earned on good items per cycle is calculated with the area of triangle ABT. Also, the interest earned on good items for the period $[T, M]$ is computed with the area of the rectangle $CDMT$. Thus, the interest earned on good items is given by:

$$\frac{sI_eDT^2}{2}+sI_eDT(M-T).$$

Besides, the retailer can obtain interest on the sale of defective items from OW and RW, which is determined with the sum of the areas of rectangles $CDMt_w$ and $EFMt_s$ as exposed in Figure 10.

$$TP_{1.4}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\alpha^2}\left(e^{\alpha(T-M)}-1\right)-\frac{D}{\alpha}(T-M)\right\} \end{array}\right\}. \quad (43)$$

Box IV

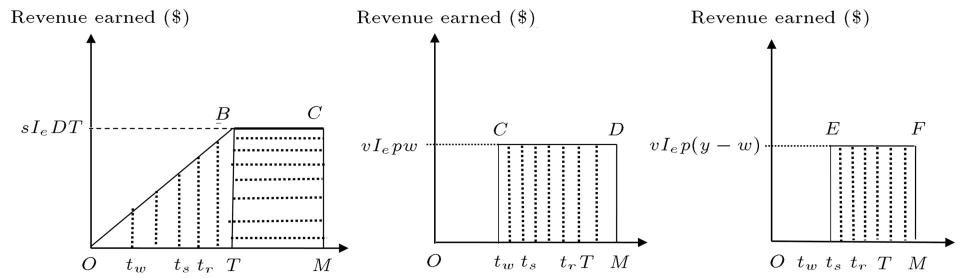

Figure 10. Graphical representation of interest earned and interest charged for $t_w < t_s < t_r < T < M$.

As a result, the interest earned from the sale of defective items from OW is calculated with $vI_epw(M-t_w)$ and the interest earned from the sale of defective items from RW is computed with $vI_ep(y-w)(M-t_s)$. Hence:

Total interest earned =

$$\frac{sI_eDT^2}{2} + sI_eDT(M-T) + vI_ep(y-w)(M-t_s) + vI_epw(M-t_w). \tag{44}$$

Since no inventory is left after time T, the interest paid is equal to zero.

Substituting the values from Eqs. (25)-(27) and (44) into Eq. (24), the total profit for Sub-case 1.5 is obtained by Eq. (45) as shown in Box V.

Case 2: $t_s < t_w < t_r$

***Sub-case 2.1:* $0 < M \le t_s < t_w < t_r$**

Here, the interest earned on good items is equal to the area of triangle OAM, which is shown in Figure 11. Thus:

The total interest earned =

$$\frac{sI_eDM^2}{2}. \tag{46}$$

The interest payable per cycle for the inventory not sold after the due period M from RW and OW is given by:

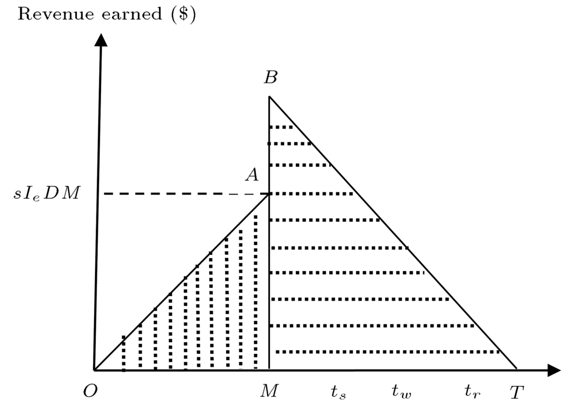

Figure 11. Graphical representation of interest earned and interest charged for $0 < M \le t_s < t_w < t_r$.

Interest paid from RW =

$$cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left(y-w+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right) + \frac{p}{\beta}(y-w)e^{\beta t_s}\left(e^{-\beta t_r}-e^{-\beta t_s}\right)\right\}. \tag{47}$$

Interest paid from OW =

$$cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M}-p\right) - \frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\}. \tag{48}$$

Substituting the values given in Eqs. (25)-(27)

$$TP_{1.5}(y)=\left\{sDT+vyp+\frac{sI_eDT^2}{2}+sI_eDT(M-T)+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \tag{45}$$

Box V

$$TP_{2.1}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left(y-w+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)+\frac{p}{\beta}(y-w)e^{\beta t_s}\left(e^{-\beta t_r}-e^{-\beta t_s}\right)\right\} \\ +cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M}-p\right)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \quad (49)$$

Box VI

and (46)-(48) into Eq. (24), the total profit for Sub-case 2.1 is obtained by Eq. (49) as shown in Box VI.

Sub-case 2.2: $t_s < M \leq t_w < t_r$

Here, the interest earned on good items per cycle is determined by the area of triangle OAM. Therefore, the interest earned on good items is $\frac{sI_eDM^2}{2}$. Furthermore, the retailer can gain interest on the sale of defective items from RW, which is given by the area of rectangle $CDMt_s$ as shown in Figure 12. Hence, the interest earned from the sale of defective items from RW is $vI_ep(y-w)(M-t_s)$. Thus:

Total interest earned =

$$\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s). \quad (50)$$

Similarly, the interest payable per cycle for the inventory not sold after the due period M from RW and OW is given by:

Interest paid from RW =

$$cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left((y-w)(1-p)e^{\beta t_s}+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right\}. \quad (51)$$

Interest paid from OW =%pagebreak[3]

$$cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M}-p\right)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\}. \quad (52)$$

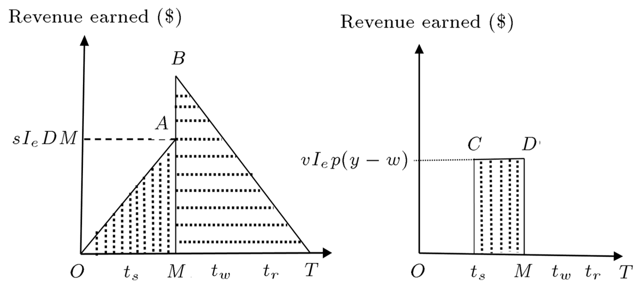

Figure 12. Graphical representation of interest earned and interest charged for $t_s < M \leq t_w < t_r$.

