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Group classification of the time-fractional Kaup-Kupershmidt equation

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Abstract. Finding the symmetries of the nonlinear fractional differential equations plays an important role in study of fractional differential equations. In this manuscript, firstly, we are interested in finding the Lie point symmetries of the time-fractional Kaup-Kupershmidt equation. Afterwards, by using the infinitesimal generators, we determine their corresponding invariant solutions.

1. Introduction

The method of group analysis of differential equations was introduced by Sophus Lie about one hundred years ago [1-3]. Lie group theory is an efficient method that we use for analysis of Partial Differential Equations (PDEs). Lie symmetries method is an effective method to solve the problems of mathematical physics.

The Fractional Differential Equations (FDEs) have been studied by scientists since about thirty years ago. Many phenomena in nature can be described using the FDEs. The fractional differential equations arise in many fields of sciences such as, electrochemistry, physics, biology, mechanics, signal processing, and viscoelastic materials [4-12].

Many articles have been presented to define fractional derivatives. The most important ones are the Caputo and the Riemann-Liouville derivatives. Each fractional derivative has some advantages and disadvantages. The Caputo derivative of a constant is zero, but Riemann-Liouville derivative of a constant is not. Many articles have been exhibited for finding the exact solutions of FDEs. There are many techniques and methods in these papers, which constitute the numerical and analytical solutions of FDEs. These methods include the fractional complex transform [13], the separating variables method [14], the variational iteration method [15], the first integral method [16], the homotopy analysis method [17], the homotopy perturbation Pade technique [18], the generalized differential transform method [19], the Hermit transform [20], etc. Many researchers obtained the exact solutions of many nonlinear PDEs using Lie group theory. But, the question may be asked here: Can we use this method for FDEs? Up to now, for FDEs, only a few works can be found in literature [21-27]. One of

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the difficulties of this type of problems originates in the non-local type of the fractional operators. Using the abovementioned research in our manuscript, we study the time-fractional Kaup-Kupershmidt equation, namely:

$$D_t^\alpha u - u_{5x} - 10uu_{3x} - 25u_x u_{2x} - 20u_x u^2 = 0,$$
$$t > 0, \quad x \in R, \tag{1}$$

where $0 < \alpha < 1$; u is a function of (x,t) and $u \in C^\infty(R^2)$. The Kaup-Kupershmidt equation plays an important role in the nonlinear dispersive wave. Solitary waves propagate in nonlinear dispersive media. These waves preserve a stable form due to dynamic balance between the dispersive and nonlinear influences. The exact solutions of this equation have been presented in many articles, such as [28,29].

The rest of our work is organized as follows. In Section 2, we present the analysis of the Lie Symmetry group of FDEs. Afterwards, in Section 3, we obtain the Lie point symmetries of the time-fractional Kaup-Kupershmidt equation. Finally, we obtain invariant solutions and reduced equations of this equation in Section 4. Discussion and conclusions are presented in Section 5.

2. Description of the symmetry group analysis of FDEs

Finding the exact solutions of the fractional differential equations is an important and difficult task. Therefore, much effort has been made to obtain the exact solutions of them. We recall that the symmetry is one of the most important concepts to study of the differential equations. Finding the exact solutions of differential equations using the fundamental method of the Lie symmetries has been used by many researchers. Invariance of the equations under transformation groups is the basic concept of the Lie theory. As it is known, there is the possibility of simplifying the differential equations if there are symmetries of the differential equations. We recall the works on this topic of Ovsiannikov [1], Olver [2], Ibragimov [3], Baumann [30], Bluman & Anco [31], and You & Zhang [32]. Now, we express the fractional Lie group method for finding infinitesimal functions of FPDEs. Let us assume an FPDE of the form:

$$D_t^\alpha u = F(x, t, u_{(1)}, \cdots), \qquad \alpha > 0, \tag{2}$$

where u is a function of independent variables; x, t, and D_t^α can be defined as follows.

