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Bayesian analysis of type-I right censored data using the 3-component mixture of Burr distributions

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Abstract. This study is concerned with the problem of estimating the parameters of a 3-component mixture of Burr distributions using type-I right censored data. The closed-form expressions for the Bayes estimators and their posterior risks assuming the non-informative (uniform and Jeffreys') priors under squared-error loss function, precautionary loss function, and DeGroot loss function are derived. Performance of the Bayes estimators for different sample sizes, test termination times (a point of time after which all other tests are terminated), and parametric values under different loss functions is investigated. The posterior predictive distribution for a future observation and the Bayesian predictive interval are constructed. In addition, the limiting expressions for the Bayes estimators and posterior risks are derived. Simulated data sets are designed for the comparisons and the model is finally illustrated using the real data.

1. Introduction

Finite mixtures of life distributions have proved to be of considerable interest in terms of their both methodological development and practical applications. Mixture models play a dynamic role in many real-life applications. Saleem [1] discussed that using finite mixture model became necessary when data from individual component densities or conditional distributions were not available, but were available from an overall mixture distribution. The direct applications of mixture models can be seen mostly in industrial engineering, medicine, botany, zoology, paleoanthropology, agriculture, economics, life testing, reliability and survival analysis, etc. A detailed account on type-I and type-II mixture models and their different features is given by Li [2] and Li and Sedransk [3]. As noted by Tahir and Aslam [4], the mixture of probability density functions from the same (different) family (families) is known as type-I (type-II) mixture model. In many applications, the available data can be considered as data from a mixture of two or more distributions. This idea enables us to mix statistical distributions to get a new distribution.

Several authors have extensively applied mixture modeling in different practical problems using classical and Bayesian analyses. For a detailed review of classical estimation techniques, discussion, and applications of mixture modeling, one can refer to [5-40].

Due to time and cost problems, it is sometimes impossible to continue the testing up to the last observation. The values which are greater than the fixed test termination time are taken as censored observations. Due to this limitation, studying censored data is inevitable in lifetime applications. For a detailed review of censoring, one may refer to Romeu [41], Gijbels [42], and Kalbfleisch and Prentice [43] and the references cited therein.

Being able to express a wide range of distribution

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shapes, Burr distribution is considered as a very flexible distribution. It can be fitted to empirical data of different nature. Different sets of its skewness and kurtosis can be covered by different parametric values. Household income, crop prices, insurance risk, travel time, flood levels, and failure data constitute a set of data modeled by the Burr distribution. In his work, Saleem [1] mentioned different cumulative distribution functions, suggested by Burr [44], with a broader range of values of skewness and kurtosis to model any observed data set from a unimodal distribution. Besides this, he discussed twelve forms for cumulative distribution function of the Burr distribution given in Johnson et al. [45]. Saleem [1] also noted that Burr [46,47], Burr and Cislak [48], and Rodriguez [49] gave special attention to one of these forms. A useful discussion on Burr and related distribution is presented by Tadikamalla [50]. Using graphical test, Economou and Caroni [51] explained the utility of Burr distribution. Further enhancing the work on Burr distribution, Saleem [1] presented a 2-component mixture of one-parameter Burr type-XII distribution assuming different priors under squared error loss function.

Motivated by wide application of mixture modeling, in this article, we plan to develop a mixture of Burr distributions for efficient modeling of a given lifetime data. A random variable Y is said to follow a finite mixture distribution with h components if the density function of Y can be written in the form: $f(y) = \sum_{m=1}^{h} p_m f_m(y)$, where $p_m (m = 1, 2, ..., h)$ is the mth mixing proportion such that $p_h = 1 - \sum_{m=1}^{h-1} p_m$, i.e. $\sum_{m=1}^{h} p_m = 1$, and $f_m(y)$ is the mth component density function. Under this definition, a finite 3-component mixture of Burr distributions with mixing proportions p_1 and p_2 has the probability density function (pdf), cumulative distribution function (cdf), and survival function as:

$$f(y;\mathbf{\Psi}) = p_1 f_1(y;\Psi_1) + p_2 f_2(y;\Psi_2) + (1-p_1-p_2) f_3(y;\Psi_3), \quad p_1, p_2 \geq 0,\ p_1 + p_2 \leq 1, \tag{1}$$

$$F(y;\mathbf{\Psi}) = p_1 F_1(y;\Psi_1) + p_2 F_2(y;\Psi_2) + (1-p_1-p_2) F_3(y;\Psi_3), \tag{2}$$

$$S(y;\mathbf{\Psi}) = p_1 S_1(y;\Psi_1) + p_2 S_2(y,\Psi_2) + (1-p_1-p_2) S_3(y,\Psi_3), \tag{3}$$

where, $\mathbf{\Psi} = (\lambda_1, \lambda_2, \lambda_3, p_1, p_2)$, $\Psi_m = \lambda_m$, $m = 1, 2, 3$, and:

$$f_m(y;\Psi_m) = \lambda_y (1+y)^{-(\lambda_m+1)}, \quad 0 < y < \infty,\ \lambda_m > 0,\ m = 1, 2, 3. \tag{4}$$

The cdf, $F_m(y;\Psi_m)$, of the mth component density is given by:

$$F_m(y;\Psi_m) = 1 - (1+y)^{-\lambda_m}, \quad 0 < y < \infty, \quad \lambda_m > 0,\ m = 1, 2, 3, \tag{5}$$

and $S_m(y;\Psi_m)$, the survival function of the mth component, is written as:

$$S_m(y;\Psi_m) = 1 - F_m(y;\Psi_m) = (1+y)^{-\lambda_m}, \quad 0 < y < \infty,\ \lambda_m > 0,\ m = 1, 2, 3. \tag{6}$$

The rest of the article is organized as follows. The sampling scheme for a 3-component mixture of Burr distributions is defined in Section 2. The expressions for likelihood function and posterior distributions using the non-informative priors are derived in Sections 3 and 4, respectively. In Section 5, the Bayes estimators and posterior risks using the uniform and the Jeffreys' priors under squared error loss function, precautionary loss function, and DeGroot loss function are derived. The posterior predictive distribution and the Bayesian predictive intervals are given in Section 6. The limiting expressions of the Bayes estimators and their posterior risks are derived in Section 7. The simulation study and the real-life data application are presented in Sections 8 and 9, respectively. Finally, the conclusion of this study is given in Section 10.

2. Sampling scheme for a 3-component mixture of Burr distributions

As defined in Tahir and Aslam [4,52], suppose n units from the 3-component mixture of the Burr distributions, defined in Section 1, are used in a life testing experiment with fixed test termination time t. The experiment is performed and it is observed that r out of n units fail until fixed test termination time t is over. The remaining $n - r$ units are still functioning. As defined by Mendenhall and Hader [5], there are many practical situations where only failed objects can be easily recognized as subsets of either subpopulation I or subpopulation II or subpopulation III. For example, based on cause of failure, an engineer may divide a certain failed object as a member of either subpopulation I or subpopulation II or subpopulation III. It may be pointed out that out of r failures, r_1, r_2, and r_3 failures belong to subpopulation I, subpopulation II, and subpopulation III, respectively, depending upon the reason of failure. Thus, the number of uncensored observations is $r = r_1 + r_2 + r_3$, and the remaining $n - r$ observations are censored observations. Define y_{lk}, $0 < y_{lk} \leq t$, to be the failure time of the kth unit belonging to the lth subpopulation, where $l = 1, 2, 3$ and $k = 1, 2, ..., r_l$.