Substituting Eqs. (25)-(27) and (50)-(52) in Eq. (24), the total profit for Sub-case 2.2 is obtained by Eq. (53) as shown in Box VII.

Sub-case 2.3: $t_s < t_w < M \leq t_r$

Here, the interest earned on good items per cycle is computed by the area of triangle OAM. Thus, the interest earned on good items is $\frac{sI_eDM^2}{2}$. Besides, the retailer earns interest on the sale of defective items from RW and OW, which is equal to the sum of the areas of rectangles $EFMt_w$ and $CDMt_s$ as shown in Figure 13. Consequently, the interest earned from the sale of defective items from OW and RW is calculated with $vI_epw(M-t_w)$ and $vI_ep(y-w)(M-t_s)$, respectively. Thus:

The total interest earned =

$$\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w). \quad (54)$$

The interest payable per cycle for the inventory not sold after the due period M from RW and OW is given by:

Interest paid from RW =

$$TP_{2.2}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left((y-w)(1-p)e^{\beta t_s}+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right\} \\ +cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M}-p\right)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \quad (53)$$

Box VII

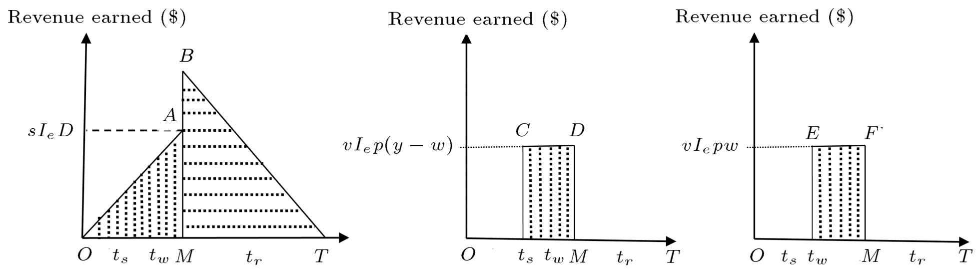

Figure 13. Graphical representation of interest earned and interest charged for $t_s < t_w < M \le t_r$.

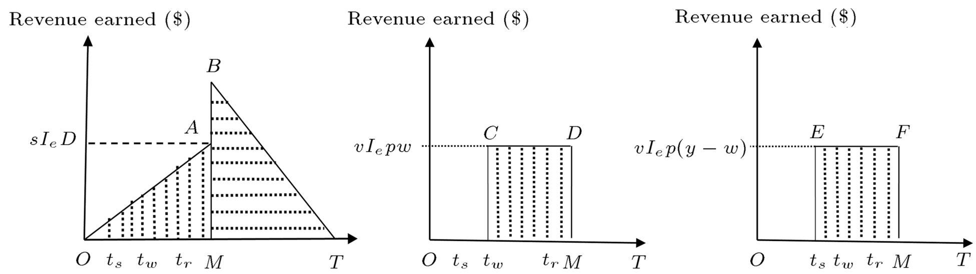

Figure 14. Graphical representation of interest earned and interest charged for $t_s < t_w < t_r < M \le T$.

$$cI_p\left\{\frac{D}{\beta}(M-t_r)\right.$$
$$\left.+\frac{1}{\beta}\left((y-w)(1-p)e^{\beta t_s}+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right\}. \quad (55)$$

Interest paid from OW =

$$cI_p\left\{\frac{w}{\alpha}\left(1-pe^{\alpha t_w}\right)e^{-\alpha M}\right.$$
$$\left.-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\}. \quad (56)$$

Substituting the values from Eqs. (25)-(27) and (54)-(56) in Eq. (24), the total profit for Sub-case 2.3 is obtained by Eq. (57) as shown in Box VIII.

Sub-case 2.4: $\boldsymbol{t_s < t_w < t_r < M \le T}$

The interest earned in this sub-case is calculated in an analogous manner like that in Sub-case 2.3, which is shown in Figure 14. Hence:

Total interest earned =

$$\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w). \quad (58)$$

Since RW is exhausted at t_r, the interest paid occurs by the inventory not sold from OW and is given as:

Interest paid from OW =

$$cI_p\left\{\frac{D}{\alpha^2}\left(e^{\alpha(T-M)}-1\right)-\frac{D}{\alpha}(T-M)\right\}. \quad (59)$$

$$TP_{2.3}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left((y-w)(1-p)e^{\beta t_s}+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right\} \\ +cI_p\left\{\frac{w}{\alpha}\left(1-pe^{\alpha t_w}\right)e^{-\alpha M}-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \tag{57}$$

Box VIII

$$TP_{2.4}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\alpha^2}\left(e^{\alpha(T-M)}-1\right)-\frac{D}{\alpha}(T-M)\right\} \end{array}\right\}. \tag{60}$$

Box IX

Substituting Eqs. (25)-(27) and (58)-(59) into Eq. (24), the total profit for Sub-case 2.4 is obtained by Eq. (60) as shown in Box IX.

