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A differential quadrature procedure for free vibration of rectangular plates involving free corners

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Abstract. The Differential Quadrature Method (DQM) is one of the most powerful approximation methods for analyzing the free vibration of rectangular plates. It is easy to use and straightforward to implement. However, in spite of its many advantages, the conventional DQM has some limitations in determining the natural frequencies of rectangular plates involving free corners. This is because it is very difficult to implement the free corner boundary condition in conventional DQM. As a result, the method may exhibit some convergence problems and this may lead to erroneous and oscillatory results for natural frequencies of rectangular plates involving free corners. To overcome this difficulty, this paper presents a simple DQM formulation in which all the natural boundary conditions, including the free corner boundary condition, are implemented in an easy manner. Its accuracy and efficiency are demonstrated through the vibration analysis of rectangular plates with different combinations of free edges and free corners. Numerical results prove that the proposed method can produce much better accuracy than the conventional DQM while exhibiting a monotonic convergence behavior with respect to the number of sampling points. Furthermore, unlike the conventional DQM, solutions of the proposed method are not very sensitive to the sampling point distribution.

1. Introduction

Rectangular plates are important structural components that are extensively used in various fields of engineering such as civil, mechanical, aerospace, marine, and structural engineering. Therefore, proper understanding of the vibration characteristics of such structural elements is crucial for the structural designers.

In general, there are two kinds of methods that can be used to solve the free vibration problem of rectangular plates, namely, the analytical and numerical methods. The analytical methods often provide better information about vibration characteristics of rectangular plates. But, their applications are limited to plate problems with simple boundary conditions such as Levy-type boundary conditions [1,2]. This limitation is caused by the complexities introduced by the satisfaction of the free edges and free corner boundary conditions. To overcome the limitations of the analytical methods, various approximate or numerical methods such as the Ritz method [3-6], the extended Kantorovich approach [7], the finite element method [8,9], the BEM-based meshless method [10], the moving least squares differential quadrature method [11], semi-analytic differential quadrature method [12], the finite difference method [13], the spectral element method [14], and the discrete singular convolution method [15,16] have been developed by researchers

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to study the behavior of rectangular or other shaped plates with general boundary conditions.

Among the approximate methods used for solving the present problem, the Differential Quadrature Method (DQM) is one of the most convenient methods to obtain natural frequencies of rectangular plates [17-20]. It is simple to use and also straightforward to implement. However, in spite of its many useful features, the conventional DQM has its own drawbacks in implementation in the differential equations with multiple boundary conditions at the boundary points, especially for fourth-order governing differential equations of classical beam and plate problems. For instance, in solving the vibration problem of rectangular plates with general boundary conditions, Malik and Bert [21] indicated that the solutions of conventional DQM for rectangular plates having adjacent free edges (or free corners) might not exhibit convergence trend and erroneous results might be obtained. To solve this difficulty, Shu and Du [22] proposed an approach referred to as the direct Coupling of Boundary Conditions with discrete Governing Equations (CBCGE) for implementing the general boundary conditions for the free vibration analysis of rectangular plates. The CBCGE approach was shown to work well for the rectangular plates without free corners, but it encountered some issues when applied to the rectangular plates involving free corners. In this case, the numerical solutions of the CBCGE approach were highly sensitive to the sampling point distribution. For instance, the solutions of the CBCGE approach with conventional non-uniform sampling points were quite erroneous. To overcome this difficulty, Shu and Du [22] proposed the use of stretched sampling points where the sampling points were stretched toward the plate boundaries. Although rather accurate solutions were obtained using the proposed stretched sampling points, the obtained solutions did not show a monotonic convergence with increasing number of sampling points and, in some cases, the natural frequencies were found to behave oscillatory.

It can be seen that a simple and general formulation based on the conventional DQM that can easily handle the plate problem with general boundary conditions is still missing. Therefore, this paper intends to present a simple and accurate DQM formulation in which all the natural boundary conditions, including the free corner boundary conditions, are satisfied in an easy and accurate manner. To demonstrate its accuracy and stability, the proposed methodology is applied to solve the vibration problem of rectangular plates with various combinations of free edges and free corners. Comparison of obtained results with those in recent literature shows that the proposed methodology is capable of producing highly accurate solutions while exhibiting a monotonic convergence behavior with respect to the number of DQM sampling points. Furthermore, the proposed approach can produce much better accuracy than the conventional DQM formulations for rectangular plates involving free corners.

2. Differential quadrature method

The DQM is a numerical solution technique for initial and/or boundary value problems [23]. It was first developed by Richard Bellman and his associates in the early 1970's [24,25]. Since its introduction, the DQM has been successfully applied to a variety of engineering problems [26-45]. Most of these applications are related to static and dynamic analyses of structural components like beams, plates, and shells. Newer applications include the use of DQM for solving moving load problems [46,47] and fluid-structure interaction problems [48,49]. The results of many research works show that the DQM is computationally efficient and is applicable to a large class of initial and/or boundary value problems. However, as we discussed in introduction, the implementation of multiple boundary conditions is not an easy task when applying the DQM to higher-order partial differential equations. To overcome this limitation, this paper is devoted to present a simple and accurate DQM formulation in which the multiple boundary conditions are implemented in an easy and simple manner.

The DQM is based on the idea that the derivative of a function with respect to a coordinate direction at any discrete point can be expressed by a weighted linear sum of the function values at all the discrete points chosen in that direction. For instance, the rth-order X-derivative of the function $W(X,Y)$ at a sample point (X_i,Y) can be expressed as [23]:

$$\frac{\partial^r W(X_i,Y)}{\partial X^r}=\sum_{j=1}^{n}A_{ik}^{(r)}W(X_k,Y),$$

$$i=1,2,\cdots,n, \tag{1}$$

where n is the number of sample points in the X-direction, and $A_{ik}^{(r)}$ is the rth-order X-derivative weighting coefficient associated with the $X=X_i$ point.

It follows from Eq. (1) of which the quadrature rules may be written collectively in matrix form as:

$$\frac{\partial^r}{\partial X^r}\{\mathbf{W}(Y)\}=\left[\mathbf{A^{(r)}}\right]\{\mathbf{W}(Y)\}, \tag{2}$$

where:

$$\{\mathbf{W}(Y)\}=\left[W(X_1,Y)\;\;W(X_2,Y)\;\;\cdots\;\;W(X_n,Y)]^T\right], \tag{3}$$

$$\left[\mathbf{A}^{(r)}\right] = \begin{bmatrix} A_{11}^{(r)} & A_{12}^{(r)} & \cdots & A_{1n}^{(r)} \\ A_{21}^{(r)} & A_{22}^{(r)} & \cdots & A_{2n}^{(r)} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}^{(r)} & A_{n2}^{(r)} & \cdots & A_{nn}^{(r)} \end{bmatrix}. \tag{4}$$

The weighting coefficients of the first-order derivative can be obtained from the following algebraic formulations [50]:

$$A_{ik}^{(1)} = \begin{cases} \frac{\Pi(X_i)}{(X_i - X_k)\Pi(X_k)} & i \neq k, \quad i,k = 1,2,\cdots,n \\ -\sum\limits_{j=1, j\neq i}^{n} A_{ij}^{(1)} & i = k, \quad i = 1,2,\cdots,n \end{cases} \tag{5}$$

where $\Pi(X)$ is defined as:

$$\Pi(X_i) = \prod_{j=1, j\neq i}^{n} (X_i - X_j). \tag{6}$$

The weighting coefficients of the higher-order derivatives can be calculated from the following recurrence relationship [26]:

$$A_{ik}^{(r)} = \begin{cases} r\left[A_{ii}^{(r-1)} A_{ik}^{(1)} - \frac{A_{ik}^{(r-1)}}{X_i - X_k}\right] \\ \quad i \neq k, \quad i,k = 1,2,\cdots,n \\ -\sum\limits_{j=1, j\neq i}^{n} A_{ij}^{(r)} \\ \quad i = k, \quad i = 1,2,\cdots,n \end{cases} \tag{7}$$