3. The likelihood function

For a 3-component mixture of the Burr distributions, the likelihood function for the data collected through sampling procedure explained in section 2 can be written as:

$$L(\boldsymbol{\Psi}|\mathbf{y})\propto \{\boldsymbol{\Pi}_{k=1}^{r_1} p_1 f_1(y_{1k})\}\{\boldsymbol{\Pi}_{k=1}^{r_2} p_2 f_2(y_{2k})\}$$

$$\{\boldsymbol{\Pi}_{k=1}^{r_3}(1-p_1-p_2)f_3(y_{3k})\}\{S(t)\}^{n-r}. \quad (7)$$

On simplification, the likelihood function of the 3-component mixture of the Burr distribution becomes:

$$L(\boldsymbol{\Psi}|\mathbf{y})\propto\Bigg[\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$\exp\Bigg\{-\lambda_1\Bigg((n-r-i)\ln(1+t)$$

$$+\sum_{k=1}^{r_1}\ln(1+y_{1k})\Bigg)\Bigg\}$$

$$\exp\Bigg\{-\lambda_2\Bigg((i-j)\ln(1+t)+\sum_{k=1}^{r_2}\ln(1+y_{2k})\Bigg)\Bigg\}$$

$$\exp\Bigg\{-\lambda_3\Bigg((j)\ln(1+t)+\sum_{k=1}^{r_3}\ln(1+y_{3k})\Bigg)\Bigg\}$$

$$\lambda_1^{r_1}\lambda_2^{r_2}\lambda_3^{r_3}p_1^{n-r-i+r_1}p_2^{i-j+r_2}(1-p_1-p_2)^{j+r_3}\Bigg], \quad (8)$$

where, $\mathbf{y}=(y_{11},y_{12},...,y_{1r_1},y_{21},y_{22},...,y_{2r_2},y_{31},y_{32},...,y_{3r_3})$ is the set of observed failure times for the uncensored observations and $\boldsymbol{\Psi}=(\lambda_1,\lambda_2,\lambda_3,p_1,p_2)$.

4. Posterior distributions assuming the non-informative priors

When no or little prior information is available, the Uniform Prior (UP) and the Jeffreys' Prior (JP) are the most commonly used priors for Bayesian estimation. In this section, the posterior distributions of parameters given in data $\mathbf{y}$ are derived assuming the UP and the JP.

4.1. Posterior distribution assuming the uniform prior

Bayes [53] and Geisser [54] proposed that one might consider the uniform distribution for the unknown parameters of interest (see Tahir et al. [4] and Tahir and Aslam [52]). We assume the improper UP for the unknown component parameter λ_m, i.e. $\lambda_m \sim$ Uniform$(0,\infty)$, $m = 1,2,3$, and the UP over the interval (0,1) for the unknown proportion parameter p_s, i.e. $p_s \sim$ Uniform$(0,1)$, $s = 1,2$. Assuming the independence of parameters, the joint prior distribution of parameters $\lambda_1,\lambda_2,\lambda_3,p_1$, and p_2 is denoted by $\pi_1(\boldsymbol{\Psi}) \propto 1$ (see [4,52]). Thus, the joint posterior distribution of parameters $\lambda_1,\lambda_2,\lambda_3,p_1$, and p_2 for the given data $\mathbf{y}$, assuming the UP, is defined by Eqs. (9) and (10) as shown in Box I, where:

$$A_{11}=r_1+1, \quad A_{21}=r_2+1 \quad A_{31}=r_3+1,$$

$$B_{11}=(n-r-i)\ln(1+t)+\sum_{k=1}^{r_1}\ln(1+y_{1k}),$$

$$B_{21}=(i-j)\ln(1+t)+\sum_{k=1}^{r_2}\ln(1+y_{2k}),$$

$$B_{31}=(j)\ln(1+t)+\sum_{k=1}^{r_3}\ln(1+y_{3k}),$$

$$A_{01}=n-r-i+r_1+1, \quad B_{01}=i-j+r_2+1,$$

$$C_{01}=j+r_3+1,$$

$$\Omega_1=\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$B(A_{01},B_{01},C_{01})B_{11}^{-A_{11}}B_{21}^{-A_{21}}B_{31}^{-A_{31}}.$$

The marginal posterior distributions of parameters $\lambda_1,\lambda_2,\lambda_2,p_1$, and p_2 using the UP are derived as:

$$q_1(\boldsymbol{\Psi}|\mathbf{y})=\frac{L(\boldsymbol{\Psi}|\mathbf{y})\pi_1(\boldsymbol{\Psi})}{\int_{\boldsymbol{\Psi}}L(\boldsymbol{\Psi}|\mathbf{y})\pi_1(\boldsymbol{\Psi})d\boldsymbol{\Psi}}, \quad (9)$$

$$q_1(\boldsymbol{\Psi}|\mathbf{y})=\frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp(-B_{11}\lambda_1)\exp(-B_{21}\lambda_2)\exp(-B_{31}\lambda_3)p_1^{A_{01}-1}p_2^{B_{01}-1}(1-p_1-p_2)^{C_{01}-1}}{\Omega_1\lambda_1^{1-A_{11}}\lambda_2^{1-A_{21}}\lambda_3^{1-A_{31}}}. \quad (10)$$

Box I

$$g_1(\lambda_1|\mathbf{y}) = \frac{\Gamma(A_{21})\Gamma(A_{31})}{\Omega_1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{21}^{-A_{21}} B_{31}^{-A_{31}} B(A_{01}, C_{01}) B(B_{01}, A_{01} + C_{01}) \lambda_1^{A_{11}-1} \exp(-B_{11}\lambda_1), \quad \lambda_1 > 0, \tag{11}$$

$$g_1(\lambda_2|\mathbf{y}) = \frac{\Gamma(A_{11})\Gamma(A_{31})}{\Omega_1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{11}^{-A_{11}} B_{31}^{-A_{31}} B(A_{01}, C_{01}) B(B_{01}, A_{01} + C_{01}) \lambda_2^{A_{21}-1} \exp(-B_{21}\lambda_2), \quad \lambda_2 > 0, \tag{12}$$

$$g_1(\lambda_3|\mathbf{y}) = \frac{\Gamma(A_{21})\Gamma(A_{31})}{\Omega_1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{11}^{-A_{11}} B_{21}^{-A_{21}} B(A_{01}, C_{01}) B(B_{01}, A_{01} + C_{01}) \lambda_3^{A_{31}-1} \exp(-B_{31}\lambda_3), \quad \lambda_3 > 0, \tag{13}$$

$$g_1(p_1|\mathbf{y}) = \frac{\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})}{\Omega_1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{11}^{-A_{11}} B_{21}^{-A_{21}} B_{31}^{-A_{31}} B(B_{01}, C_{01}) p_1^{A_{01}-1} (1-p_1)^{B_{01}+C_{01}+1}, \quad 0 < p_1 < 1, \tag{14}$$

$$g_1(p_2|\mathbf{y}) = \frac{\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})}{\Omega_1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{11}^{-A_{11}} B_{21}^{-A_{21}} B_{31}^{-A_{31}} B(A_{01}, C_{01}) p_2^{B_{01}-1} (1-p_2)^{A_{01}+C_{01}-1}, \quad 0 < p_2 < 1. \tag{15}$$

4.2. Posterior distribution assuming the Jeffreys' prior

Jeffreys [55,56] proposed a rule for obtaining the non-informative prior as $p(\lambda) \propto \sqrt{|I(\lambda)|}$ if λ is an h-vector valued component parameter, where $I(\lambda) = (I_{uv})_{h\times h}$ is an $h \times h$ Fisher's information matrix, in which the (u,v)-th element is $-E\left[\frac{\partial^2 \ln L(\lambda|\mathbf{y})}{\partial\lambda_u \partial\lambda_v}\right]$; $u, v = 1, 2, ..., h$. The prior distributions of the proportion parameters p_1 and p_2 are assumed to be the uniform distributions over the interval $(0,1)$, i.e. $p_s \sim \text{Uniform}(0,1)$, $s = 1,2$. Assuming the independence of parameters, the joint prior distribution of parameters $\lambda_1, \lambda_2, \lambda_3, p_1$, and p_2 is $\pi_2(\mathbf{\Psi}) \propto \frac{1}{\lambda_1\lambda_2\lambda_3}$. The joint posterior distribution of parameters $\lambda_1, \lambda_2, \lambda_3, p_1$, and p_2 for the given data $\mathbf{y}$ using the JP is given by Eqs. (16) and (17) as shown in Box II, where:

$$A_{12} = r_1, \qquad A_{22} = r_2,$$

$$A_{32} = r_3,$$

$$B_{12} = (n-r-i)\ln(1+t) + \sum_{k=1}^{r_1} \ln(1+y_{1k}),$$

$$B_{22} = (i-j)\ln(1+t) + \sum_{k=1}^{r_2} \ln(1+y_{2k}),$$

$$B_{32} = (j)\ln(1+t) + \sum_{k=1}^{r_3} \ln(1+y_{3k}),$$

$$A_{02} = n-r-i+r_1+1,$$

$$B_{02} = i-j+r_2+1,$$

$$C_{02} = j+r_3+1,$$

$$\Omega_2 = \Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32}) \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B(A_{02}, B_{02}, C_{02}) B_{12}^{-A_{12}} B_{22}^{-A_{22}} B_{32}^{-A_{32}}.$$