Definition 1 [4,6]. D_t^α is the Riemann-Liouville fractional derivative operator defined by:

$$D_t^\alpha u = \begin{cases} \frac{\partial^m u}{\partial t^m}, \\ \quad \alpha = m \in N, \\ \\ \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t \frac{u(\tau,x)}{(t-\tau)^{\alpha+1-m}} d\tau; \\ \quad m-1 \leq \alpha < m, m \in N. \end{cases} \tag{3}$$

Similar to the discussion on PDEs [2,33], we can write:

$$D_{\bar{t}}^\alpha \bar{u} = D_t^\alpha u + \varepsilon \left[\eta_t^{(\alpha)}(x, t, u, u_{(\alpha)}, u_{(1)}, \cdots)\right] + O(\varepsilon^2), \tag{4}$$

here, $\eta_t^{(\alpha)}$ is given by the prolongation formula [22]:

$$\eta_t^{(\alpha)} = D_t^\alpha(\eta) + \xi^x D_t^\alpha(u_x) - D_t^\alpha(\xi^x u_x) + D_t^\alpha\left(D_t(\xi^t) u\right) - D_t^{\alpha+1}\left(\xi^t u\right) + \xi^t D_t^{\alpha+1} u, \tag{5}$$

where D_t is the total derivative operator defined as:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + u_{xxt} \frac{\partial}{\partial u_{xx}} + \cdots. \tag{6}$$

Simplifying Eq. (5) using the Leibnitz formula [34]:

$$D_t^\alpha[f(t)g(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} f(t) D_t^n g(t), \quad \alpha > 0, \tag{7}$$

where:

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)},$$
$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{8}$$

we can write [35]:

$$\eta_t^{(\alpha)} = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \left(\eta_u - \alpha D_t(\xi^t)\right) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n(\eta_u)}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\xi^t)\right] D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}(u_x) D_t^n(\xi^x). \tag{9}$$

We have a definition as follows.

Definition 2. The equations for finding coefficients of the infinitesimal operator X are given below:

$$X^{(\alpha)}\left[D_t^\alpha u - F(x,t,u,u_{(1)},\cdots)\right]_{D_t^\alpha u = F(x,t,u_{(1)},\cdots)} = 0, \tag{10}$$

where:

$$X^{(\alpha)} = \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x,t,u,u_{(1)})\frac{\partial}{\partial u_i} + \cdots + \eta^{(k)}_{i_1,i_2\cdots,i_k}(x,t,u,u_{(1)},\cdots,u_{(k)})\frac{\partial}{\partial u_{i_1,i_2\cdots,i_k}} + \eta_t^{(\alpha)}(x,t,u,\cdots,u_{(\alpha)},\cdots)\frac{\partial}{\partial u_t^{(\alpha)}}. \tag{11}$$

Expanding Eq. (10) using Eq. (11) and preceding relations, we obtain the determining equations. As a result, these obtained equations yield Lie symmetries.

3. Application of fractional Lie symmetries to the time-fractional Kaup-Kupershmidt equation

Here, we employ this method for the time-fractional Kaup-Kupershmidt equation:

$$D_t^\alpha u - u_{5x} - 10uu_{3x} - 25u_xu_{2x} - 20u_xu^2 = 0, \quad t>0, \quad 0<\alpha<1. \tag{12}$$

We search the infinitesimal generator of Eq. (12).