Let $Y_1, Y_2, \cdots, Y_m$ be a set of sampling points in the Y-direction. Using the quadrature rule, the sth-order Y-derivative of the vector $\{W(Y)\}$ at a sample point $Y = Y_i$ can be expressed as [51]:

$$\frac{d^s}{dY^s}\{\mathbf{W}(Y_i)\} = \sum_{j=1}^{m} B_{ik}^{(s)}\{\mathbf{W}(Y_k)\},$$

$$i = 1,2,\cdots,m, \tag{8}$$

where $B_{ik}^{(s)}$ is the sth-order Y-derivative weighting coefficient associated with the $Y = Y_i$ point, and:

$$\{\mathbf{W}(Y_i)\}$$

$$= \left[W(X_1,Y_i) \quad W(X_2,Y_i) \quad \cdots \quad W(X_n,Y_i)\right]^T,$$

$$i = 1,2,\cdots,m. \tag{9}$$

$B_{ik}^{(1)}$ and $B_{ik}^{(s)}$ are given by:

$$B_{ik}^{(1)} = \begin{cases} \frac{\Pi(Y_i)}{(Y_i - Y_k)\Pi(Y_k)} \\ \quad i \neq k, \quad i,k = 1,2,\cdots,m \\ -\sum\limits_{j=1, j\neq i}^{m} B_{ij}^{(1)} \\ \quad i = k, \quad i = 1,2,\cdots,m \end{cases} \tag{10}$$

$$B_{ik}^{(s)} = \begin{cases} s\left[B_{ii}^{(s-1)} B_{ik}^{(1)} - \frac{B_{ik}^{(s-1)}}{Y_i - Y_k}\right] \\ \quad i \neq k, \quad i,k = 1,2,\cdots,m \\ -\sum\limits_{j=1, j\neq i}^{m} B_{ij}^{(s)} \\ \quad i = k, \quad i = 1,2,\cdots,m \end{cases} \tag{11}$$

where:

$$\Pi(Y_i) = \prod_{j=1, j\neq i}^{m} (Y_i - Y_j). \tag{12}$$

It is noted that the quadrature rule (Eq. (8)) can be written for all the i values in the following matrix form:

$$\frac{d^r}{dY^r}\left\{\tilde{\mathbf{W}}\right\} = \left[\tilde{\mathbf{B}}^{(s)}\right]\{\tilde{\mathbf{W}}\}, \tag{13}$$

where:

$$\left\{\tilde{\mathbf{W}}\right\}$$

$$= \left[\{W(Y_1)\}^T \quad \{W(Y_2)\}^T \quad \cdots \quad \{W(Y_m)\}^T]^T\right], \tag{14}$$

$$\left[\tilde{\mathbf{B}}^{(s)}\right] = \begin{bmatrix} B_{11}^{(s)}[\mathbf{I}^x] & B_{12}^{(s)}[\mathbf{I}^x] & \cdots & B_{1m}^{(s)}[\mathbf{I}^x] \\ B_{21}^{(s)}[\mathbf{I}^x] & B_{22}^{(s)}[\mathbf{I}^x] & \cdots & B_{2m}^{(s)}[\mathbf{I}^x] \\ \vdots & \vdots & \vdots & \vdots \\ B_{m1}^{(s)}[\mathbf{I}^x] & B_{m2}^{(s)}[\mathbf{I}^x] & \cdots & B_{mm}^{(s)}[\mathbf{I}^x] \end{bmatrix}, \tag{15}$$

where $[\mathbf{I}^x]$ is an identity matrix of order $n \times n$.

3. Governing equation and boundary conditions

The governing differential equation for free vibration of an isotropic thin rectangular plate with length a and width b can be expressed as:

$$W_{,XXXX} + 2\lambda^2 W_{,XXYY} + \lambda^4 W_{,YYYY} = \Omega^2 W, \tag{16}$$

where a subscript comma indicates partial differentiation; $W(X,Y)$ is the dimensionless mode function of the lateral deflection; $X = x/a$ and $Y = y/b$ are dimensionless coordinates; $\lambda = a/b$ is the aspect ratio; and $\Omega = \omega a^2\sqrt{\rho h/D}$ is the dimensionless frequency parameter, wherein ω, ρ, h, and D are, respectively, the circular frequency, mass density, thickness, and bending stiffness of the plate. The boundary conditions of the rectangular plate are:

(I) Simply-supported edge (S):

$$W = W_{,XX} = 0 \quad \text{at} \quad X=0 \quad \text{and/or} \quad X=1, \tag{17}$$

$$W = W_{,YY} = 0 \quad \text{at} \quad Y=0 \quad \text{and/or} \quad Y=1. \tag{18}$$

(II) Clamped edge (C):

$$W = W_{,X} = 0 \quad \text{at} \quad X=0 \quad \text{and/or} \quad X=1, \tag{19}$$

$$W = W_{,Y} = 0 \quad \text{at} \quad Y=0 \quad \text{and/or} \quad Y=1. \tag{20}$$

(III) Free edge (F):

$$W_{,XXX} + (2-\mu)\lambda^2 W_{,XYY} = W_{,XX} + \mu\lambda^2 W_{,YY} = 0, \quad \text{at} \quad X=0 \quad \text{and/or} \quad X=1, \tag{21}$$

$$W_{,YYY} + \frac{2-\mu}{\lambda^2} W_{,YXX} = W_{,YY} + \frac{\mu}{\lambda^2} W_{,XX} = 0, \quad \text{at} \quad Y=0 \quad \text{and/or} \quad Y=1, \tag{22}$$

wherein μ is the Poisson's ratio. For a free corner formed by the intersection of two free edges, the following additional condition must also be satisfied at the corner [1]:

$$W_{,XY} = 0. \tag{23}$$

4. Proposed differential quadrature methodology

The proposed methodology first reduces the original plate problem (governed by Eq. (16)) to two simple beam problems. Each beam problem is then solved using the DQM while the corresponding boundary conditions can be implemented separately. This significantly simplifies the solution procedure and its implementation as compared with the conventional procedure where the plate problem is directly solved using the DQM. The details of the proposed methodology are given in the following sub-sections.

4.1. Procedure for the solution of the first beam problem

4.1.1. DQM approximation of X-derivatives

Satisfying Eq. (16) at any sample point $X = X_i$, one has:

$$W_{,XXXX}(X_i,Y) + 2\lambda^2 W_{,XXYY}(X_i,Y) + \lambda^4 W_{,YYYY}(X_i,Y) = \Omega^2 W(X_i,Y), \quad i=1,2,\cdots,n. \tag{24}$$

Substituting the quadrature rule, given in Eq. (1), with Eq. (24) gives:

$$\sum_{j=1}^{n} A_{ij}^{(4)} W(X_j,Y) + 2\lambda^2 \sum_{j=1}^{n} A_{ij}^{(2)} W_{,YY}(X_j,Y) + \lambda^4 W_{,YYYY}(X_i,Y) = \Omega^2 W(X_i,Y), \quad i=1,2,\cdots,n. \tag{25}$$

Eq. (25) can also be expressed in matrix notation as:

$$[\mathbf{A}^{(4)}]\{\mathbf{W}(Y)\} + 2\lambda^2[\mathbf{A}^{(2)}]\{\mathbf{W}_{,YY}(Y)\} + \lambda^4[\mathbf{I}^x]\{\mathbf{W}_{,YYYY}(Y)\} = \Omega^2[\mathbf{I}^x]\{\mathbf{W}(Y)\}, \tag{26}$$

where $[\mathbf{A}^{(4)}]$ and $[\mathbf{A}^{(2)}]$ are the fourth-order and second-order DQM weighting coefficient matrices, respectively; $[\mathbf{I}^x]$ is an identity matrix of order $n \times n$; and the vector $\{\mathbf{W}(Y)\}$ has already been defined in Eq. (3).