The marginal posterior distributions of parameters $\lambda_1, \lambda_2, \lambda_3, p_1$, and p_2 using the JP are worked out as:

$$g_2(\lambda_1|\mathbf{y}) = \frac{\Gamma(A_{22})\Gamma(A_{32})}{\Omega_2} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{22}^{-A_{22}} B_{32}^{-A_{32}} B(A_{02}, C_{02}) B(B_{02}, A_{02} + C_{02}) \lambda_1^{A_{12}-1} \exp(-B_{12}\lambda_1), \quad \lambda_1 > 0, \tag{18}$$

$$g_2(\lambda_2|\mathbf{y}) = \frac{\Gamma(A_{12})\Gamma(A_{32})}{\Omega_2} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{12}^{-A_{12}} B_{32}^{-A_{32}} B(A_{02}, C_{02}) B(B_{02}, A_{02} + C_{02}) \lambda_2^{A_{22}-1} \exp(-B_{22}\lambda_2), \quad \lambda_2 > 0, \tag{19}$$

$$q_2(\boldsymbol{\Psi}|\mathbf{y}) = \frac{L(\boldsymbol{\Psi}|\mathbf{y})\pi_2(\boldsymbol{\Psi})}{\int_{\Psi} L(\boldsymbol{\Psi}|\mathbf{y})\pi_2(\boldsymbol{\Psi})d\boldsymbol{\Psi}}, \tag{16}$$

$$q_2(\boldsymbol{\Psi}|\mathbf{y}) = \frac{\sum_{i=1}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp(-B_{12}\lambda_1)\exp(-B_{22}\lambda_2)\exp(-B_{32}\lambda_3)p_1^{A_{02}-1}p_2^{B_{01}-1}(1-p_1-p_2)^{C_{02}-1}}{\Omega_2\lambda_1^{1-A_{12}}\lambda_2^{1-A_{22}}\lambda_3^{1-A_{32}}}. \tag{17}$$

Box II

Table 1. Bayes estimators and posterior risks under SELF, PLF, and DLF.

Loss function	Bayes estimators	Posterior risks
SELF $= L(\lambda,d) = (\lambda-d)^2$	$\hat{d} = E_{\lambda\|\mathbf{y}}(\lambda)$	$\rho(\hat{d}) = E_{\lambda\|\mathbf{y}}(\lambda^2) - \{E_{\lambda\|\mathbf{y}(\lambda)}\}^2$
PLF $= L(\lambda,d) = \frac{(\lambda-d)^2}{d}$	$\hat{d} = \{E_{\lambda\|\mathbf{y}}(\lambda^2)\}^{\frac{1}{2}}$	$\rho(\hat{d}) = 2\{E_{\lambda\|\mathbf{y}}(\lambda^2)\}^{\frac{1}{2}} - 2E_{\lambda\|\mathbf{y}}(\lambda)$
DLF $= L(\lambda,d) = \left(\frac{\lambda-d}{d}\right)^2$	$\hat{d} = \{E_{\lambda\|\mathbf{y}}(\lambda)\}^{-1}E_{\lambda\|\mathbf{y}}(\lambda^2)$	$\rho(\hat{d}) = 1 - \{E_{\lambda\|\mathbf{y}}(\lambda)\}^2\{E_{\lambda\|\mathbf{y}}(\lambda^2)\}^{-1}$

$$g_2(\lambda_3|\mathbf{y}) = \frac{\Gamma(A_{22})\Gamma(A_{32})}{\Omega_2}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{12}^{-A_{12}}B_{22}^{-A_{22}}B(A_{02},C_{02})B(B_{02},A_{02} + C_{02})\lambda_3^{A_{32}-1}\exp(-B_{32}\lambda_3), \quad \lambda_3 > 0, \tag{20}$$

$$g_2(p_1|\mathbf{y}) = \frac{\Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32})}{\Omega_2}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{12}^{-A_{12}}B_{22}^{-A_{22}}B_{32}^{-A_{32}}B(B_{02},C_{02}) p_1^{A_{02}-1}(1-p_1)^{B_{02}+C_{02}-1}, \quad 0 < p_1 < 1, \tag{21}$$

$$g_2(p_2|\mathbf{y}) = \frac{\Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32})}{\Omega_2}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{12}^{-A_{12}}B_{22}^{-A_{22}}B_{32}^{-A_{32}}B(A_{02},C_{02}) p_2^{B_{02}-1}(1-p_2)^{A_{02}+C_{02}-1}, \quad 0 < p_2 < 1. \tag{22}$$

5. Bayesian estimation

In this section, we present the derivation of the Bayes estimators and posterior risks using the UP and the JP under three different loss functions, namely, Squared Error Loss Function (SELF), Precautionary Loss Function (PLF), and DeGroot Loss Function (DLF). The SELF, defined as:

$$L = (\lambda, d) = (\lambda - d)^2,$$

was introduced by Legendre [57] to develop the least square theory. Norstrom [58] discussed an asymmetric PLF and also introduced a special case of a general class of PLFs, which was defined as $L(\lambda,d) = \frac{(\lambda-d)^2}{d}$. The DLF was presented by DeGroot [59] and was defined as $L(\lambda,d) = \left(\frac{\lambda-d}{d}\right)^2$ (see [4,52]). For a given prior, the Bayes estimators and posterior risks under SELF, PLF, and DLF are given in Table 1.

5.1. *Expressions for the Bayes estimators and posterior risks assuming the UP and the JP under SELF*

The expressions for the Bayes estimators and posterior risks are derived as follows:

$$\hat{\lambda}_{1v} = \frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-(A_{1v}+1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v}) B(B_{0v},A_{0v}+C_{0v}), \tag{23}$$

$$\hat{\lambda}_{2v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v}) B(B_{0v},A_{0v}+C_{0v}), \tag{24}$$

$$\hat{\lambda}_{3v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} (ij) B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-(A_{3v}+1)} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}), \quad (25)$$

$$\hat{p}_{1v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(B_{0v}, C_{0v}) B(A_{0v}+1, B_{0v}+C_{0v}), \quad (26)$$

$$\hat{p}_{2v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}+1, A_{0v}+C_{0v}), \quad (27)$$

$$\rho(\hat{\lambda}_{1v}) = \frac{\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} B_{1v}^{-(A_{1v}+2)} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) - \left\{ \frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{1v}^{-(A_{1v}+1)} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \right\}^2, \quad (28)$$

$$\rho(\hat{\lambda}_{2v}) = \frac{\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-(A_{2v}+2)} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) - \left\{ \frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-(A_{2v}+1)} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \right\}^2, \quad (29)$$

$$\rho(\hat{\lambda}_{3v}) = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-(A_{3v}+2)} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) - \left\{ \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-(A_{3v}+1)} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \right\}^2, \quad (30)$$

$$\rho(\hat{p}_{1v}) = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(B_{0v}, C_{0v}) B(A_{0v}+2, B_{0v}+C_{0v}) - \left\{ \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(B_{0v}, C_{0v}) B(A_{0v}+1, B_{0v}+C_{0v}) \right\}^2, \quad (31)$$

$$\rho(\hat{p}_{2v}) = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}+2, A_{0v}+C_{0v})$$

$$-\left\{\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\right\}^2, \quad (32)$$

where $v = 1$ for the UP and $v = 2$ for the JP.