Theorem 1. Lie symmetries of the time fractional Kaup-Kupershmidt equation (Eq. (12)) are:

1. If $\alpha \neq \frac{1}{2}, \frac{4}{5}$, then we have:
   $$\xi^x = c_1\alpha x + c_2, \qquad \xi^t = 5c_1 t, \qquad \eta_u = -2c_1\alpha u,$$
   where c_1 and c_2 are two arbitrary constants. Therefore, the infinitesimal generators are given by:
   $$X_{1,1} = \frac{\partial}{\partial x}, \qquad X_{1,2} = \alpha x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u}.$$
2. If $\alpha = \frac{1}{2}$, then we have:
   $$\xi^x = c_1 x + c_2, \qquad \xi^t = 10c_1 t, \qquad \eta_u = -2c_1 u,$$
   where c_1 and c_2 are two arbitrary constants. Therefore, the infinitesimal generators are given by:
   $$X_{2,1} = \frac{\partial}{\partial x}, \qquad X_{2,2} = x\frac{\partial}{\partial x} + 10t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$
3. If $\alpha = \frac{4}{5}$, then we have:
   $$\xi^x = 4c_1 x + c_2, \qquad \xi^t = 25c_1 t, \qquad \eta_u = -8c_1 u,$$
   where c_1 and c_2 are three arbitrary constants. Therefore, the infinitesimal generators are given by:
   $$X_{3,1} = \frac{\partial}{\partial x}, \qquad X_{3,2} = 4x\frac{\partial}{\partial x} + 25t\frac{\partial}{\partial t} - 8u\frac{\partial}{\partial u}.$$

Proof. Let us assume the one-parameter Lie group of infinitesimal transformations in x, t, u given by:

$$x^* = x + \varepsilon\xi^x(x,t,u) + O(\varepsilon^2),$$
$$t^* = t + \varepsilon\xi^t(x,t,u) + O(\varepsilon^2),$$
$$u^* = u + \varepsilon\eta_u(x,t,u) + O(\varepsilon^2),$$

where ε is the group parameter, and the Lie algebra of Kaup-Kupershmidt equation is spanned by vector fields:

$$X = \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta_u(x,t,u)\frac{\partial}{\partial u}, \tag{13}$$

where:

$$\xi^x = \left.\frac{dx^*}{d\varepsilon}\right|_{\varepsilon=0}, \qquad \xi^t = \left.\frac{dt^*}{d\varepsilon}\right|_{\varepsilon=0}, \qquad \eta_u = \left.\frac{du^*}{d\varepsilon}\right|_{\varepsilon=0}. \tag{14}$$

Applying $X^{(\alpha)}$ to Eq. (12), we have:

$$X^{(\alpha)}\left[D_t^\alpha u - u_{5x} - 10uu_{3x} - 25u_xu_{2x} - 20u_xu^2\right]_{D_t^\alpha u - u_{5x} - 10uu_{3x} - 25u_xu_{2x} - 20u_xu^2 = 0} = 0, \tag{15}$$

where $X^{(\alpha)}$ is given by Eq. (11). Expanding Eq. (15), and solving the obtained system using the Maple, we obtain the Lie point symmetries for the time-fractional Kaup-Kupershmidt equation. If $\alpha \neq \frac{1}{2}, \frac{4}{5}$, then we have:

$$\xi^x = c_1\alpha x + c_2, \qquad \xi^t = 5c_1 t, \qquad \eta_u = -2c_1\alpha u.$$

Therefore, the infinitesimal generators are given by:

$$X_{1,1} = \frac{\partial}{\partial x}, \qquad X_{1,2} = \alpha x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u}.$$

We now apply this argument again, with $\alpha = \frac{1}{2}$, to obtain:

$$\xi^x = c_1 x + c_2, \qquad \xi^t = 10c_1 t, \qquad \eta_u = -2c_1 u.$$

Therefore, the infinitesimal generators are given by:

$$X_{2,1} = \frac{\partial}{\partial x}, \qquad X_{2,2} = x\frac{\partial}{\partial x} + 10t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

In the same manner, for $\alpha = \frac{4}{5}$, we can obtain:

$$\xi^x = 4c_1 x + c_2, \qquad \xi^t = 25c_1 t, \qquad \eta_u = -8c_1 u.$$

Therefore, the infinitesimal generators are given by:

$$X_{3,1} = \frac{\partial}{\partial x}, \qquad X_{3,2} = 4x\frac{\partial}{\partial x} + 25t\frac{\partial}{\partial t} - 8u\frac{\partial}{\partial u}.$$

The proof is completed.