Eq. (26) represents a system of coupled ordinary differential equations of the fourth-order, which can be further discretized using the DQM. However, it is possible to impose the X-direction boundary conditions of the plate before applying the DQM to this system. The details are given in the next subsection.

4.1.2. DQM analogs of the boundary conditions in the X-direction

The boundary conditions of the rectangular plate in the X-direction are given in Eqs. (17), (19), and (21). The corresponding quadrature analogs are detailed below:

(I) *Simply supported end condition at $X = X_p$ ($p = 1$ or n)*: From Eqs. (1) and (17), the quadrature analogs of the boundary conditions are obtained as follows:

$$W(X_p,Y) = 0, \qquad \sum_{j=1}^{n} A_{pj}^{(2)} W(X_j,Y) = 0. \tag{27}$$

(II) *Clamped end condition at $X = X_p$ ($p = 1$ or n)*: From Eqs. (1) and (19), the quadrature analogs of the boundary conditions are simply written as:

$$W(X_p, Y) = 0,$$

$$\sum_{j=1}^{n} A_{pj}^{(1)} W(X_j, Y) = 0. \tag{28}$$

(III) *Free end condition at $X = X_p$ ($p = 1$ or n)*: From Eqs. (1) and (21), the quadrature analogs of the boundary conditions are written as:

$$\sum_{j=1}^{n} A_{pj}^{(3)} W(X_j, Y) + (2-\mu)\lambda^2 \sum_{j=1}^{n} A_{pj}^{(1)} W_{,YY}(X_j, Y) = 0, \tag{29}$$

$$\sum_{j=1}^{n} A_{pj}^{(2)} W(X_j, Y) + \mu\lambda^2 W_{,YY}(X_p, Y) = 0. \tag{30}$$

4.1.3. Implementation of boundary conditions in the X-direction

At this step, the boundary conditions of the plate in the X-direction can be applied to Eq. (26). This can be done simply by direct substitution of the boundary analog equations corresponding to simply-supported, clamped, and free edges (given in Eqs. (27)-(30)) with Eq. (26). By doing so, we obtain the following system of ordinary differential equations:

$$\begin{bmatrix} [\mathbf{A}^{(4)}]_{bl} \\ [\mathbf{A}^{(4)}]_{d} \\ [\mathbf{A}^{(4)}]_{br} \end{bmatrix} \{\mathbf{W}(Y)\} + 2\lambda^2 \begin{bmatrix} [\mathbf{A}^{(2)}]_{bl} \\ [\mathbf{A}^{(2)}]_{d} \\ [\mathbf{A}^{(2)}]_{br} \end{bmatrix} \{\mathbf{W}_{,YY}(Y)\} + \lambda^4 \begin{bmatrix} [\mathbf{I}^{x}]_{bl} \\ [\mathbf{I}^{x}]_{d} \\ [\mathbf{I}^{x}]_{br} \end{bmatrix} \{\mathbf{W}_{,YYYY}(Y)\} = \Omega^2 \begin{bmatrix} [\mathbf{I}^{x}]_{bl} \\ [\mathbf{I}^{x}]_{d} \\ [\mathbf{I}^{x}]_{br} \end{bmatrix} \{\mathbf{W}(Y)\}, \tag{31}$$

where the subscripts *bl*, *d*, and *br* denote left boundary points, domain points, and right boundary points, respectively. $[\mathbf{A}^{(4)}]_{bl}$, $[\mathbf{A}^{(4)}]_{br}$, $[\mathbf{A}^{(2)}]_{bl}$, and $[\mathbf{A}^{(2)}]_{br}$ are matrices depending on the boundary conditions of the plate in the X-direction (see Appendices A and B for details); $[\mathbf{I}^x]_{bl}$ and $[\mathbf{I}^x]_{br}$ are zero matrices; and:

$$[\mathbf{A}^{(4)}]_d = \begin{bmatrix} A_{31}^{(4)} & A_{32}^{(4)} & \cdots & A_{3n}^{(4)} \\ A_{41}^{(4)} & A_{42}^{(4)} & \cdots & A_{4n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots \\ A_{(n-2)1}^{(4)} & A_{(n-2)2}^{(4)} & \cdots & A_{(n-2)n}^{(4)} \end{bmatrix},$$

$$[\mathbf{A}^{(2)}]_d = \begin{bmatrix} A_{31}^{(2)} & A_{32}^{(2)} & \cdots & A_{3n}^{(2)} \\ A_{41}^{(2)} & A_{42}^{(2)} & \cdots & A_{4n}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ A_{(n-2)1}^{(2)} & A_{(n-2)2}^{(2)} & \cdots & A_{(n-2)n}^{(2)} \end{bmatrix}, \tag{32}$$

$$[\mathbf{I}^x]_d = \begin{bmatrix} I_{31}^{x} & I_{32}^{x} & \cdots & I_{3n}^{x} \\ I_{41}^{x} & I_{42}^{x} & \cdots & I_{4n}^{x} \\ \vdots & \vdots & \vdots & \vdots \\ I_{(n-2)1}^{x} & I_{(n-2)2}^{x} & \cdots & I_{(n-2)n}^{x} \end{bmatrix}. \tag{33}$$

After eliminating the degrees of freedom related to Dirichlet-type boundary conditions (if any exist), Eq. (31) takes the form:

$$\left[\hat{\mathbf{A}}^{(4)}\right]\left\{\hat{\mathbf{W}}(Y)\right\} + 2\lambda^2 \left[\hat{\mathbf{A}}^{(2)}\right]\left\{\hat{\mathbf{W}}_{,YY}(Y)\right\} + \lambda^4 \left[\hat{\mathbf{I}}^x\right]\left\{\hat{\mathbf{W}}_{,YYYY}(Y)\right\} = \Omega^2 \left[\hat{\mathbf{I}}^x\right]\left\{\hat{\mathbf{W}}(Y)\right\}. \tag{34}$$

Eq. (34) involves the quadrature analog equations of the plate boundary conditions in the X-direction. If we denote the order of this matrix equation by n_f, it can be easily verified that $n_f = n - n_s - n_c$, wherein n_s is the number of simply supported edges in the X-direction and n_c is the number of clamped edges in the X-direction. Therefore, in general, $n_f \leq n$, where n is the size of matrix equation (26):

4.2. Procedure for the solution of the second beam problem

4.2.1. DQM approximation of Y-derivatives

Satisfying Eq. (34) at any sample point $Y = Y_i$, one has:

$$\left[\hat{\mathbf{A}}^{(4)}\right]\left\{\hat{\mathbf{W}}(Y_i)\right\} + 2\lambda^2 \left[\hat{\mathbf{A}}^{(2)}\right]\left\{\hat{\mathbf{W}}_{,YY}(Y_i)\right\} + \lambda^4 \left[\hat{\mathbf{I}}^x\right]\left\{\hat{\mathbf{W}}_{,YYYY}(Y_i)\right\} = \Omega^2 \left[\hat{\mathbf{I}}^x\right]\left\{\hat{\mathbf{W}}(Y_i)\right\},$$

$$i = 1, 2, \cdots, m. \tag{35}$$

Substituting the quadrature rule, given in Eq. (8), with Eq. (35) gives:

$$\left[\hat{\mathbf{A}}^{(4)}\right]\left\{\hat{\mathbf{W}}(Y_i)\right\}+2\lambda^2\left[\hat{\mathbf{A}}^{(2)}\right]\sum_{j=1}^{m}B_{ij}^{(2)}\left\{\hat{\mathbf{W}}(Y_j)\right\}$$

$$+\lambda^4\left[\hat{\mathbf{I}}^x\right]\sum_{j=1}^{m}B_{ij}^{(4)}\left\{\hat{\mathbf{W}}(Y_j)\right\}=\Omega^2\left[\hat{\mathbf{I}}^x\right]\left\{\hat{\mathbf{W}}(Y_i)\right\},$$