Similarly, the Bayes estimators and posterior risks using the UP and the JP under PLF and DLF can also be derived. For the sake of brevity, we have not presented these expressions; but, these expressions are available in the studies by the first author.

6. Posterior predictive distribution and Bayesian predictive interval

The posterior predictive distribution contains the information about the future observation, $X = Y_{n+1}$, of a random variable given in the data, $\mathbf{y}$, already observed. Dependent on the observed data, it is defined as the distribution of a new independent and identical future observation drawn from the same population. It is normally used in a Bayesian framework. Using the entire posterior distribution of the parameter(s), a probability distribution over an interval is derived as a posterior predictive distribution of the future observation conditional on the observed data. To be more specific, by marginalizing the posterior distribution over the parameter(s), posterior predictive distribution of future observation can be derived. Arnold and Press [60], Al-Hussaini et al. [61], Al-Hussaini and Ahmad [62], Bolstad [63], and Bansal [64] have given a detailed discussion on prediction and predictive distribution under the Bayesian paradigm. We, now, present the derivation of posterior predictive distribution and Bayesian predictive interval.

The posterior predictive distribution of a future observation, $X = Y_{n+1}$, for the given data, $\mathbf{y}$, assuming the UP and the JP, is written as:

$$f(x|\underset{\sim}{y}) = \int_{p_2}\int_{p_1}\int_{\lambda_3}\int_{\lambda_2}\int_{\lambda_1} f(x|\mathbf{\Psi})q_v(\mathbf{\Psi}|\mathbf{y})\, d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2, \quad (33)$$

where:

$$f(x|\mathbf{\Psi}) = p_1 f_1(x;\Psi_1) + p_2 f_2(x;\Psi_2) + (1-p_1-p_2)f_3(x;\Psi_3),$$

$$f_m(x;\Psi_m) = \lambda_m(1+x)^{-(\lambda_m+1)}, \quad 0 < x < \infty, \quad \lambda_m > 0, \quad m = 1,2,3,$$

and $q_v(\mathbf{\Psi}|\mathbf{y})$ is defined as shown in Box III. Thus, the posterior predictive distribution given in Eq. (33), assuming the UP and the JP of a future observation, $X = Y_{n+1}$, for the given data, $\mathbf{y}$, is given by:

$$f(x|\mathbf{y}) = \frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{(1+x)\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}(B_{1v}+\ln(1+x))^{-(A_{1v}+1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v}+1,C_{0v})B(B_{0v},A_{0v}+C_{0v}+1)$$
$$+\frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{(1+x)\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}\Big(B_{2v}+\ln(1+x)\Big)^{-(A_{2v}+1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v}+1)$$
$$+\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{(1+x)\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}\Big(B_{3v}+\ln(1+x)\Big)^{-(A_{3v}+1)}B(A_{0v}+1,C_{0v}+1)B(B_{0v},A_{0v}+C_{0v}+1). \quad (34)$$

To construct a Bayesian predictive interval, suppose L and U are the two endpoints of the Bayesian predictive

$$q_v(\mathbf{\Psi}|\mathbf{y}) = \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp(-B_{1v}\lambda_1)\exp(-B_{2v}\lambda_2)\exp(-B_{3v}\lambda_3)p_1^{A_{0v}-1}p_2^{B_{0v}-1}(1-p_1-p_2)^{C_{0v}-1}}{\Omega_v\lambda_1^{1-A_{1v}}\lambda_2^{1-A_{2v}}\lambda_3^{1-A_{3v}}}.$$

Box III

interval. These two endpoints can be obtained using the posterior predictive distribution defined in Eq. (34). A 100(1− ∝)% Bayesian predictive interval (L, U) can be obtained by solving the following equations:

$$\int_0^L f(x|\mathbf{y})dx = \frac{\alpha}{2} = \int_U^\infty f(x|\mathbf{y})dx,$$

or:

$$\frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{A_{1v}\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$\left\{B_{1v}^{-A_{1v}} - (B_{1v}+\ln(1+L))^{-A_{1v}}\right\}$$

$$B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v}+1, C_{0v})B(B_{0v}, A_{0v}+C_{0v}$$

$$+1) + \frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{A_{2v}\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}$$

$$\binom{i}{j}B_{1v}^{-A_{1v}}\left\{B_{2v}^{-A_{2v}} - (B_{2v}+\ln(1+L))^{-A_{2v}}\right\}$$

$$B_{3v}^{-A_{3v}}B(A_{0v}, C_{0v})B(B_{0v}+1, A_{0v}+C_{0v}+1)$$

$$+\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{A_{3v}\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}\left\{B_{3v}^{-A_{3v}} - (B_{3v}+\ln(1+L))^{-A_{3v}}\right\}$$

$$B(A_{0v}, C_{0v}+1)B(B_{0v}, A_{0v}+C_{0v}+1) = \frac{\alpha}{2}$$

and:

$$\frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{A_{1v}\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}$$

$$(B_{1v}+\ln(1+U))^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}$$

$$B(A_{0v}+1, C_{0v})B(B_{0v}, A_{0v}+C_{0v}+1)$$

$$+\frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{A_{2v}\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}$$

$$\binom{i}{j}B_{1v}^{-A_{1v}}(B_{2v}+\ln(1+U))^{-A_{2v}}B_{3v}^{-A_{3v}}$$

$$B(A_{0v}, C_{0v})B(B_{0v}+1, A_{0v}+C_{0v}+1)$$

$$+\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{A_{3v}\Omega_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}$$

$$\binom{i}{j}B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}(B_{3v}+\ln(1+U))^{-A_{3v}}$$

$$B(A_{0v}, C_{0v}+1)B(B_{0v}, A_{0v}+C_{0v}+1) = \frac{\alpha}{2}.$$

7. Limiting expressions for complete dataset

When test termination time t tends to ∞, uncensored observations r tends to sample size n and r_l tends to n_l, $l = 1, 2, 3$. Consequently, all the censored observations become uncensored and the amount of information contained in the sample increases and results in the reduction of the posterior risks of the Bayes estimators. Thus, efficiency of the Bayes estimators increases, because all the observations are incorporated in our sample (see [4,52]). When t tends to ∞, the limiting expressions for the Bayes estimators and posterior risks are given in Tables 2-7.

Table 2. Limiting expressions for the Bayes estimators assuming the UP and the JP under SELF.

Bayes estimators	
UP	**JP**
$\lim_{t\to\infty}\hat{\lambda}_1 = \frac{n_1+1}{\sum_{k=1}^{n_1}\ln(1+y_{1k})}$	$\lim_{t\to\infty}\hat{\lambda}_1 = \frac{n_1}{\sum_{k=1}^{n_1}\ln(1+y_{1k})}$
$\lim_{t\to\infty}\hat{\lambda}_2 = \frac{n_2+1}{\sum_{k=1}^{n_2}\ln(1+y_{2k})}$	$\lim_{t\to\infty}\hat{\lambda}_2 = \frac{n_2}{\sum_{k=1}^{n_2}\ln(1+y_{2k})}$
$\lim_{t\to\infty}\hat{\lambda}_3 = \frac{n_3+1}{\sum_{k=1}^{n_3}\ln(1+y_{3k})}$	$\lim_{t\to\infty}\hat{\lambda}_3 = \frac{n_3}{\sum_{k=1}^{n_3}\ln(1+y_{3k})}$
$\lim_{t\to\infty}\hat{p}_1 = \frac{n_1+1}{n+3}$	$\lim_{t\to\infty}\hat{p}_1 = \frac{n_1+1}{n+3}$
$\lim_{t\to\infty}\hat{p}_2 = \frac{n_2+1}{n+3}$	$\lim_{t\to\infty}\hat{p}_2 = \frac{n_2+1}{n+3}$

Table 3. Limiting expressions for the posterior risks assuming the UP and the JP under SELF.