4. Invariant solutions and the reduced equations of the time-fractional Kaup-Kupershmidt equation

The time-fractional Kaup-Kupershmidt equation is expressed by the coordinates (x,t,u); thus, we want to reduce it using new coordinates. By introducing invariants (r,z), we obtain the new coordinates corresponding to the infinitesimal symmetry generator and we can reduce the mentioned equation [36]. Consider a Lie point symmetry:

$$X=\xi^x(x,t,u)\frac{\partial}{\partial x}+\xi^t(x,t,u)\frac{\partial}{\partial t}+\eta_u(x,t,u)\frac{\partial}{\partial u},$$

of the time-fractional Kaup-Kupershmidt equation:

$$D_t^\alpha u-u_{5x}-10uu_{3x}-25u_xu_{2x}-20u_xu^2=0,$$

$$t>0,\quad 0<\alpha<1.$$

Under the one-parameter group generated by X, the invariant solutions are obtained as follows. Two linearly independent invariants $r=\varphi(x,t)$ and $z=\psi(x,t)$ can be calculated by solving the first-order quasi-linear PDE:

$$X(J)=\xi^x(x,t,u)\frac{\partial(J)}{\partial x}+\xi^t(x,t,u)\frac{\partial(J)}{\partial t}+\eta_u(x,t,u)\frac{\partial(J)}{\partial u}=0,$$

or its characteristic equations:

$$\frac{dx}{\xi^x(x,t,u)}=\frac{dt}{\xi^t(x,t,u)}=\frac{du}{\eta_u(x,t,u)}.$$

Then, we write one of the invariants as a function of the other, for example:

$$z=f(r),\tag{16}$$

and solve Eq. (16) for u. Finally, the expression of u is substituted in Eq. (12) and a fractional ODE is obtained for the unknown function f. With this procedure, we can reduce the number of independent variables by one. Now, we obtain the corresponding invariants and present the reduced nonlinear fractional ordinary differential equations. Finally, we obtain the corresponding group invariant solutions of the fractional Kaup-Kupershmidt equation as follows:

Case 1: $0<\alpha<1$, $\alpha\neq\frac{1}{2},\frac{4}{5}$, $X_{1,1}=\partial_x$.

In this case, the corresponding invariants are given by:

$$r=t,\qquad z=u.\tag{17}$$

A solution to our equation becomes:

$$z=f(r)\Rightarrow u=f(t).\tag{18}$$

We substitute Eq. (18) into Eq. (12) in order to determine $f(r)$. Then, $f(r)$ fulfils the following differential equation:

$$\frac{d^\alpha f(t)}{dt^\alpha}=0.\tag{19}$$

The solution of the Eq. (19), by using Laplace transform, is given by [6]:

$$f(t)=\frac{k}{\Gamma(\alpha)}t^{\alpha-1},\tag{20}$$

where k is a constant and $\Gamma(\alpha)$ is given by Eq. (8).

Case 2: $0<\alpha<1$, $\alpha\neq\frac{1}{2},\frac{4}{5}$, $X_{1,2}=\alpha x\frac{\partial}{\partial x}+5t\frac{\partial}{\partial t}-2\alpha u\frac{\partial}{\partial u}$.