$$i=1,2,\cdots,m. \tag{36}$$

Eq. (36) can be written for all the i-values in the compact form:

$$\left[\tilde{\mathbf{K}}\right]\left\{\tilde{\mathbf{W}}\right\}=\Omega^2\left[\tilde{\mathbf{M}}\right]\left\{\tilde{\mathbf{W}}\right\}, \tag{37}$$

where $[\tilde{\mathbf{K}}]$ is the stiffness matrix, $[\tilde{\mathbf{M}}]$ is the mass matrix, and $\{\tilde{\mathbf{W}}\}$ is the unknown coefficient vector. The $n_f \times n_f$ sub-matrices $[\tilde{\mathbf{K}}_{ij}]$ and $[\tilde{\mathbf{M}}_{ij}]$ are 2 given by:

$$\left[\tilde{\mathbf{K}}_{ij}\right]=I_{ij}^y\left[\hat{\mathbf{A}5}^4\right]+2\lambda^2B_{ij}^{(2)}\left[\hat{\mathbf{A}}^{(2)}\right]+\lambda^4B_{ij}^{(4)}\left[\hat{I}^x\right],$$

$$i,j=1,2,\cdots,m, \tag{38}$$

$$\left[\tilde{\mathbf{M}}_{ij}\right]=I_{ij}^y\left[\hat{\mathbf{I}}^x\right], \tag{39}$$

where I_{ij}^y is the element of $m\times m$ identity matrix, and:

$$\left\{\tilde{\mathbf{W}}\right\}$$

$$=\left[\left\{\hat{\mathbf{W}}(Y_1)\right\}^T\ \left\{\hat{\mathbf{W}}(Y_2)\right\}^T\cdots\ \left\{\hat{\mathbf{W}}(Y_m)\right\}^T\right]^T. \tag{40}$$

It is noted that the size of mass and stiffness matrices in Eq. (37) is $mn_f \times mn_f$, where $n_f = n - n_s - n_c$. The eigenvalue problem (Eq. 37) can be solved for the eigenvalues Ω, if the boundary conditions of the plate problem in Y-direction are also applied. The procedure will be detailed in the next section.

4.2.2. DQM analogs of the boundary conditions in the Y-direction

The boundary conditions of the rectangular plate in Y-direction are given in Eqs. (18), (20), and (22). The corresponding quadrature analogs are detailed below:

(I) *Simply supported end condition at $Y=Y_q$ ($q=1$ or m)*: From Eqs. (8) and (18), the quadrature analogs of the boundary conditions are obtained as follows:

$$\left\{\hat{\mathbf{W}}(Y_q)\right\}=\{\mathbf{0}\}_{n_f\times 1},$$

$$\left\{\hat{\mathbf{W}}_{,YY}(Y_q)\right\}=\sum_{j=1}^{m}B_{qj}^{(2)}\left\{\hat{\mathbf{W}}(Y_j)\right\}=\{\mathbf{0}\}_{n_f\times 1}. \tag{41}$$

(II) *Clamped end condition at $Y=Y_q$ ($q=1$ or m)*: From Eqs. (8) and (20), the quadrature analogs of the boundary conditions are simply written as:

$$\left\{\hat{\mathbf{W}}(Y_q)\right\}=\{0\}_{n_f\times 1},$$

$$\left\{\hat{\mathbf{W}}_{,Y}(Y_q)\right\}=\sum_{j=1}^{m}B_{qj}^{(1)}\left\{\hat{\mathbf{W}}(Y_j)\right\}=\{\mathbf{0}\}_{n_f\times 1}. \tag{42}$$

(III) *Free end condition at $Y=Y_q$ ($q=1$ or m)*: Substituting the quadrature rule (given in Eq. (2)) with Eq. (22) gives:

$$\{\mathbf{W}_{,YYY}(Y_q)\}+\frac{2-\mu}{\lambda^2}[\mathbf{A}^{(2)}]\{\mathbf{W}_{,Y}(Y_q)\}$$

$$=\{\mathbf{0}\}_{n\times 1}, \tag{43}$$

$$\{\mathbf{W}_{,YY}(Y_q)\}+\frac{\mu}{\lambda^2}[\mathbf{A}^{(2)}]\{\mathbf{W}(Y_q)\}=\{\mathbf{0}\}_{n\times 1}. \tag{44}$$

By eliminating the degrees of freedom related to Dirichlet-type boundary conditions (if any exist), Eqs. (43) and (44) may be rewritten as:

$$\left\{\hat{\mathbf{W}}_{,YYY}(Y_q)\right\}+\frac{2-\mu}{\lambda^2}\left[\tilde{\mathbf{A}}^{(2)}\right]\left\{\hat{\mathbf{W}}_{,Y}(Y_q)\right\}$$

$$=\{\mathbf{0}\}_{n_f\times 1}, \tag{45}$$

$$\left\{\hat{\mathbf{W}}_{,YY}(Y_q)\right\}+\frac{\mu}{\lambda^2}\left[\tilde{\mathbf{A}}^{(2)}\right]\left\{\hat{\mathbf{W}}(Y_q)\right\}$$

$$=\{\mathbf{0}\}_{n_f\times 1}. \tag{46}$$

Now, using the quadrature rule given in Eq. (8), Eqs. (45) and (46) may be written as:

$$\sum_{j=1}^{m}B_{qj}^{(3)}\left\{\hat{\mathbf{W}}(Y_j)\right\}$$

$$+\frac{2-\mu}{\lambda^2}\left[\tilde{\mathbf{A}}^{(2)}\right]\sum_{j=1}^{m}B_{qj}^{(1)}\left\{\hat{\mathbf{W}}(Y_j)\right\}$$

$$=\{\mathbf{0}\}_{n_f\times 1}, \tag{47}$$

$$\sum_{j=1}^{m}B_{qj}^{(2)}\left\{\hat{\mathbf{W}}(Y_j)\right\}+\frac{\mu}{\lambda^2}\left[\tilde{\mathbf{A}}^{(2)}\right]\left\{\hat{\mathbf{W}}(Y_q)\right\}$$

$$=\{\mathbf{0}\}_{n_f\times 1}. \tag{48}$$

(IV) *Free corner boundary condition at (X_p, Y_q)*: As pointed out earlier, the condition $W_{,XY}=0$ must also be applied at free corners. First, we note that:

$$\frac{\partial W(X_p,Y)}{\partial X}=\sum_{j=1}^{n}A_{pj}^{(1)}W(X_j,Y)$$

$$=\left[A_{p1}^{(1)}\quad A_{p2}^{(2)}\quad\cdots\quad A_{pn}^{(1)}\right]\begin{Bmatrix}W(X_1,Y)\\W(X_2,Y)\\\vdots\\W(X_n,Y)\end{Bmatrix}$$

$$=[\mathbf{A}_p]_{1\times n}\{\mathbf{W}(Y)\}_{n\times 1},\tag{49}$$

also:

$$\frac{\partial^2 W(X_p,Y_q)}{\partial X\partial Y}=\frac{\partial}{\partial Y}\left(\frac{\partial W(X_p,Y)}{\partial X}\right)_{Y=Y_q}$$

$$=[\mathbf{A}_p]\{\mathbf{W}_{,Y}(Y_q)\}.\tag{50}$$

Now, using the quadrature rule given in Eq. (8), the quadrature analog of the corner boundary condition is obtained as:

$$\frac{\partial^2 W(X_p,Y_q)}{\partial X\partial Y}=[\mathbf{A}_p]\sum_{j=1}^{m}B_{qj}^{(1)}\{\mathbf{W}(Y_j)\}=0.\tag{51}$$

If the degrees of freedom related to Dirichlet-type boundary conditions (in the X-direction) are also eliminated, Eq. (51) can be expressed as:

$$\frac{\partial^2 W(X_p,Y_q)}{\partial X\partial Y}=\left[\hat{\mathbf{A}}_p\right]\sum_{j=1}^{m}B_{qj}^{(1)}\left\{\hat{\mathbf{W}}(Y_j)\right\}=0.\tag{52}$$

4.2.3. Implementation of boundary conditions in the Y-direction

The procedure for implementing the boundary conditions of the plate problem in Y-direction is similar to that presented in Section 4.1.3. But, the procedure for plates with free corners differs slightly from that of plates without free corners, as we will show in the following subsections.