Sub-case 2.5: $t_s < t_w < t_r < T < M$

Here, the interest obtained on good items per cycle is computed by the area of triangle OBT. Also, the interest gained on good items from the period $[T, M]$ is calculated by the area of the rectangle BCMT. Hence, the interest gained on good items is determined with:

$$\frac{sI_eDT^2}{2}+sI_eDT(M-T).$$

In addition, the retailer can get interest on the sale of defective items from OW and RW, which is equal to sum of the areas of rectangles $CDMt_w$ and $EFMt_s$ that are displayed in Figure 15. Therefore, the interest earned from the sale of defective items from OW is $vI_epw(M-t_w)$ and the interest earned from the sale of

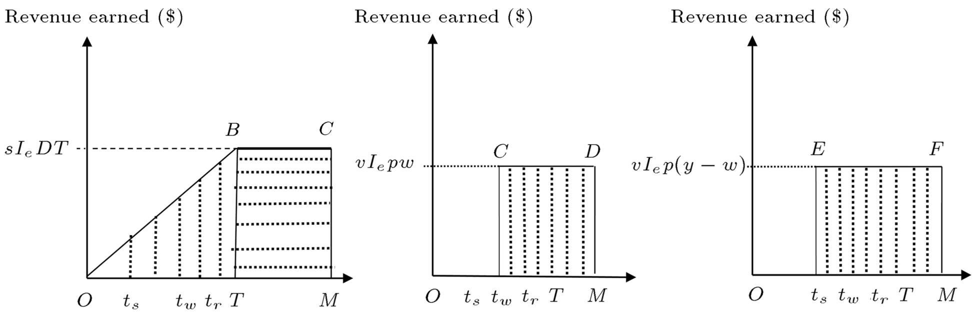

Figure 15. Graphical representation of interest earned and interest charged for $t_s < t_w < t_r < T < M$.

defective items from RW is $vI_e p(y-w)(M-t_s)$. Thus:

Total interest earned =

$$\frac{sI_eDT^2}{2}+sI_eDT(M-T)+vI_ep(y-w)(M-t_s)$$
$$+vI_epw(M-t_w). \tag{61}$$

As the inventory gets exhausted at time T, the interest paid is zero.

Substituting the values given in Eqs. (25)-(27) and (61) in Eq. (24), the total profit for Sub-case 2.5 is obtained by Eq. (62) as shown in Box X.

Case 3: $t_s < t_r < t_w$

***Sub-case 3.1:* $0 < M \leq t_s < t_r < t_w$**

The retailer gains interest on the revenue caused by the sale of good quality items up to M, which is presented in Figure 16. Although, the account must be paid at M and for that, the money must be arranged at a specified rate of interest in order to obtain a financing for the remaining stocks for the period M to T. Therefore:

The interest earned on good items =

$$\frac{sI_eDM^2}{2}. \tag{63}$$

Similarly, the interest payable per cycle for the inventory not sold after the due period M from RW and OW is given by:

Interest paid from RW =

$$cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left(y-w+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right.$$
$$\left.+\frac{p}{\beta}(y-w)e^{\beta t_s}\left(e^{-\beta t_r}-e^{-\beta t_s}\right)\right\}. \tag{64}$$

Interest paid from OW =

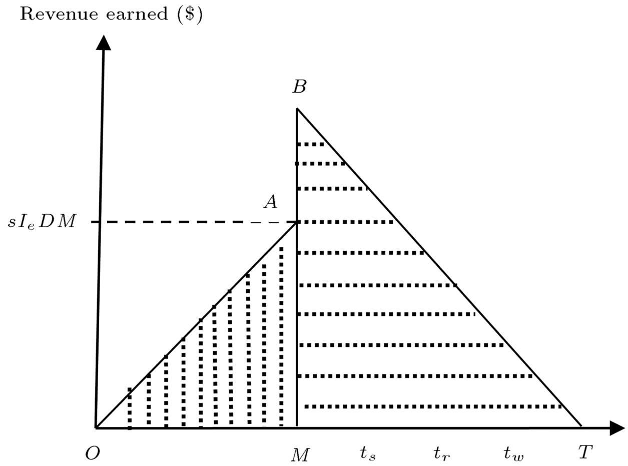

Figure 16. Graphical representation of interest earned and interest charged for $0 < M \leq t_s < t_r < t_w$.

$$cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M}-p\right)\right.$$
$$\left.-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\}. \tag{65}$$

Substituting the values from Eqs. (25)-(27) and (63)-(65) in Eq. (24), the total profit for Sub-case 3.1 is obtained by Eq. (66) as shown in Box XI.

***Sub-case 3.2:* $t_s < M \leq t_r < t_w$**

The interest gained on good items per cycle is obtained with the area of triangle OAM. Thus, the interest gained on good items is $\frac{sI_eDM^2}{2}$. Additionally, the retailer can win interest on the sale of defective items from RW, which is determined with the area of rectangle $CDMt_s$ that is illustrated in Figure 17.

Thus, the interest earned from the sale of defective items from RW is $vI_ep(y-w)(M-t_s)$.

Interest earned =

$$\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s). \tag{67}$$

Also, the interest payable per cycle for the inventory

$$TP_{2.5}(y)=\left\{sDT+vyp+\frac{sI_eDT^2}{2}+sI_eDT(M-T)+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \tag{62}$$

Box X

$$TP_{3.1}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left(y-w+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)+\frac{p}{\beta}(y-w)e^{\beta t_s}\left(e^{-\beta t_r}-e^{-\beta t_s}\right)\right\} \\ +cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M}-p\right)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \quad (66)$$

Box XI

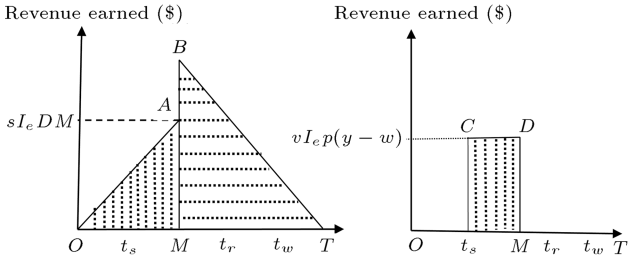

Figure 17. Graphical representation of interest earned and interest charged for $t_s < M \leq t_r < t_w$.

not sold after the due period M from RW and OW is given by:

Interest paid from RW =

$$cI_p\left\{\frac{D}{\beta}(M-t_r)+\frac{1}{\beta}\left((y-w)(1-p)e^{\beta t_s}+\frac{D}{\beta}\right)\left(e^{-\beta M}-e^{-\beta t_r}\right)\right\}. \quad (68)$$

Interest paid from OW =

$$cI_p\left\{\frac{w}{\alpha}\left(e^{-\alpha M}-p\right)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha w}{D}\left(1-pe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\}. \quad (69)$$

Substituting the values from Eqs. (25)-(27) and (67)-(69) in Eq. (24), the total profit for Sub-case 3.2 is obtained by Eq. (70) as shown in Box XII.