Posterior risks	
UP	**JP**
$\lim_{t\to\infty}\rho(\hat{\lambda}_1) = \frac{n_1+1}{\left(\sum_{k=1}^{n_1}\ln(1+y_{1k})\right)^2}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_1) = \frac{n_1}{\left(\sum_{k=1}^{n_1}\ln(1+y_{1k})\right)^2}$
$\lim_{t\to\infty}\rho(\hat{\lambda}_2) = \frac{n_2+1}{\left(\sum_{k=1}^{n_2}\ln(1+y_{2k})\right)^2}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_2) = \frac{n_2}{\left(\sum_{k=1}^{n_2}\ln(1+y_{2k})\right)^2}$
$\lim_{t\to\infty}\rho(\hat{\lambda}_3) = \frac{n_3+1}{\left(\sum_{k=1}^{n_3}\ln(1+y_{3k})\right)^2}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_3) = \frac{n_3}{\left(\sum_{k=1}^{n_3}\ln(1+y_{3k})\right)^2}$
$\lim_{t\to\infty}\rho(\hat{p}_1) = \frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$	$\lim_{t\to\infty}\rho(\hat{p}_1) = \frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$
$\lim_{t\to\infty}\rho(\hat{p}_2) = \frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$	$\lim_{t\to\infty}\rho(\hat{p}_2) = \frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$

Table 4. Limiting expressions for the Bayes estimators assuming the UP and the JP under PLF.

Bayes estimators	
UP	**JP**
$\lim_{t\to\infty}\hat{\lambda}_1=\frac{(n_1+1)^{1/2}(n_1+2)^{1/2}}{\left(\sum_{k=1}^{n_1}\ln(1+y_{1k})\right)^{1/2}}$	$\lim_{t\to\infty}\hat{\lambda}_1=\frac{(n_1)^{1/2}(n_1+1)^{1/2}}{\left(\sum_{k=1}^{n_1}\ln(1+y_{1k})\right)^{1/2}}$
$\lim_{t\to\infty}\hat{\lambda}_2=\frac{(n_2+1)^{1/2}(n_2+2)^{1/2}}{\left(\sum_{k=1}^{n_2}\ln(1+y_{2k})\right)^{1/2}}$	$\lim_{t\to\infty}\hat{\lambda}_2=\frac{(n_2)^{1/2}(n_2+1)^{1/2}}{\left(\sum_{k=1}^{n_2}\ln(1+y_{2k})\right)^{1/2}}$
$\lim_{t\to\infty}\hat{\lambda}_3=\frac{(n_3+1)^{1/2}(n_3+2)^{1/2}}{\left(\sum_{k=1}^{n_3}\ln(1+y_{3k})\right)^{1/2}}$	$\lim_{t\to\infty}\hat{\lambda}_3=\frac{(n_3)^{1/2}(n_3+1)^{1/2}}{\left(\sum_{k=1}^{n_3}\ln(1+y_{3k})\right)^{1/2}}$
$\lim_{t\to\infty}\hat{p}_1=\frac{(n_1+1)^{1/2}(n_1+2)^{1/2}}{(n+3)^{1/2}(n+4)^{1/2}}$	$\lim_{t\to\infty}\hat{p}_1=\frac{(n_1+1)^{1/2}(n_1+2)^{1/2}}{(n+3)^{1/2}(n+4)^{1/2}}$
$\lim_{t\to\infty}\hat{p}_2=\frac{(n_2+1)^{1/2}(n_2+2)^{1/2}}{(n+3)^{1/2}(n+4)^{1/2}}$	$\lim_{t\to\infty}\hat{p}_2=\frac{(n_2+1)^{1/2}(n_2+2)^{1/2}}{(n+3)^{1/2}(n+4)^{1/2}}$

8. Simulation study

It is obvious that the analytical comparisons among the Bayes estimators (under different priors and loss functions) are not possible; a simulation study is conducted to serve this purpose. The performance of Bayes estimators has been scrutinized under different priors, loss functions, parametric values, sample sizes, and test termination times. We calculated the Bayes estimates and posterior risks of five parameters $\lambda_1, \lambda_2, \lambda_3, p_1$, and p_2 of a 3-component mixture of Burr distributions given in Eqs. (1) and (4) through a Monte Carlo simulation using the following steps.

1. A random sample of the mixtures is generated as follows:
   (i) For each observation, a random number, u, is generated from the uniform distribution over the interval (0,1);
   (ii) If $u < p_1$, then a random variate, y, is generated by using Eq. (4) as $y = F_1^{-1}(u)$ (the cdf of Burr distribution with parameter λ_1);
   (iii) If $p_1 < u < p_2$, then a random variate, y, is generated by using Eq. (4) as $y = F_2^{-1}(u)$ (the cdf of Burr distribution with parameter λ_2);

Table 6. Limiting expressions for the Bayes estimators assuming the UP and the JP under DLF.

Bayes estimators	
UP	**JP**
$\lim_{t\to\infty}\hat{\lambda}_1=\frac{n_1+2}{\sum_{k=1}^{n_1}\ln(1+y_{1k})}$	$\lim_{t\to\infty}\hat{\lambda}_1=\frac{n_1+1}{\sum_{k=1}^{n_1}\ln(1+y_{1k})}$
$\lim_{t\to\infty}\hat{\lambda}_2=\frac{n_2+2}{\sum_{k=1}^{n_2}\ln(1+y_{2k})}$	$\lim_{t\to\infty}\hat{\lambda}_2=\frac{n_2+1}{\sum_{k=1}^{n_2}\ln(1+y_{2k})}$
$\lim_{t\to\infty}\hat{\lambda}_3=\frac{n_3+2}{\sum_{k=1}^{n_3}\ln(1+y_{3k})}$	$\lim_{t\to\infty}\hat{\lambda}_3=\frac{n_3+1}{\sum_{k=1}^{n_3}\ln(1+y_{3k})}$
$\lim_{t\to\infty}\hat{p}_1=\frac{n_1+2}{n+4}$	$\lim_{t\to\infty}\hat{p}_1=\frac{n_1+2}{n+4}$
$\lim_{t\to\infty}\hat{p}_2=\frac{n_2+2}{n+4}$	$\lim_{t\to\infty}\hat{p}_2=\frac{n_2+2}{n+4}$

Table 7. Limiting expressions for the posterior risks assuming the UP and the JP under DLF.

Posterior risks	
UP	**JP**
$\lim_{t\to\infty}\rho(\hat{\lambda}_1)=\frac{1}{n_1+2}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_1)=\frac{1}{n_1+1}$
$\lim_{t\to\infty}\rho(\hat{\lambda}_2)=\frac{1}{n_2+2}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_2)=\frac{1}{n_2+1}$
$\lim_{t\to\infty}\rho(\hat{\lambda}_3)=\frac{1}{n_3+2}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_3)=\frac{1}{n_3+1}$
$\lim_{t\to\infty}\rho(\hat{p}_1)=\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$	$\lim_{t\to\infty}\rho(\hat{p}_1)=\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$
$\lim_{t\to\infty}\rho(\hat{p}_2)=\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$	$\lim_{t\to\infty}\rho(\hat{p}_2)=\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$

   (iv) If $u > p_2$, then a random variate, y, is generated by using Eq. (4) as $y = F_3^{-1}(u)$ (the cdf of Burr distribution with parameter λ_3).
2. A sample censored at a fixed test termination time, t, is selected. The observations which are greater than a fixed test termination time, t, are taken as censored ones;
3. Using the steps 1 and 2 for the fixed values of

Table 5. Limiting expressions for the posterior risks assuming the UP and the JP under PLF.