4.2.3.1. Procedure for plates without free corners

Substituting the boundary analog equations corresponding to simply-supported, clamped, and free edges (given in Section 4.2.2) with Eq. (37) leads to the following eigenvalue equation:

$$\begin{bmatrix}\left[\tilde{\mathbf{K}}_{bl}\right]\\\left[\tilde{\mathbf{K}}_{d}\right]\\\left[\tilde{\mathbf{K}}_{br}\right]\end{bmatrix}\left\{\tilde{\mathbf{W}}\right\}=\Omega^2\begin{bmatrix}\left[\tilde{\mathbf{M}}_{bl}\right]\\\left[\tilde{\mathbf{M}}_{d}\right]\\\left[\tilde{\mathbf{M}}_{br}\right]\end{bmatrix}\left\{\tilde{\mathbf{W}}\right\},\tag{53}$$

where the subscripts bl, d, and br denote left boundary points, domain points, and right boundary points, respectively. $[\tilde{\mathbf{K}}_{bl}]$ and $[\tilde{\mathbf{K}}_{br}]$ are matrices depending on boundary conditions of the plate in Y-direction (see Appendices C and D for details); $[\tilde{\mathbf{M}}_{bl}]$ and $[\tilde{\mathbf{M}}_{br}]$ are zero matrices, and:

$$\left[\tilde{\mathbf{K}}_d\right]=\begin{bmatrix}\left[\tilde{\mathbf{K}}_{31}\right]&\left[\tilde{\mathbf{K}}_{32}\right]&\cdots&\left[\tilde{\mathbf{K}}_{3m}\right]\\\left[\tilde{\mathbf{K}}_{41}\right]&\left[\tilde{\mathbf{K}}_{42}\right]&\cdots&\left[\tilde{\mathbf{K}}_{4m}\right]\\\vdots&\vdots&\vdots&\vdots\\\left[\tilde{\mathbf{K}}_{(m-2)1}\right]&\left[\tilde{\mathbf{K}}_{(m-2)2}\right]&\cdots&\left[\tilde{\mathbf{K}}_{(m-2)m}\right]\end{bmatrix},$$

$$\left[\tilde{\mathbf{M}}_d\right]=\begin{bmatrix}\left[\tilde{\mathbf{M}}_{31}\right]&\left[\tilde{\mathbf{M}}_{32}\right]&\cdots&\left[\tilde{\mathbf{M}}_{3m}\right]\\\left[\tilde{\mathbf{M}}_{41}\right]&\left[\tilde{\mathbf{M}}_{42}\right]&\cdots&\left[\tilde{\mathbf{M}}_{4m}\right]\\\vdots&\vdots&\vdots&\vdots\\\left[\tilde{\mathbf{M}}_{(m-2)1}\right]&\left[\tilde{\mathbf{M}}_{(m-2)2}\right]&\cdots&\left[\tilde{\mathbf{M}}_{(m-2)m}\right]\end{bmatrix}.\tag{54}$$

For some cases, Eq. (53) can be directly solved for the eigenvalues. However, in general, the eigenvalue problem (Eq. (53)) is highly ill-conditioned and cannot be easily solved for the eigenvalues. A way for overcoming this issue is to eliminate the degrees of freedom corresponding to the Dirichlet-type boundary conditions. By doing so, the ill-conditioned eigenvalue problem (Eq. (53)) is converted to a well-conditioned eigenvalue problem. It is noted that the size of resulting eigenvalue equations is $m_f n_f\times m_f n_f$, where $m_f=m-m_s-m_c$ and $n_f=n-n_s-n_c$; wherein n_s is the number of simply supported edges in the X-direction, n_c is the number of clamped edges in the X-direction, m_s is the number of simply supported edges in the Y-direction, and m_c is the number of clamped edges in the Y-direction. It is also noted that the resultant mass matrix of the eigenvalue problem (Eq. (53)) involves some zero rows and hence is singular. But, such eigenvalue problem can be easily solved using the QZ algorithm [52], of which the programs and subroutines are available in most linear algebra software packages such as MATLAB and LAPACK.

4.2.3.2. Procedure for plates with free corners

The solution procedure for plates involving free corners is similar to that presented in Section 4.2.3.1. But, it involves an additional step. In this case, as it was pointed out earlier, the additional boundary analog equation (52) must also be imposed on the system of discrete equations (53).

In the very first glance, it may appear that the free corner boundary analog equation is arbitrarily substituted in the system of discrete (Eqs. (53)). In this regard, there are different choices for replacement of the quadrature analog equations of the governing differential equation by the quadrature analog equation of the free corner boundary condition. Noting that the plate has a free corner at (X_p, Y_q), a natural way is to impose the free corner boundary condition at this point (free corner point). In other words, the quadrature analog equation of the governing differential equation at (X_p, Y_q) can be replaced by the quadrature analog equation of the free corner boundary condition.

5. Numerical results

To demonstrate the stability, rate of convergence, and accuracy of the proposed DQM methodology, natural frequencies of rectangular plates with different boundary conditions are evaluated and the results are tabulated in Tables 1-5. To simplify the notation, the edge conditions for plates are denoted by letters S (simply supported), C (clamped), and F (free). For instance, SCSF denotes that the plate has a simply supported edge at $X = 0$, a clamped edge at $Y = 0$, a simply supported edge at $X = 1$, and a free edge at $Y = 1$.

For the DQM solution of the present problem, we considered a grid of $n \times m$ sampling points obtained by taking n and m points in $0 \leq X \leq 1$ and $0 \leq Y \leq 1$, respectively. Moreover, the DQM sampling points are taken non-uniformly spaced and are given by the following equations:

$$X_1 = 0, \qquad X_2 = \delta, \qquad X_{n-1} = 1-\delta, \qquad X_n = 1,$$

$$X_i = \frac{1}{2}\left[1 - \cos\left(\frac{(i-2)\pi}{n-3}\right)\right],$$

$$i = 3, 4, \cdots, n-2, \tag{55}$$

and similar equations for the Y-direction sampling points. Here, X_2 and X_{n-1} are discrete points very close to the boundary points (adjacent δ-points). The parameter δ shows the closeness of the adjacent point and the respective boundary point. In order to achieve accurate solutions by using this type of sampling points, the magnitude of δ should be as small as possible ($\leq 10^{-3}$). In this study, the magnitude of parameter δ is assumed to be $\delta = 10^{-3}$.

Table 1 shows the convergence study of the first five dimensionless natural frequencies of Levy-type square plates (i.e., plates with two opposite sides simply supported). The number of sampling points in the X and Y directions (i.e., n and m) are taken to be the same (i.e., we assumed that $n = m$). The analytical solutions of Leissa [1] are also shown in this table for comparison purposes. It can be clearly seen from Table 1 that the present results converge very quickly and agree very closely with the exact solution values of Leissa [1], even with all the available significant digits.

The first five non-dimensional frequency parameters for square plates involving free corners are tabulated in Tables 2 and 3. These results are obtained by considering an equal number of sampling points in both X and Y directions ($n = m$). The results are also compared with the results obtained by the conventional Ritz method [1], new Ritz formulation [5], Generalized Differential Quadrature e Method (GDQM) [22], and the FE-Ritz method [53]. It is noted that the results of Eftekhari and Jafari in [5,53] are believed to be highly accurate since both the geometric and natural boundary conditions of the plate are strongly satisfied in the algorithms presented in these references. Comparing the results of Table 2 with those of Table 1, it can be seen that the rate of convergence of the proposed DQM methodology for plates with free corners is not as high as those for plates without free corners. It can also be seen from Tables 2 and 3 that, in most cases, the present solutions converge to values less than those given in [1,5,22,53]. Noting that the results given in [1,5,53] are upper bounds of the analytical values, it can be concluded that the present solutions are often closer to the exact values of the natural frequencies than those in [1,5,53]. The results of Shu and Du [22] are found to be somewhat oscillatory. For instance, while some results of them [22] are very close to those of Eftekhari and Jafari in [5,53], some others do not show the same trend and do not agree well with the results of these references. This oscillatory behavior is due to the lack of satisfaction of the free corner boundary conditions in the GQDM formulation of Shu and Du [22].