Sub-case 3.3: $t_s < t_r < M \leq t_w$

The interest gained in this sub-case is calculated in the same way as that in Sub-case 3.2 (see Figure 18). Thaus:

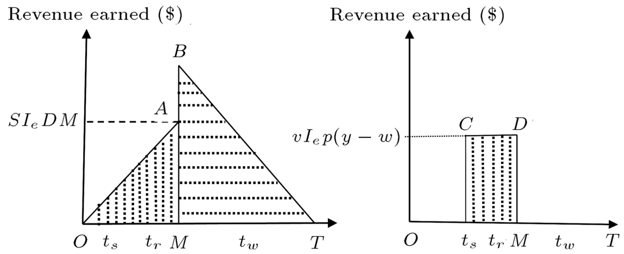

Figure 18. Graphical representation of interest earned and interest charged for $t_s < t_r < M \leq t_w$.

The total interest earned =

$$\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s). \quad (71)$$

The RW is exhausted at time t_r; therefore, the interest to be paid occurs only due to the inventory left from OW and is given by:

Interest paid from OW =

$$cI_p\left\{\frac{D}{\alpha}(M-T)+\frac{w}{\alpha}\left(e^{-\alpha M}-p\right)+\frac{D}{\alpha^2}\left(e^{-\alpha(M-t_r)}-1\right)\right\}. \quad (72)$$

Substituting the values from Eqs. (25)-(27) and (71)-(72) in Eq. (24), the total profit for Sub-case 3.3 is obtained by Eq. (73) as shown in Box XIII.

Sub-case 3.4: $t_s < t_r < t_w < M \leq T$

The interest earned on good items per cycle is calculated with the area of triangle OAM. As a result, the interest earned on good items is $\frac{sI_eDM^2}{2}$. As well, the retailer can get interest on the sale of defective items from RW and OW, which is determined with the sum of the areas of rectangles $CDMt_w$ and $EFMt_s$

$$TP_{3.2}(y) = \left\{ sDT + vyp + \frac{sI_eDM^2}{2} + vI_ep(y-w)(M-t_s) \right\}$$
$$- \left\{ \begin{array}{l} k + cy + dy + h_r \left\{ \frac{y-w}{\beta}(1-p) - \frac{D}{\beta^2} \left\{ \ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1 \right) \right\} \right\} \\ +h_o \left\{ \frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2} \ln\left\{ \frac{\alpha}{D}\left(w - wpe^{\alpha t_w}\right) e^{-\alpha t_r} + 1 \right\} \right\} \\ +cI_p \left\{ \frac{D}{\beta}(M-t_r) + \frac{1}{\beta}\left((y-w)(1-p)e^{\beta t_s} + \frac{D}{\beta} \right) \left(e^{-\beta M} - e^{-\beta t_r} \right) \right\} \\ +cI_p \left\{ \frac{w}{\alpha}\left(e^{-\alpha M} - p \right) - \frac{D}{\alpha^2} \ln\left\{ \frac{\alpha w}{D}\left(1 - pe^{\alpha t_w}\right) e^{-\alpha t_r} + 1 \right\} \right\} \end{array} \right\}. \quad (70)$$

Box XII

$$TP_{3.3}(y) = \left\{ sDT + vyp + \frac{sI_eDM^2}{2} + vI_ep(y-w)(M-t_s) \right\}$$
$$- \left\{ \begin{array}{l} k + cy + dy + h_r \left\{ \frac{y-w}{\beta}(1-p) - \frac{D}{\beta^2} \left\{ \ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1 \right) \right\} \right\} \\ +h_o \left\{ \frac{w}{\alpha}(1-p) - \frac{D}{\alpha^2} \ln\left\{ \frac{\alpha}{D}\left(w - wpe^{\alpha t_w}\right) e^{-\alpha t_r} + 1 \right\} \right\} \\ +cI_p \left\{ \frac{D}{\alpha}(M-T) + \frac{w}{\alpha}\left(e^{-\alpha M} - p \right) + \frac{D}{\alpha^2}\left(e^{-\alpha(M-t_r)} - 1 \right) \right\} \end{array} \right\}. \quad (73)$$

Box XIII

that are portrayed in Figure 19. Hence, the interest received from the sale of defective items from OW is $vI_epw(M-t_w)$ and the interest gotten from the sale of defective items from RW $= vI_ep(y-w)(M-t_s)$. As a result:

The interest earned =

$$\frac{sI_eDM^2}{2} + vI_ep(y-w)(M-t_s) + vI_epw(M-t_w). \quad (74)$$

The interest payable per cycle for the inventory not sold after the due period M from OW is given by:

Interest paid from OW =

$$cI_p \left\{ \frac{D}{\alpha}(M-T) + \frac{1}{\alpha}\left(we^{-\alpha t_r} + \frac{D}{\alpha} - pwe^{\alpha(t_w - t_r)} \right) \left(e^{-\alpha(M-t_r)} - e^{-\alpha(T-t_r)} \right) \right\}. \quad (75)$$