Posterior risks	
UP	**JP**
$\lim_{t\to\infty}\rho(\hat{\lambda}_1)=\frac{2(n_1+1)}{\sum_{k=1}^{n_1}\ln(1+y_{1k})}\left\{\frac{(n_1+2)^{1/2}}{(n_1+1)^{1/2}}-1\right\}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_1)=\frac{2n_1}{\sum_{k=1}^{n_1}\ln(1+y_{1k})}\left\{\frac{(n_1+1)^{1/2}}{(n_1)^{1/2}}-1\right\}$
$\lim_{t\to\infty}\rho(\hat{\lambda}_2)=\frac{2(n_2+1)}{\sum_{k=1}^{n_2}\ln(1+y_{2k})}\left\{\frac{(n_2+2)^{1/2}}{(n_2+1)^{1/2}}-1\right\}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_2)=\frac{2n_2}{\sum_{k=1}^{n_2}\ln(1+y_{2k})}\left\{\frac{(n_2+1)^{1/2}}{(n_2)^{1/2}}-1\right\}$
$\lim_{t\to\infty}\rho(\hat{\lambda}_3)=\frac{2(n_3+1)}{\sum_{k=1}^{n_3}\ln(1+y_{3k})}\left\{\frac{(n_3+2)^{1/2}}{(n_3+1)^{1/2}}-1\right\}$	$\lim_{t\to\infty}\rho(\hat{\lambda}_3)=\frac{2n_3}{\sum_{k=1}^{n_3}\ln(1+y_{3k})}\left\{\frac{(n_3+1)^{1/2}}{(3_2)^{1/2}}-1\right\}$
$\lim_{t\to\infty}\rho(\hat{p}_1)=\frac{2(n_1+1)}{(n+3)}\left\{\frac{(n_1+2)^{1/2}(n_1+1)^{-1/2}}{(n+4)^{1/2}(n+3)^{-1/2}}-1\right\}$	$\lim_{t\to\infty}\rho(\hat{p}_1)=\frac{2(n_1+1)}{(n+3)}\left\{\frac{(n_1+2)^{1/2}(n_1+1)^{-1/2}}{(n+4)^{1/2}(n+3)^{-1/2}}-1\right\}$
$\lim_{t\to\infty}\rho(\hat{p}_2)=\frac{2(n_2+1)}{(n+3)}\left\{\frac{(n_2+2)^{1/2}(n_2+1)^{-1/2}}{(n+4)^{1/2}(n+3)^{-1/2}}-1\right\}$	$\lim_{t\to\infty}\rho(\hat{p}_2)=\frac{2(n_2+1)}{(n+3)}\left\{\frac{(n_2+2)^{1/2}(n_2+1)^{-1/2}}{(n+4)^{1/2}(n+3)^{-1/2}}-1\right\}$

Table 8. Bayes Estimate (BE) and Posterior Risk (PR) using the UP with $\lambda_1 = 6$, $\lambda_2 = 5$, $\lambda_3 = 4$, $p_1 = 0.5$, and $p_2 = 0.3$.

l	n	Loss functions		UP				
				$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.4	50	SELF	BE	6.997260	6.691200	6.452920	0.489451	0.303943
			PR	4.583960	7.799900	13.13170	0.006500	0.005796
		PLF	BE	7.348310	7.177890	7.604400	0.496734	0.314146
			PR	0.608363	1.018170	1.685550	0.013223	0.018654
		DLF	BE	7.602530	7.528110	8.423640	0.505724	0.323751
			PR	0.081876	0.143088	0.214871	0.026505	0.059566
	100	SELF	BE	6.538970	5.869170	5.618080	0.496233	0.303230
			PR	2.186560	3.444070	5.482010	0.003609	0.003254
		PLF	BE	6.777700	6.231320	5.917420	0.498822	0.306856
			PR	0.322962	0.555434	0.854331	0.007218	0.010590
		DLF	BE	6.803540	6.629790	6.353780	0.505517	0.309445
			PR	0.048157	0.089911	0.145986	0.014627	0.035049
	200	SELF	BE	6.287410	5.487840	4.825760	0.497838	0.301859
			PR	1.124150	1.727490	2.467200	0.001986	0.001804
		PLF	BE	6.410030	5.681020	5.079700	0.499043	0.304370
			PR	0.174347	0.301519	0.474581	0.003956	0.005892
		DLF	BE	6.505640	5.754450	5.421110	0.502216	0.308710
			PR	0.026876	0.052632	0.091868	0.007846	0.019183
0.7	50	SELF	BE	6.722710	6.012760	5.576240	0.492014	0.302771
			PR	2.550230	3.742480	5.365850	0.004976	0.004285
		PLF	BE	6.813420	6.272540	6.039290	0.497186	0.310368
			PR	0.36593	0.585562	0.874647	0.010155	0.014127
		DLF	BE	6.956670	6.507270	6.640960	0.501614	0.318774
			PR	0.054030	0.092403	0.141637	0.020482	0.044960
	100	SELF	BE	6.324060	5.480920	4.857800	0.496323	0.301978
			PR	1.237760	1.747910	2.352140	0.002645	0.002288
		PLF	BE	6.416690	5.696410	5.019710	0.499082	0.305470
			PR	0.188655	0.303863	0.438772	0.005299	0.007493
		DLF	BE	6.524860	5.873160	5.230060	0.501384	0.308068
			PR	0.029086	0.052747	0.085033	0.010579	0.024499
	200	SELF	BE	6.152260	5.292670	4.468900	0.498232	0.300575
			PR	0.606736	0.861029	1.061440	0.001359	0.001177
		PLF	BE	6.245260	5.362640	4.599630	0.499281	0.303342
			PR	0.098646	0.159784	0.231530	0.002727	0.003912
		DLF	BE	6.288950	5.436710	4.687870	0.500282	0.305337
			PR	0.015778	0.029579	0.049803	0.005462	0.012845

parameters, test termination time, and sample size, 1000 samples are generated;

4. The Bayes estimates and posterior risks of parameters $\lambda_1, \lambda_2, \lambda_3, p_1$, and p_2 are calculated based on 1000 Monte Carlo repetitions by solving Eqs. (23)-(32).

The above steps 1-4 are used for each of the sample sizes $n = 50, 100, 200$ and each choice of the vector of the parameters $(\lambda_1, \lambda_2, \lambda_3, p_1, p_2) = \{(6, 5, 4, 0.5, 0.3)$ $(8, 7, 6, 0.5, 0.3)\}$ taking $t = 0.4, 0.7$ The choice of the test termination time is made in such a way that the censoring rate in resulting sample is approximately 10% to 25%.

From Tables 8-11, it is observed that component parameters $\lambda_1, \lambda_2, \lambda_3$ and the proportion parameter p_2 are over-estimated assuming the UP and the JP under SELF, PLF, and DLF at different sample sizes and

Table 9. Bayes Estimate (BE) and Posterior Risk (PR) using the JP with $\lambda_1 = 6$, $\lambda_2 = 5$, $\lambda_3 = 4$, $p_1 = 0.5$, and $p_2 = 0.3$.

l	n	Loss functions		UP				
				$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.4	50	SELF	BE	6.886140	5.918820	5.517360	0.484936	0.304609
			PR	4.259220	6.490770	10.56210	0.006302	0.005679
		PLF	BE	7.069440	6.492790	6.093830	0.491957	0.311387
			PR	0.570876	0.962736	1.461740	0.012899	0.018329
		DLF	BE	7.476850	7.149820	6.910650	0.497321	0.320983
			PR	0.080904	0.147252	0.237096	0.026447	0.059106
	100	SELF	BE	6.502440	5.630090	4.806170	0.490801	0.302200
			PR	2.134230	3.234210	4.423630	0.003535	0.003225
		PLF	BE	6.737250	5.781700	5.313280	0.494329	0.308441
			PR	0.318232	0.527867	0.827861	0.007176	0.010588
		DLF	BE	6.808520	6.159180	5.810010	0.498101	0.313483
			PR	0.047383	0.090698	0.151750	0.014603	0.034361
	200	SELF	BE	6.309370	5.383100	4.516600	0.495464	0.301536
			PR	1.097750	1.642300	2.201910	0.001937	0.001772
		PLF	BE	6.365650	5.534190	4.784880	0.497933	0.303982
			PR	0.170060	0.296000	0.455499	0.003905	0.005886
		DLF	BE	6.511610	5.666990	4.985940	0.499075	0.307124
			PR	0.026479	0.052774	0.093847	0.007799	0.019238
0.7	50	SELF	BE	6.378500	5.639130	4.849220	0.490331	0.302340
			PR	2.365360	3.492550	4.609880	0.004974	0.004282
		PLF	BE	6.495760	5.848090	5.333080	0.495593	0.309936
			PR	0.352190	0.561249	0.830935	0.010133	0.014048
		DLF	BE	6.736510	6.135530	5.741040	0.500727	0.316377
			PR	0.054126	0.096276	0.153533	0.020386	0.045291
	100	SELF	BE	6.207900	5.388300	4.515750	0.495220	0.301770
			PR	1.176680	1.675780	2.079240	0.002623	0.002262
		PLF	BE	6.346240	5.520510	4.653160	0.496980	0.305032
			PR	0.186157	0.296435	0.417369	0.005297	0.007468
		DLF	BE	6.372050	5.622880	5.064280	0.500538	0.309661
			PR	0.029411	0.053728	0.089035	0.010608	0.024308
	200	SELF	BE	6.141450	5.234200	4.264300	0.497463	0.300380
			PR	0.604560	0.848853	0.994630	0.001353	0.001174
		PLF	BE	6.171870	5.245570	4.422600	0.499386	0.302403
			PR	0.096162	0.154823	0.222537	0.002713	0.003891
		DLF	BE	6.222020	5.368450	4.499850	0.500209	0.304641
			PR	0.015665	0.029549	0.050108	0.005442	0.012821