For this case, the corresponding invariants are given below:

$$r=tx^{-\frac{5}{\alpha}},\qquad z=x^2u.\tag{21}$$

Then, a solution to our equation has the form:

$$z=f(r)\Rightarrow u=x^{-2}f(tx^{-\frac{5}{\alpha}}),\tag{22}$$

and we substitute it into Eq. (12) to determine $f(r)$. Thus, $f(r)$ is satisfied in the following equation:

$$\begin{aligned}&\alpha^5\frac{\partial^\alpha f}{\partial r^\alpha}+k_1f(r)+k_2f(r)^2+k_3f(r)^3+k_4rf'(r)\\&+k_5rf(r)f'(r)+k_6rf(r)^2f'(r)+k_7r^2f'(r)^2\\&+k_8r^2f''(r)+k_9r^2f(r)f''(r)+k_{10}r^3f'(r)f''(r)\\&+k_{11}r^3f^{(3)}(r)+k_{12}r^3f(r)f^{(3)}(r)+k_{13}r^4f^{(4)}(r)\\&+k_{14}r^5f^{(5)}(r)=0,\end{aligned}$$

where $k_i=h_i(\alpha)$ $(i=1,2,\ldots,14)$ are constants.

Case 3: $\alpha=\frac{1}{2}$, $X_{22}=x\frac{\partial}{\partial x}+10t\frac{\partial}{\partial t}-2u\frac{\partial}{\partial u}$.

The corresponding invariants for $\alpha=\frac{1}{2}$ and $X_{2,2}$ can be written as:

$$r=tx^{-10},\qquad z=x^2u.\tag{23}$$

As a result, we obtain:

$$z=f(r)\Rightarrow u=x^{-2}f(tx^{-10}),\tag{24}$$

by substituting Eq. (24) into Eq. (12), we are able to determine $f(r)$ as solution to the following differential equation:

$$\frac{1}{20}\frac{\partial^\alpha f}{\partial r^\alpha} + 5000r^5 f^{(5)}(r) + 60000r^4 f^{(4)}(r) + 1250r^3 f'(r)f''(r) + 500r^3 f(r)f^{(3)}(r) + 192750r^3 f^{(3)}(r) + 1875r^2 f'(r)^2 + 2200r^2 f(r)f''(r) + 171150r^2 f''(r) + 10rf(r)^2 f'(r) + 1530rf(r)f'(r) + 26172rf'(r) + 2f(r)^3 + 27f(r)^2 + 36f(r) = 0.$$

Case 4: $\alpha = \frac{4}{5}$, $X_{3,2} = 4x\frac{\partial}{\partial x} + 25t\frac{\partial}{\partial t} - 8u\frac{\partial}{\partial u}$.

The invariants in this case have the following forms:

$$r = tx^{-\frac{25}{4}}, \qquad z = x^2 u. \tag{25}$$

As a result, we obtain:

$$z = f(r) \Rightarrow u = x^{-2} f(tx^{-\frac{25}{4}}). \tag{26}$$

By substituting Eq. (26) into Eq. (12), we conclude that $f(r)$ has to satisfy the following differential equation:

$$\frac{1024}{5}\frac{\partial^\alpha f}{\partial r^\alpha} + 1953125r^5 f^{(5)}(r) + 25781250r^4 f^{(4)}(r) + 1250000r^3 f'(r)f''(r) + 500000r^3 f(r)f^{(3)}(r) + 94078125r^3 f^{(3)}(r) + 2250000r^2 f'(r)^2 + 2620000r^2 f(r)f''(r) + 100936875r^2 f''(r) + 25600rf(r)^2 f'(r) + 2464800rf(r)f'(r) + 21929445rf'(r) + 8192f(r)^3 + 110592f(r)^2 + 147456f(r) = 0.$$

5. Conclusion

In the present study, we investigated the efficiency of the classical Lie symmetry group analysis for the fractional differential equations. The fractional Lie symmetries method was use for application to the time-fractional Kaup-Kupershmidt equation with Riemann-Liouville derivative, and we found the Lie point symmetries group of this equation. As an application of the infinitesimal symmetries, we have shown that time-fractional Kaup-Kupershmidt equation can be obtained as a nonlinear ODE of fractional order. Finally, some group invariant solutions have been obtained in explicit form as well.

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Biographies

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