Substituting the values from Eqs. (25)-(27) and (74)-

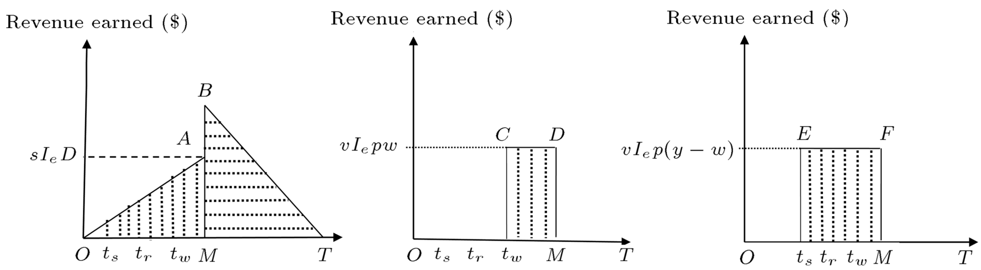

Figure 19. Graphical representation of interest earned and interest charged for $t_s < t_r < t_w < M \leq T$.

$$TP_{3.4}(y)=\left\{sDT+vyp+\frac{sI_eDM^2}{2}+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \\ +cI_p\left\{\frac{D}{\alpha}(M-T)+\frac{1}{\alpha}\left(we^{-\alpha t_r}+\frac{D}{\alpha}-pwe^{\alpha(t_w-t_r)}\right)\left(e^{-\alpha(M-t_r)}-e^{-\alpha(T-t_r)}\right)\right\} \end{array}\right\}. \qquad (76)$$

Box XIV

(75) in Eq. (24), the total profit for the Sub-case 3.4 is obtained by Eq. (76) as shown in Box XIV.

Sub-case 3.5: $t_s < t_r < t_w < T < M$

Here, the interest acquired on good items per cycle is obtained with the area of triangle OBT. Likewise, the interest gained on good items for the period $[T, M]$ is given by the area of rectangle $BCMT$. Consequently, the interest received on good items is:

$$\frac{sI_eDT^2}{2}+sI_eDT(M-T).$$

In addition, the retailer can gain interest on the sale of defective items from OW and RW, which is equal to the sum of the areas of rectangles $DCMt_w$ and $EFMt_s$ as shown in Figure 20. Therefore, the interest earned from the sale of defective items from OW is determined with $vI_epw(M-t_w)$ and the interest received from the sale of defective items from RW is obtained with $vI_ep(y-w)(M-t_s)$. Thus:

Total interest earned =
$$\frac{sI_eDT^2}{2}+sI_eDT(M-T)+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w). \qquad (77)$$

As the inventory gets exhausted at T, the interest paid is equal to zero.

Substituting the values from Eqs. (25)-(27) and (77) in Eq. (24), the total profit for Sub-case 3.5 is obtained by Eq. (78) as shown in Box XV. From all different equations of profit functions, it is found that:

From Eqs. (32), (50), and (68):

$TP_{1.1}(y) = TP_{2.1}(y) = TP_{3.1}(y)$,

From Eqs. (40) and (58):

$TP_{1.3}(y) = TP_{2.3}(y)$,

From Eqs. (43) and (61):

$TP_{1.4}(y) = TP_{2.4}(y)$,

From Eqs. (46), (64), and (81):

$TP_{1.5}(y) = TP_{2.5}(y) = TP_{3.5}(y)$,

From Eqs. (54) and (72):

$TP_{2.2}(y) = TP_{3.2}(y)$.

Hence, eight different cases exist for the retailer's profit per cycle, which can be expressed as:

$\mathrm{TPU}(y) =$

$$\mathrm{TPU}_1(y)=\mathrm{TPU}_{1.1}(y)=\mathrm{TPU}_{2.1}(y)=\mathrm{TPU}_{3.1}(y), \qquad (79a)$$

$$\mathrm{TPU}_2(y)=\mathrm{TPU}_{1.2}(y), \qquad (79b)$$

$$\mathrm{TPU}_3(y)=\mathrm{TPU}_{1.3}(y)=\mathrm{TPU}_{2.3}(y), \qquad (79c)$$

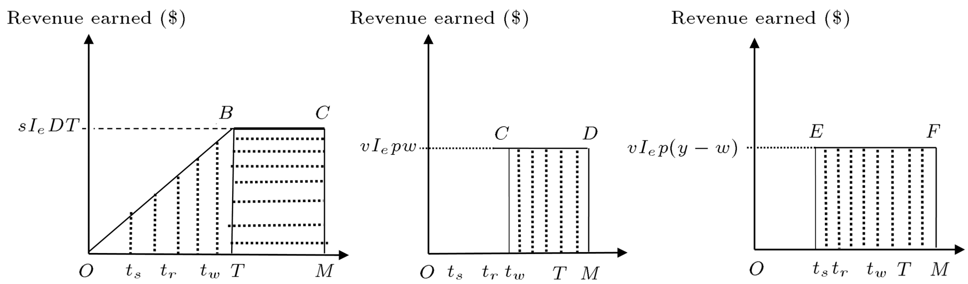

Figure 20. Graphical representation of interest earned and interest charged for $t_s < t_r < t_w < T < M$.

$$TP_{3.5}(y)=\left\{sDT+vyp+\frac{sI_eDT^2}{2}+sI_eDT(M-T)+vI_ep(y-w)(M-t_s)+vI_epw(M-t_w)\right\}$$
$$-\left\{\begin{array}{l} k+cy+dy+h_r\left\{\frac{y-w}{\beta}(1-p)-\frac{D}{\beta^2}\left\{\ln\left(\frac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\frac{w}{\alpha}(1-p)-\frac{D}{\alpha^2}\ln\left\{\frac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\}. \tag{78}$$