test termination times. The proportion parameter p_1 is under-estimated assuming the UP and the JP under SELF and PLF; but, under DLF, it is over-estimated (under-estimated) using the UP (JP) at varying sample sizes and test termination times. The extent of under-estimation of component and proportion parameters using the UP and the JP under SELF, PLF, and DLF is lower for larger sample sizes in a fixed test termination time. Also, the extent of over-estimation of component and proportion parameters is higher for smaller test termination times. The extent of over-estimation (under-estimation) of component and proportion parameters is higher for smaller values of component parameters at varying test termination times and sample sizes. The differences of the Bayes estimates of component and proportion parameters

Table 10. Bayes Estimate (BE) and Posterior Risk (PR) using the UP with $\lambda_1 = 8$, $\lambda_2 = 7$, $\lambda_3 = 6$, $p_1 = 0.5$, and $p_2 = 0.3$.

l	n	Loss functions		UP				
				$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.4	50	SELF	BE	9.101440	8.492790	8.341060	0.488651	0.303173
			PR	5.452830	8.613770	13.44860	0.005265	0.004551
		PLF	BE	9.387530	9.211350	9.021220	0.494763	0.309361
			PR	0.557451	0.929613	1.361310	0.010654	0.014711
		DLF	BE	9.499820	9.608110	9.681080	0.502075	0.316453
			PR	0.059466	0.100828	0.151129	0.021450	0.047514
	100	SELF	BE	8.541580	7.919530	7.329280	0.494873	0.301242
			PR	2.553910	4.068990	5.802390	0.002794	0.002410
		PLF	BE	8.770320	7.970950	7.656010	0.496402	0.306334
			PR	0.292496	0.479138	0.731082	0.005641	0.007980
		DLF	BE	8.780040	8.437000	8.049320	0.500752	0.309444
			PR	0.033399	0.059586	0.095045	0.011334	0.025918
	200	SELF	BE	8.312190	7.393980	6.726530	0.497286	0.301108
			PR	1.284600	1.937140	2.730210	0.001458	0.001268
		PLF	BE	8.419090	7.594670	6.823740	0.498471	0.302694
			PR	0.154238	0.259959	0.389658	0.002943	0.004202
		DLF	BE	8.404300	7.695930	7.070130	0.500196	0.304976
			PR	0.018188	0.034037	0.056276	0.005889	0.013827
0.7	50	SELF	BE	8.829450	8.229030	7.711900	0.490367	0.302182
			PR	3.654050	5.515350	7.741740	0.004717	0.003997
		PLF	BE	8.911510	8.430550	8.160940	0.495091	0.309217
			PR	0.387904	0.608145	0.872169	0.009580	0.013098
		DLF	BE	9.114080	8.784570	8.431350	0.500294	0.315044
			PR	0.043606	0.072338	0.107143	0.019265	0.042162
	100	SELF	BE	8.316650	7.616550	6.771110	0.495054	0.300922
			PR	1.668830	2.460720	3.105320	0.002461	0.002082
		PLF	BE	8.448940	7.723530	7.025970	0.497581	0.304828
			PR	0.195833	0.306976	0.430101	0.004952	0.006867
		DLF	BE	8.590020	7.941750	7.215240	0.500142	0.307822
			PR	0.023232	0.039958	0.061849	0.009933	0.022480
	200	SELF	BE	8.243320	7.280110	6.379610	0.497326	0.300765
			PR	0.842157	1.176740	1.465400	0.001257	0.001065
		PLF	BE	8.255690	7.414080	6.545220	0.498936	0.302285
			PR	0.099966	0.158283	0.221581	0.002522	0.003528
		DLF	BE	8.305860	7.510540	6.733040	0.500118	0.304104
			PR	0.012069	0.021128	0.033458	0.005047	0.011618

from assumed values reduce to zero with an increase in sample size at different test termination times and same is the case for larger test termination times at varying sample sizes.

It can be seen that the posterior risks of Bayes estimators of parameters assuming the UP and the JP under SELF, PLF, and DLF decrease with an increase in sample size for a fixed test termination time. The same observation is made for large test termination times at different sample sizes. Also, the posterior risks of Bayes estimators of component parameters under SELF are small, but the posterior risks of Bayes estimators of component parameters under DLF and the posterior risks of Bayes estimators of proportion parameters under SELF, PLF, and DLF are larger for smaller values of component parameters at different

Table 11. Bayes Estimate (BE) and Posterior Risk (PR) using the JP with $\lambda_1 = 8$, $\lambda_2 = 7$, $\lambda_3 = 6$, $p_1 = 0.5$, and $p_2 = 0.3$.

l	n	Loss functions		UP				
				$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
0.4	50	SELF	BE	8.641140	7.993400	7.302960	0.486916	0.303231
			PR	4.932750	7.831050	11.54690	0.005218	0.004507
		PLF	BE	9.069250	8.355610	8.051480	0.494535	0.309745
			PR	0.538713	0.881620	1.337550	0.010562	0.014658
		DLF	BE	9.389360	9.167340	8.575960	0.498551	0.315971
			PR	0.059978	0.105442	0.164111	0.021526	0.047375
	100	SELF	BE	8.463760	7.524270	6.767160	0.493152	0.301301
			PR	2.515380	3.785490	5.310540	0.002779	0.002408
		PLF	BE	8.537750	7.926030	7.135560	0.496273	0.305022
			PR	0.283743	0.477354	0.705170	0.005616	0.007888
		DLF	BE	8.741370	8.115590	7.531480	0.498788	0.309560
			PR	0.033168	0.060628	0.098511	0.011309	0.025798
	200	SELF	BE	8.310270	7.296180	6.442230	0.495855	0.301205
			PR	1.268940	1.885900	2.553720	0.001446	0.001259
		PLF	BE	8.340030	7.499810	6.596130	0.498042	0.302429
			PR	0.150276	0.255460	0.377614	0.002915	0.004164
		DLF	BE	8.448250	7.586900	6.780790	0.499004	0.304982
			PR	0.017835	0.033476	0.056154	0.005836	0.013655
0.7	50	SELF	BE	8.531000	7.624840	6.894530	0.489826	0.302358
			PR	3.516130	5.075680	6.841800	0.004713	0.003995
		PLF	BE	8.659990	7.913270	7.389920	0.495120	0.309025
			PR	0.387680	0.603757	0.862288	0.009579	0.013006
		DLF	BE	8.957510	8.200370	7.768060	0.499207	0.315682
			PR	0.044697	0.075846	0.115101	0.019314	0.042009
	100	SELF	BE	8.200430	7.296860	6.414750	0.495139	0.300922
			PR	1.655160	2.356370	2.984730	0.002460	0.002080
		PLF	BE	8.333100	7.501960	6.719570	0.497522	0.304692
			PR	0.195334	0.306449	0.430046	0.004949	0.006866
		DLF	BE	8.487190	7.660860	6.887420	0.499444	0.308210
			PR	0.023485	0.040872	0.064067	0.009949	0.022428
	200	SELF	BE	8.077800	7.239740	6.283810	0.497620	0.300338
			PR	0.808702	1.166490	1.434430	0.001256	0.001062
		PLF	BE	8.165370	7.298070	6.341300	0.498864	0.302261
			PR	0.099229	0.157256	0.218314	0.002520	0.003525
		DLF	BE	8.207960	7.294920	6.446890	0.499748	0.304457
			PR	0.012188	0.021579	0.034558	0.005055	0.011619

sample sizes and test termination times. However, the posterior risks of Bayes estimators of component parameters under PLF do not follow a pattern.