To better see the convergence behavior and accuracy of the proposed DQM, the variation of the percent error in quadrature solutions (defined as $|\Omega_{\mathrm{DQM}} - \Omega_{\mathrm{Ritz}\ [5]}/\Omega_{\mathrm{Ritz}\ [5]} \times 100$) with respect to the number of sampling points is shown in Figure 1. It can be seen from Figure 1 that the solutions obtained by the present DQM, in most cases, show a monotonic convergence with increasing number of sampling points. However, the speed of convergence is very slow in some cases, particularly in the results for the fundamental frequencies of plates with CCFF, CSFF, and CFFF boundary conditions. Therefore, the proposed DQM requires a large computational cost to obtain sufficient accuracy for these cases. In solving the free vibration problem of plates with irregular geometries, Bert and Malik [54] have also reported this convergence problem and have found that this

Table 1. Convergence and comparison of natural frequencies of Levy-type square plates.

Plate	$n = m$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSSS	11	19.7392	49.3489	49.3489	78.9580	98.6268
	13	19.7392	49.3480	49.3480	78.9568	98.6948
	15	19.7392	49.3480	49.3480	78.9568	98.6961
	17	19.7392	49.3480	49.3480	78.9568	98.6960
	19	19.7392	49.3480	49.3480	78.9568	98.6960
	Exact [1]	19.7392	49.3480	49.3480	78.9568	98.6960
SCSS	11	23.6463	51.6753	58.6481	86.1364	100.2023
	13	23.6463	51.6743	58.6463	86.1345	100.2686
	15	23.6463	51.6743	58.6464	86.1345	100.2698
	17	23.6463	51.6743	58.6464	86.1345	100.2698
	19	23.6463	51.6743	58.6464	86.1345	100.2698
	Exact [1]	23.6463	51.6743	58.6464	86.1345	100.2698
SCSC	11	28.9509	54.7441	69.3317	94.5904	102.1501
	13	28.9509	54.7431	69.3271	94.5854	102.2150
	15	28.9509	54.7431	69.3270	94.5853	102.2162
	17	28.9509	54.7431	69.3270	94.5853	102.2162
	19	28.9509	54.7431	69.3270	94.5853	102.2162
	Exact [1]	28.9509	54.7431	69.3270	94.5853	102.2162
SSSF	11	11.6846	27.7570	41.2023	59.0790	61.8910
	13	11.6845	27.7564	41.1969	59.0662	61.8600
	15	11.6845	27.7563	41.1967	59.0655	61.8606
	17	11.6845	27.7563	41.1967	59.0655	61.8606
	19	11.6845	27.7563	41.1967	59.0655	61.8606
	Exact [1]	11.6845	27.7563	41.1967	59.0655	61.8606
SCSF	11	12.6874	33.0666	41.7068	63.0375	72.4757
	13	12.6874	33.0652	41.7021	63.0161	72.3948
	15	12.6874	33.0651	41.7019	63.0149	72.3977
	17	12.6874	33.0651	41.7019	63.0148	72.3976
	19	12.6874	33.0651	41.7019	63.0148	72.3976
	Exact [1]	12.6874	33.0651	41.7019	63.0148	72.3976
SFSF	11	9.6314	16.1353	36.7264	38.9483	46.7564
	13	9.6314	16.1348	36.7257	38.9450	46.7390
	15	9.6314	16.1348	36.7256	38.9450	46.7382
	17	9.6314	16.1348	36.7256	38.9450	46.7381
	19	9.6314	16.1348	36.7256	38.9450	46.7381
	Exact [1]	9.6314	16.1348	36.7256	38.9450	46.7381

Table 2. Convergence and comparison of natural frequencies of square plates involving free corners.

Plate	$n = m$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSFF	11	3.365	17.317	19.291	38.216	51.074
	13	3.365	17.316	19.289	38.208	51.035
	15	3.365	17.316	19.289	38.208	51.035
	17	3.366	17.316	19.290	38.208	51.035
	19	3.366	17.316	19.290	38.208	51.035
	21	3.366	17.316	19.290	38.208	51.035
	New Ritz [5]	3.3670	17.316	19.293	38.211	51.035
	FE-Ritz [53]	3.3670	17.3164	19.2929	38.2112	51.0354
	Conventional Ritz [1]	3.3687	17.407	19.367	38.291	51.324
	GDQM[a] [22]	3.363	17.317	19.293	38.218	51.032
CSFF	11	5.522	19.201	24.464	42.954	52.748
	13	5.463	19.168	24.524	42.964	52.714
	15	5.428	19.146	24.564	42.984	52.716
	17	5.405	19.130	24.592	43.003	52.716
	19	5.390	19.118	24.612	43.019	52.715
	21	5.379	19.109	24.626	43.032	52.714
	23	5.371	19.102	24.637	43.043	52.714
	25	5.365	19.096	24.645	43.051	52.713
	27	5.361	19.092	24.651	43.058	52.712
	29	5.358	19.088	24.655	43.063	52.711
	31	5.355	19.085	24.659	43.067	52.711
	33	5.354	19.083	24.662	43.071	52.710
	New Ritz [5]	5.351	19.075	24.671	43.088	52.707
	FE-Ritz [53]	5.3511	19.0752	24.6705	43.0876	52.7075
	Conventional Ritz [1]	5.364	19.171	24.768	43.191	53.000
	GDQM[a] [22]	5.402	19.219	25.005	43.372	52.702
CCFF	11	7.102	23.891	26.486	47.574	62.845
	13	7.048	23.883	26.514	47.547	62.727
	15	7.017	23.885	26.531	47.551	62.719
	17	6.997	23.888	26.544	47.562	62.714
	19	6.982	23.892	26.552	47.573	62.711
	21	6.971	23.894	26.559	47.584	62.709
	23	6.962	23.896	26.563	47.592	62.708
	25	6.955	23.897	26.567	47.600	62.707
	27	6.950	23.898	26.570	47.606	62.707
	29	6.946	23.899	26.572	47.611	62.706
	31	6.942	23.900	26.574	47.615	62.706
	33	6.939	23.900	26.575	47.619	62.706
	New Ritz [5]	6.919	23.904	26.585	47.651	62.706
	FE-Ritz [53]	6.9195	23.9040	26.5851	47.6519	62.7063
	Conventional Ritz [1]	6.942	24.034	26.681	47.785	63.039
	GDQM[a] [22]	6.982	24.193	26.683	47.909	62.489

[a]: Solutions with stretched sampling points.

Table 3. Convergence and comparison of natural frequencies of square plates involving free corners.