Box XV

$$\mathrm{TPU}_4(y)=\mathrm{TPU}_{1.4}(y)=\mathrm{TPU}_{2.4}(y), \tag{79d}$$

$$\mathrm{TPU}_5(y)=\mathrm{TPU}_{1.5}(y)=\mathrm{TPU}_{2.5}(y)=\mathrm{TPU}_{3.5}(y), \tag{79e}$$

$$\mathrm{TPU}_6(y)=\mathrm{TPU}_{2.2}(y)=\mathrm{TPU}_{3.2}(y), \tag{79f}$$

$$\mathrm{TPU}_7(y)=\mathrm{TPU}_{3.3}(y), \tag{79g}$$

$$\mathrm{TPU}_8(y)=\mathrm{TPU}_{3.4}(y). \tag{79h}$$

The main objective is to obtain the optimal value of y, which maximizes the total profit function $\mathrm{TPU}_i(y)$. In order to determine the optimal value of y, which maximizes the total profit per unit time, the necessary and sufficient conditions for optimality are:

$$\frac{d(\mathrm{TPU}_i(y))}{dy}=0, \tag{80}$$

and:

$$\frac{d^2(\mathrm{TPU}_i(y))}{dy^2}\le 0.$$

4. Special cases

a. When $M=0$, $I_e=0$, and $I_p=0$, i.e. there is no trade credit, the total profit per unit time becomes:

$$TP(y)=\frac{1}{T}\left[\begin{array}{l} sDT+vyp \\ -\left\{\begin{array}{l} k+cy+dy+h_r\left\{\dfrac{y-w}{\beta}(1-p)\right. \\ \left.-\dfrac{D}{\beta^2}\left\{\ln\left(\dfrac{\beta}{D}(y-w)\left(1-pe^{\beta t_s}\right)+1\right)\right\}\right\} \\ +h_o\left\{\dfrac{w}{\alpha}(1-p)\right. \\ \left.-\dfrac{D}{\alpha^2}\ln\left\{\dfrac{\alpha}{D}\left(w-wpe^{\alpha t_w}\right)e^{-\alpha t_r}+1\right\}\right\} \end{array}\right\} \end{array}\right],$$

which is the same profit function as that in the model of Jaggi et al. [19];

b. When $\alpha=\beta=0$, $M=0$, $I_e=0$, and $I_p=0$, i.e. there is no deterioration and no trade credit, the total profit per unit time becomes:

$$\mathrm{TPU}(y)=D\left(s-v+\frac{h_r(y-w)^2}{xy}+\frac{h_ow^2}{xy}\right)+\frac{D}{(1-p)}\left(v-\frac{k}{y}-c-d-\frac{h_r(y-w)^2}{xy}-\frac{h_ow^2}{xy}\right)-\frac{h_r(y-w)^2(1-p)}{2y}-h_o\left(w(1-p)-\frac{w^2(1-p)}{2y}\right),$$

which is the same profit function as that given by Chung et al. [18].

c. When $h_r=h_w$, $M=0$, $I_e=0$, and $I_p=0$, i.e. there is no trade credit and storage capacity of OW is unlimited, the total profit per unit time becomes:

$$\mathrm{TPU}(y)=sD+\frac{hD}{\alpha}+\frac{\alpha y\left(vp-c-d-\frac{h}{\alpha}(1-p)\right)-k\alpha}{\ln\left(y+\frac{D}{\alpha}-pye^{\alpha t_s}\right)-\ln\left(\frac{D}{\alpha}\right)},$$

which is the same profit function as that obtained by Moussawi-Haidar et al. [8].

d. When $h_r=h_w$, $\alpha=\beta=0$, $M=0$, $I_e=0$, and $I_p=0$, i.e. there is no deterioration and no trade credit, and storage capacity of OW is unlimited, the total profit per unit time becomes,

$$\mathrm{TPU}(y)=D\left(s-v+\frac{hy}{x}\right)+\frac{D}{(1-p)}\left(v-\frac{k}{y}-c-d-\frac{hy}{x}\right)-\frac{hy(1-p)}{2},$$

which is the same profit function as that obtained by Salameh and Jaber [4].

e. When $h_r=h_w$, $s=c$, $\alpha=0$, $\beta=0$, $p=0$, and storage capacity of OW is unlimited, the proposed model is same as that of Goyal [9].

5. Solution procedure

In order to find the optimal value of y^*, which maximizes the total profit function, the following algorithm is proposed:

Step 1. Determine $y^* = y_1$ from Eq. (80). Now, using the value of y_1, calculate the values of t_w, t_s, t_r, and T. If $0 < M \leq t_w < t_s < t_r$ or $0 < M \leq t_s < t_w < t_r$ or $0 < M \leq t_s < t_r < t_w$, then the optimal value of total profit is derived from Eq. (79a);

Step 2. Determine $y* = y_2$ from Eq. (80). Now, using the value of y_2, calculate the values of t_w, t_s, t_r, and T. If $t_w < M \leq t_s < t_r$, then the optimal value of total profit is obtained from Eq. (79b);

Step 3. Determine $y* = y_3$ from Eq. (80). Now, using the value of y_3, calculate the values of t_w, t_s, t_r, and T. If $t_w < t_s < M \leq t_r$ or $t_s < t_w < M \leq t_r$, then the optimal value of total profit is determined from Eq. (79c);

Step 4. Determine $y* = y_4$ from Eq. (80). Now, using the value of y_4, calculate the values of t_w, t_s, t_r, and T. If $t_w < t_s < t_r < M \leq T$ or $t_s < t_w < t_r < M \leq T$, then the optimal value of total profit is calculated from Eq. (79d);

Step 5. Determine $y* = y_5$ from Eq. (80). Now, using the value of y_5, calculate the values of t_w, t_s, t_r, and T. If $t_w < t_s < t_r < T < M$ or $t_s < t_w < t_r < T < M$ or $t_s < t_r < t_w < T < M$, then the optimal value of total profit is computed from Eq. (79e);

Step 6. Determine $y* = y_6$ from Eq. (80). Now, using the value of y_6, calculate the values of t_w, t_s, t_r, and T. If $t_s < M \leq t_w < t_r$ or $t_s < M \leq t_r < t_w$, then the optimal value of total profit is derived from Eq. (79f);

Step 7. Determine $y* = y_7$ from Eq. (80). Now, using the value of y_7, calculate the values of t_w, t_s, t_r, and T. If $t_s < t_r < M \leq t_w$, then the optimal value of total profit is obtained from Eq. (79g);

Step 8. Determine $y* = y_8$ from Eq. (80). Now, using the value of y_8, calculate the values of t_w, t_s, t_r, and T. If $t_s < t_r < t_w < M \leq T$, then the optimal value of total profit is calculated from Eq. (79h).