As far as the problem of selecting a suitable prior is concerned, it can be seen that the JP emerges as a more efficient prior due to less associated posterior risk than that of the UP under both SELF and PLF; but, we cannot identify which prior is suitable under DLF. On the other hand, the DLF is observed performing superior to PLF and SELF for estimating the component parameters; whereas, for estimating the proportion parameters, SELF is observed performing better than PLF and DLF. The selection of the best prior and loss function does not depend on test termination time and sample size. However, it is to be noted that selection of the best prior (loss function) for a given loss function

Table 12. Bayesian predictive interval (L, U) using the UP and the JP with $\lambda_1 = 6$, $\lambda_2 = 5$, $\lambda_3 = 4$, $p_1 = 0.5$, and $p_2 = 0.3$.

l	n	UP		JP	
		L	U	L	U
0.4	50	0.008340	1.078420	0.008993	1.266310
	100	0.008950	0.970155	0.009284	1.040020
	200	0.009286	0.893051	0.009451	0.922440
0.7	50	0.008642	0.929006	0.009194	1.045020
	100	0.009114	0.868725	0.009398	0.917158
	200	0.009420	0.838663	0.009563	0.861010

Table 13. Bayesian predictive interval (L, U) using the UP and the JP with $\lambda_1 = 8$, $\lambda_2 = 7$, $\lambda_3 = 6$, $p_1 = 0.5$, and $p_2 = 0.3$.

l	n	UP		JP	
		L	U	L	U
0.4	50	0.006151	0.611745	0.006568	0.683404
	100	0.006570	0.572961	0.006786	0.601959
	200	0.006816	0.550113	0.006925	0.563462
0.7	50	0.006337	0.565627	0.006722	0.620488
	100	0.006675	0.545038	0.006874	0.569459
	200	0.006867	0.533859	0.006969	0.545406

(prior) is made based on posterior risks associated with it.

The results in Tables 12 and 13 are the 90% Bayesian predictive intervals assuming the UP and the JP. It is observed that the Bayesian predictive intervals become narrower with an increase in sample size for a fixed test termination time. The same observation can be made with larger test termination times at a fixed sample size. The Bayesian predictive intervals become narrower (wider) for larger (smaller) component parametric values in each sample size and test termination time considered in the simulation study. Also, the Bayesian predictive intervals using the JP are wider than the predictive intervals using the UP.

9. A real-life example

Davis [65] reported a mixture data, $\mathbf{x} = (x_{11}, x_{12}, ..., x_{1r_1}, x_{21}, x_{22}, ..., x_{2r_2}, x_{31}, x_{32}, ..., x_{3r_3})$, on lifetimes (in thousand hours) of many components used in aircraft sets. A part of these data have also been used in [4]. To illustrate the proposed methodology, we take the data on three components, namely, R105 RESISTOR USED IN PE218 CONVERTER, Z303 NETWORK USED IN RF UNIT, and V7 TRANSMITTER TUBE. Davis showed that the data $\mathbf{x}$ could be modeled by a mixture of exponential distributions. The transformation $y = \exp(x) - 1$ of an exponential random data $(\mathbf{x})$ yields the Burr random data $(\mathbf{y})$. This transformation allows us to use the Davis mixture data for applying the proposed Bayesian analysis. It is unknown that which component fails until a failure occurs at or before the test termination time (1 hour). The tests are conducted 582 times. The data summary required to evaluate the Bayes estimates and posterior risks is given by:

$$n = 582, \quad r_1 = 252, \quad r_2 = 54, \quad r_3 = 175$$

$$r = r_1 + r_2 + r_3 = 481, \quad n - r = 101,$$

$$\sum_{k=1}^{r_1} \ln(1 + y_{1k}) = \sum_{k=1}^{r_1} x_{1k} = 90.60,$$

$$\sum_{k=1}^{r_2} \ln(1 + y_{2k}) = \sum_{k=1}^{r_2} x_{2k} = 23.20,$$

$$\sum_{k=1}^{r_3} \ln(1 + y_{3k}) = \sum_{k=1}^{r_3} x_{3k} = 46.125.$$

Since $n - r = 101$, we have almost 17.35% type-I right censored sample. The Bayes estimates and their posterior risks are shown in Table 14.

From Table 14, it is noticed that the results obtained through real-life data are compatible with the simulated results. The performance of the Bayes estimators using the JP is seen as the best in comparison with the UP under all the loss functions considered in this study. It is also observed that DLF (SELF) is better than PLF and SELF (PLF and DLF) for estimating component (proportion) parameters.

10. Conclusion

A 3-component mixture of Burr distributions is developed to model lifetime data. Type-I right censoring sampling scheme is considered. Assuming the availability of the non-informative priors and different loss functions, expressions of the Bayes estimators and their posterior risks are derived. To judge the relative performance of the Bayes estimators and also to deal with the problem of selecting the priors and loss functions at different sample sizes and test termination times, a comprehensive simulation and real-life study have been conducted. The simulation study revealed some important and interesting properties of the Bayes estimators. From numerical results, we observed that an increase in sample size or test termination time provided improved Bayes estimators. The effect of test termination time, sample size, and parametric values on the Bayes estimators is in the form of over-estimation or under-estimation. To be more specific, the smaller (larger) sample size results

Table 14. Bayes Estimates (BEs) and Posterior Risks (PRs) assuming the UP and the JP under SELF and DLF with Davis real-life mixture data.

Prior	Loss functions		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
SELF	UP	BE	1.75303781	0.93662706	3.32539143	0.52658110	0.16052665
		PR	0.04102084	0.07309535	0.11130084	0.00098376	0.00080169
	JP	BE	1.76753121	0.89014676	3.30209669	0.52349272	0.16349037
		PR	0.04066844	0.06212724	0.11052237	0.00095621	0.00077923
PLF	UP	BE	1.76469896	0.97486696	3.34208453	0.52751438	0.16300459
		PR	0.02332229	0.07647981	0.03338620	0.00186654	0.00495588
	JP	BE	1.77899832	0.92438546	3.31878968	0.52440522	0.16585636
		PR	0.02293422	0.06847741	0.03338597	0.00182501	0.00473198
DLF	UP	BE	1.77643767	1.01466809	3.35886142	0.52844930	0.16552078
		PR	0.01317235	0.07691287	0.00996468	0.00353524	0.03017225
	JP	BE	1.79053982	0.95994113	3.33556705	0.52531932	0.16825660
		PR	0.01285010	0.07270693	0.01003438	0.00347712	0.02832712

in larger (smaller) extent of over-estimation or under-estimation at a fixed test termination time. On the other hand, the extent of over-estimation or under-estimation of parameters is quite smaller (larger) with relatively larger (smaller) test termination times for a fixed sample size. Also, the extent of over-estimation or under-estimation of parameters is less for larger values of component parameters and vice versa. However, as sample size (test termination time) increases (decreases), the posterior risks of Bayes estimators of parameters decrease (increase) for a fixed test termination time (sample size). As the cut-off test termination time tends to infinity, the limiting expressions (for complete dataset) of the Bayes estimators and posterior risks are greatly simplified. Moreover, the posterior risks of the Bayes estimators (for complete dataset) are expected to reduce further as there is no more effect of test termination time. Finally, we conclude that for a Bayesian analysis of mixture data, the JP paired with SELF and both the UP and the JP paired with DLF are preferable choices for estimating proportion and component parameters, respectively. When PLF is considered, the JP is the suitable prior for estimating component parameters. Also, the results obtained through real-life data coincide with the simulated results.

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