Plate	$n = m$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SFFF	11	6.638	14.902	25.378	25.997	48.466
	13	6.639	14.901	25.371	25.995	48.445
	15	6.640	14.901	25.370	25.996	48.447
	17	6.641	14.901	25.371	25.996	48.448
	19	6.641	14.901	25.371	25.997	48.448
	21	6.642	14.901	25.372	25.998	48.449
	23	6.642	14.901	25.372	25.998	48.449
	New Ritz [5]	6.6437	14.9015	25.3757	26.0005	48.4495
	FE-Ritz [53]	6.6437	14.9015	25.3757	26.0005	48.4495
	Conventional Ritz [1]	6.6480	15.023	25.492	26.126	48.711
	GDQM[a] [22]	6.636	14.901	25.388	26.003	48.469
CFFF	11	3.606	8.780	21.121	27.334	30.666
	13	3.545	8.710	21.174	27.307	30.715
	15	3.513	8.660	21.207	27.287	30.764
	17	3.495	8.624	21.228	27.271	30.805
	19	3.484	8.597	21.243	27.258	30.838
	21	3.477	8.576	21.254	27.247	30.863
	23	3.472	8.560	21.261	27.238	30.882
	25	3.470	8.548	21.267	27.231	30.897
	27	3.468	8.539	21.271	27.225	30.909
	29	3.467	8.531	21.274	27.221	30.918
	31	3.466	8.525	21.276	27.217	30.926
	33	3.466	8.521	21.278	27.214	30.932
	New Ritz [5]	3.4712	8.5074	21.2864	27.1990	30.9590
	FE-Ritz [53]	3.4711	8.5067	21.2850	27.1989	30.9563
	Conventional Ritz [1]	3.9417	8.5246	21.429	27.331	31.111
	GDQM[a] [22]	3.485	8.604	21.586	27.230	31.358
FFFF	11	13.458	19.595	24.269	34.802	34.817
	13	13.457	19.596	24.267	34.788	34.797
	15	13.458	19.596	24.267	34.788	34.794
	17	13.460	19.596	24.267	34.789	34.794
	19	13.461	19.596	24.268	34.791	34.794
	21	13.462	19.596	24.268	34.792	34.795
	23	13.463	19.596	24.268	34.793	34.795
	25	13.463	19.596	24.268	34.794	34.796
	27	13.464	19.596	24.269	34.795	34.796
	29	13.464	19.596	24.269	34.796	34.797
	31	13.465	19.596	24.269	34.796	34.797
	33	13.465	19.596	24.269	34.797	34.797
	New Ritz [5]	13.4682	19.5961	24.2702	34.8009	34.8009
	FE-Ritz [53]	13.4682	19.5961	24.2702	34.8009	34.8009
	Conventional Ritz [1]	13.489	19.789	24.432	35.024	35.024
	GDQM[a] [22]	13.454	19.597	24.271	34.815	34.817

[a]: Solutions with stretched sampling points.

Table 4. Convergence and comparison of natural frequencies of square plates with free corners (when non-uniform sampling points without adjacent δ-points are used in the algorithm).

Plate	Method	$n = m$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSFF	Present	15	3.403	17.316	19.388	38.319	51.039
		21	3.392	17.316	19.359	38.286	51.035
		27	3.385	17.316	19.341	38.266	51.035
		33	3.381	17.316	19.329	38.253	51.035
		37	3.379	17.316	19.324	38.247	51.035
		41	3.377	17.316	19.320	38.242	51.035
	New Ritz [5]		3.3670	17.316	19.293	38.211	51.035
	GDQM [22]	15	2.549	17.316	17.662	36.576	51.039
CCFF	Present	15	7.443	23.961	26.482	47.485	62.722
		21	7.280	23.900	26.525	47.431	62.707
		27	7.188	23.893	26.547	47.440	62.708
		33	7.125	23.894	26.564	47.463	62.709
		37	7.092	23.895	26.572	47.480	62.708
		41	7.065	23.896	26.580	47.496	62.708
	New Ritz [5]		6.919	23.904	26.585	47.651	62.706
	GDQM [22]	15	7.873	23.615	23.873	44.587	62.730
CSFF	Present	15	5.960	19.675	24.029	42.745	52.810
		21	5.767	19.525	24.190	42.714	52.802
		27	5.638	19.417	24.327	42.763	52.795
		33	5.550	19.338	24.425	42.819	52.787
		37	5.507	19.298	24.475	42.853	52.781
		41	5.472	19.265	24.514	42.884	52.776
	New Ritz [5]		5.351	19.075	24.671	43.088	52.707
	GDQM [22]	15	5.780	20.703	20.926	40.296	52.255
SFFF	Present	15	6.705	14.909	25.505	26.129	48.474
		21	6.691	14.908	25.476	26.077	48.463
		27	6.679	14.906	25.452	26.053	48.458
		33	6.671	14.905	25.435	26.040	48.456
		37	6.667	14.904	25.427	26.034	48.455
		41	6.664	14.904	25.421	26.029	48.454
	New Ritz [5]		6.6437	14.9015	25.3757	26.0005	48.4495
	GDQM [22]	15	5.161	14.725	23.082	24.156	46.296
CFFF	Present	15	3.893	9.687	20.934	27.780	30.032
		21	3.695	9.413	20.995	27.710	30.139
		27	3.583	9.192	21.067	27.635	30.311
		33	3.517	9.027	21.122	27.568	30.457
		37	3.487	8.942	21.151	27.529	30.536
		41	3.466	8.871	21.174	27.494	30.603
	New Ritz [5]		3.4712	8.5074	21.2864	27.1990	30.9590
	GDQM [22]	15	3.898	9.459	20.206	26.150	26.500
FFFF	Present	15	13.668	19.596	24.379	35.016	35.196
		21	13.616	19.596	24.342	34.978	35.037
		27	13.578	19.596	24.321	34.939	34.962
		33	13.553	19.596	24.308	34.909	34.920
		37	13.541	19.596	24.302	34.895	34.902
		41	13.532	19.596	24.298	34.883	34.887
	New Ritz [5]		13.4682	19.5961	24.2702	34.8009	34.8009
	GDQM [22]	15	10.303	19.596	22.146	30.026	30.803

Table 5. Percent error in solutions of two DQ approaches (present and GDQM) for natural frequencies of square plates with free corners (when non-uniform sampling points without adjacent δ-points are used in the algorithms).

Plate	Method	$n = m$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSFF	Present	15	1.0685	0.0025	0.4942	0.2835	0.0085
	GDQM [22]	15	24.2946	0	8.4538	4.2789	0.0078
CCFF	Present	15	7.5800	0.2397	0.3877	0.3480	0.0255
	GDQM [22]	15	13.7881	1.2090	10.2012	6.4301	0.0383
CSFF	Present	15	11.3895	3.1438	2.6028	0.7957	0.1956
	GDQM [22]	15	8.0172	8.5347	15.1798	6.4798	0.8576
SFFF	Present	15	0.9272	0.0519	0.5094	0.4930	0.0503
	GDQM [22]	15	22.3174	1.1844	9.0390	7.0941	4.4448
CFFF	Present	15	12.1652	13.8716	1.6528	2.1369	2.9933
	GDQM [22]	15	12.2955	11.1856	5.0755	3.8568	14.4029
FFFF	Present	15	1.4881	0.0005	0.4495	0.6194	1.1355
	GDQM [22]	15	23.5013	0.0005	8.7523	13.7206	11.4879