6. Numerical examples

This section presents three numerical examples in order to illustrate the proposed inventory model.

Example 1. The values of parameters are: $w = 500$ units (thus, $t_w = 0.008$ year), $D = 15000$ units/year, $\alpha = 20\%$, $\beta = 12.5\%$, $k = \$1000$/cycle, $h_r = \$7$/unit/year, $h_o = \$5$/unit/year, $x = 60000$ unit/year, $c = \$45$/unit, $s = \$70$/unit, $v = \$30$/unit, $d = \$1.0$/unit, $M = 20$ days, and the percentage of defective random variable p with p.d.f is:

$$f(p) = \begin{cases} 10 & 0 \leq p \leq 0.1 \\ 0 & \text{otherwise} \end{cases} \qquad E(p) = 0.05.$$

Two cases are considered:

(a) Let $I_e = 0.10$/year and $I_p = 0.12$/year ($sI_e > cI_p$). Results are obtained, using the proposed algorithm, as $y^* = 1311$. Substituting the optimal value of y^* in the expressions of t_s, t_r, T, and $\text{TPU}^*(y)$, we get $t_s^* = 0.0135$ year, $t_r^* = 0.051$ year, $T^* = 0.082$ year, and $\text{TPU}^*(y) = \$328198$;

(b) Let $I_e = 0.05$/year and $I_p = 0.08$/year ($sI_e < cI_p$). Results are obtained, using the proposed algorithm, as $y^* = 1408$. Substituting the optimal value of y^* in the expressions of t_s, t_r, T, and $\text{TPU}^*(y)$, we get $t_s^* = 0.0151$ year, $t_r^* = 0.057$ year, $T^* = 0.088$ year, and $\text{TPU}^*(y) = \$327362$.

Example 2. Here, the values of parameters are: $w = 800$ units (thus $t_w = 0.013$ year), $D = 15000$ units/year, $\alpha = 20\%$, $\beta = 12.5\%$, $k = \$1000$/cycle, $h_r = \$6$/unit/year, $h_o = \$6$/unit/year, $x = 60000$ unit/year, $c = \$35$/unit, $s = \$60$/unit, $v = \$25$/unit, $d = \$1.0$/unit, $M = 18$ days, and the percentage of defective random variable p with p.d.f is:

$$f(p) = \begin{cases} 10 & 0 \leq p \leq 0.1 \\ 0 & \text{otherwise} \end{cases} \qquad E(p) = 0.05.$$

Now, we consider two cases:

(a) Let $I_e = 0.08$/year and $I_p = 0.10$/year ($sI_e > cI_p$). Results are obtained, using the proposed algorithm, as $y^* = 1478$. Substituting the optimal value of y^* in the expressions of t_s, t_r, T, and $\text{TPU}^*(y)$, we get $t_s^* = 0.0113$ year, $t_r^* = 0.043$ year, $T^* = 0.093$ year, and $\text{TPU}^*(y) = \$331970$;

(b) Let $I_e = 0.04$/year and $I_p = 0.07$/year ($sI_e < cI_p$). Results are obtained, using the proposed algorithm, as $y^* = 1555$. Substituting the optimal value of y^* in the expressions of t_s, t_r, T, and $\text{TPU}^*(y)$, we get $t_s^* = 0.0126$ year, $t_r^* = 0.048$ year, $T^* = 0.098$ year, and $\text{TPU}^*(y) = \$331655$.

Example 3. Here, the values of parameters are: $w = 1200$ units (thus $t_w = 0.02$ year), $D = 15000$ units/year, $\alpha = 20\%$, $\beta = 12.5\%$, $k = \$1000$/cycle, $h_r = \$6$/unit/year, $h_o = \$6$/unit/year, $x = 60000$ unit/year, $c = \$35/unit$, $s = \$60$/unit, $v = \$25$/unit, $d = \$1.0$/unit, $M = 20$ days, and the percentage of defective random variable p with p.d.f is:

$$f(p) = \begin{cases} 10 & 0 \leq p \leq 0.1 \\ 0 & \text{otherwise} \end{cases} \qquad E(p) = 0.05.$$

Now, we consider two cases:

(a) Let $I_e = 0.10$/year and $I_p = 0.12$/year ($sI_e > cI_p$). Results are obtained, using the proposed algorithm, as $y^* = 1394$. Substituting the optimal value of y^* in the expressions of t_s, t_r, T, and $\mathrm{TPU}^*(y)$, we get $t_s^* = 0.0032$ year, $t_r^* = 0.012$ year, $T^* = 0.087$ year, and $\mathrm{TPU}^*(y) = \$332178$;

(b) Let $I_e = 0.05$/year and $I_p = 0.08$/year ($sI_e < cI_p$). Results are obtained, using the proposed algorithm, as $y^* = 1492$. Substituting the optimal value of y^* in the expressions of t_s, t_r, T, and $\mathrm{TPU}^*(y)$, we get $t_s^* = 0.0049$ year, $t_r^* = 0.018$ year, $T^* = 0.094$ year, and $\mathrm{TPU}^*(y) = \$331542$.

7. Sensitivity analysis

Sensitivity analysis was performed to study the effects of permissible delay (M), interest earned (I_e), interest paid (I_p), deterioration (α and β), percentage of defective items (p), and change in the capacity of OW (w) on the optimal lot size (y^*) and the total profit per unit time $\mathrm{TPU}^*(y)$. The observations are shown in Tables 2 to 5.

From Table 2, it is observed that when $sI_e > cI_p$, cycle lengths of RW and OW as well as the optimal order quantity decrease as the permissible delay period increases along with the increase in annual profit. This insinuates that trade credit turns beneficial for

Table 2. Effect of change in capacity of owned warehouse and trade credit period on the model ($sI_e > cI_p$).

W	M (days)	t_s	t_r	T	y^*	$\mathrm{TPU}^*(y)$	Case
400 ($t_w = 0.007$)	10	0.0166	0.063	0.088	1394	325628	$t_w < t_s < M < t_r < T$
	20	0.0152	0.058	0.082	1312	328272	$t_w < t_s < M < t_r < T$
	30	0.0147	0.056	0.081	1283	331110	$t_w < t_s < t_r < T < M$
900 ($t_w = 0.015$)	10	0.0070	0.027	0.083	1322	325280	$t_s < t_w < M < t_r < T$
	20	0.0068	0.026	0.082	1305	327897	$t_s < t_w < t_r < M < T$
	30	0.0063	0.024	0.080	1276	330737	$t_s < t_w < t_r < T < M$
1200 ($t_w = 0.020$)	10	0.0019	0.007	0.083	1315	325107	$t_s < t_r < t_w < M < T$
	20	0.0016	0.006	0.081	1298	327725	$t_s < t_r < t_w < M < T$
	30	0.0012	0.004	0.080	1270	330569	$t_s < t_r < t_w < T < M$

Table 3. Effect of change in capacity of owned warehouse and trade credit period on the model ($sI_e < cI_p$).

W	M (days)	t_s	t_r	T	y^*	$\mathrm{TPU}^*(y)$	Case
400 ($t_w = 0.007$)	10	0.0168	0.063	0.088	1406	325950	$t_w < t_s < M < t_r < T$
	20	0.0168	0.064	0.089	1409	327450	$t_w < t_s < t_r < M < T$
	30	0.0169	0.064	0.089	1412	328931	$t_w < t_s < t_r < T < M$
900 ($t_w = 0.015$)	10	0.0083	0.031	0.088	1398	325565	$t_s < t_w < M < t_r < T$
	20	0.0083	0.032	0.088	1400	327062	$t_s < t_w < t_r < M < T$
	30	0.0084	0.032	0.088	1403	328544	$t_s < t_w < t_r < T < M$
1200 ($t_w = 0.02$)	10	0.0032	0.012	0.087	1390	325391	$t_s < t_r < t_w < M < T$
	20	0.0032	0.012	0.087	1391	326887	$t_s < t_r < t_w < M < T$
	30	0.0032	0.012	0.088	1394	328368	$t_s < t_r < t_w < T < M$

Table 4. Effect of change of I_e and I_p on optimal replenishment policy ($M = 10$ days).

I_p	I_e = 0.03		0.05		0.07	
	y^*	$\mathrm{TPU}^*(y)$	y^*	$\mathrm{TPU}^*(y)$	y^*	$\mathrm{TPU}^*(y)$
0.1	928	338260	920	338382	898	338454
0.15	888	338056	880	338184	872	338312
0.2	854	337876	847	338008	840	338141

Table 5. Effect of change of percentage of defective items on the model ($w = 900$, $\alpha = 0.2$, $\beta = 0.125$, $I_e = 0.1$/year, $I_p = 0.12$/year).

Percentage of defective items p	t_s	t_w	t_r	T	y	$\mathrm{TPU}^*(y)$
0.025	0.0128	0.015	0.030	0.065	1666	562408
0.05	0.0133	0.015	0.030	0.064	1700	551454
0.075	0.0139	0.015	0.031	0.064	1734	539900

economic ordering policy. Thus, the retailer should procure less quantity and take the advantage of permissible delay in payments more often.

Now, from Table 3, it is observed that if $sI_e < cI_p$, then, as the permissible delay period increases, the optimal order quantity and, thus, the total annual profit increase. This implies that the retailer should procure more quantity to avoid higher interest charges on the inventory left after the credit period, which eventually results in higher profits.

Table 4 shows that as interest earned (I_e) by retailer increases, the optimal order quantity (y^*) decreases; but, expected profit increases, implying that when the interest earned per dollar is high, the expected total cost is low, which results in higher expected profit. Also, increase in interest paid (I_p) by retailer results in decrease in optimal order quantity, as well as the expected profit, because the expected total cost increases when the interest payable rate for the items stocked is high. Thus, a retailer should order less but more frequently when the interest payable rate per dollar is high.

Table 5 clearly shows that as the percentage of imperfect quality items (p) increases, the optimal order quantity (y^*) increases to meet the demand out of perfect quality items; but, the retailer's total profit $\mathrm{TPU}^*(y)$ decreases significantly. Thus, the retailer should be more vigilant when ordering and should carefully select the suppliers.

8. Conclusion

This paper amalgamates the concepts of two warehouses and the effect of deterioration on the retailer's lot when the items are of imperfect quality under the permissible delay of payments. The screening rate is assumed to be more than the demand rate so that the demand can be fulfilled out of the products that are found to be of perfect quality while the screening is in process. The numerical examples followed by the sensitivity analysis of various model parameters indicate that in case of highly deteriorating products, order should be made more frequently to reduce the losses due to deterioration. Also, as the defective items increase, the total profit decreases; in such a situation, the corrective measures need to be taken in order to procure good quality products. The results of sensitivity analysis also show that the presence of trade credit period is beneficent for retailer ordering policy. The retailer should order more to avoid higher interest charged after the grace period that eventually increases his/her total profit under the situation $sI_e < cI_p$; whereas, in other situation, i.e. $sI_e > cI_p$, the retailer should order less to avail the benefit of permissible delay more frequently. Also, as the rate of interest to be paid increases, the retailer should order less but more frequently. The proposed inventory model is a general framework as it includes numerous previous models.

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