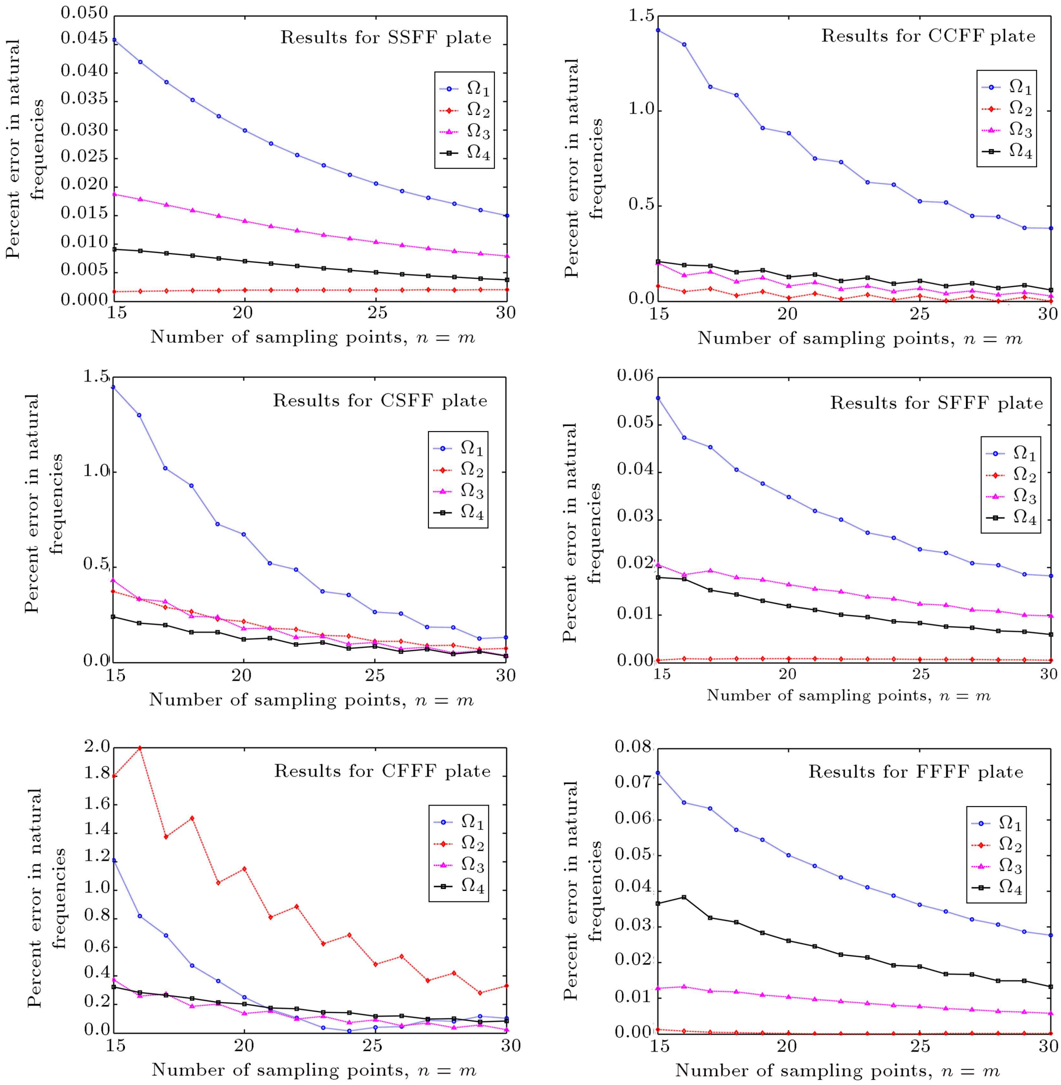

Figure 1. Convergence and accuracy of the first four natural frequencies of square plates involving free corners.

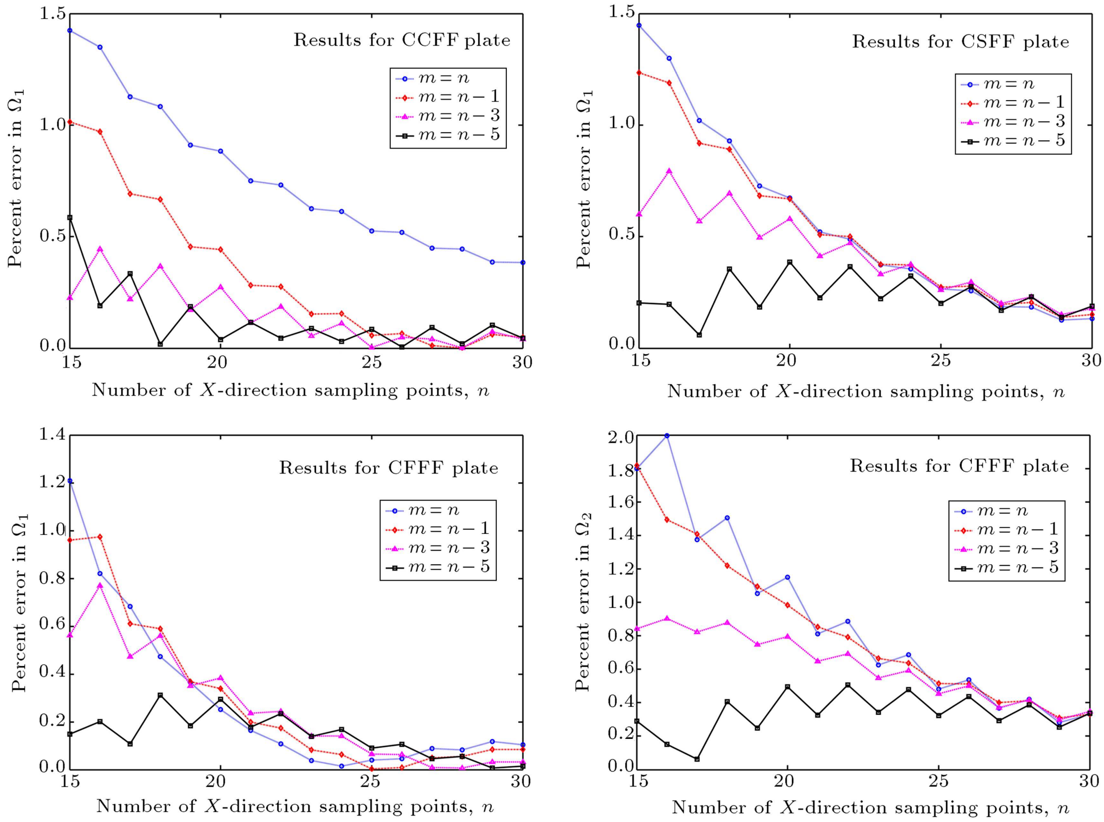

Figure 2. Convergence behavior and accuracy of natural frequencies of square plates with CCFF, CSFF, and CFFF boundary conditions.

difficulty is caused when an equal number of sampling points is considered in both coordinate directions of the plate. They have also shown that the convergence rate of the DQM is greatly enhanced by selecting an unequal number of sampling points in each coordinate direction of the plate. Our numerical experiments also confirmed this numerical observation for rectangular plates involving free corners and showed that, depending on the boundary conditions of the plate, the accuracy of numerical results could be improved when n value was considered to be smaller or larger than m value. Figure 2 presents the numerical results for plates with CCFF, CSFF, and CFFF boundary conditions. It can be seen that better accuracy and convergence rate can be achieved when n value is chosen to be larger than m value. The numerical results for plates with SSFF, SFFF and FFFF boundary conditions are shown in Figure 3. It can be seen that in these cases, better accuracy and convergence rate are achieved by selecting n value smaller than m value.

As pointed out earlier in introduction, Shu and Du [22] reported that the solutions of the GDQM are very sensitive to the sampling point distributions when the plate under investigation involves some free corners. For instance, their procedure led to erroneous results when using the following type of sampling points:

$$X_i = \frac{1}{2}\left[1 - \cos\left(\frac{(i-1)\pi}{n-1}\right)\right], \quad i = 1, 2, \cdots, n, \tag{56}$$

$$Y_i = \frac{1}{2}\left[1 - \cos\left(\frac{(i-1)\pi}{m-1}\right)\right], \quad i = 1, 2, \cdots, m. \tag{57}$$

To investigate the effect of sampling point distribution, we also solved the present problem using the above type of sampling points. Table 4 shows the convergence of solutions for the first five natural frequencies of square plates involving free corners when the coordinates of the sampling points are computed from Eqs. (56) and (57). The GDQM solution results of Shu and Du [22] are also shown for comparison purposes. Similar to Shu and Du [22], we considered an equal number of sampling points in both coordinate directions of the plate ($n = m$). It can be seen that the results of present method show a monotonic convergence with respect to the number of sampling points while the numerical results of Shu and Du [22] do not exhibit any convergence trend for these cases. This implies that the solutions of the proposed method for plates involving free corners are not highly sensitive to the sampling point distribution. Table 5 presents a fair comparison of the present DQM and the GDQM when a mesh size of 15 × 15 is used. Note that

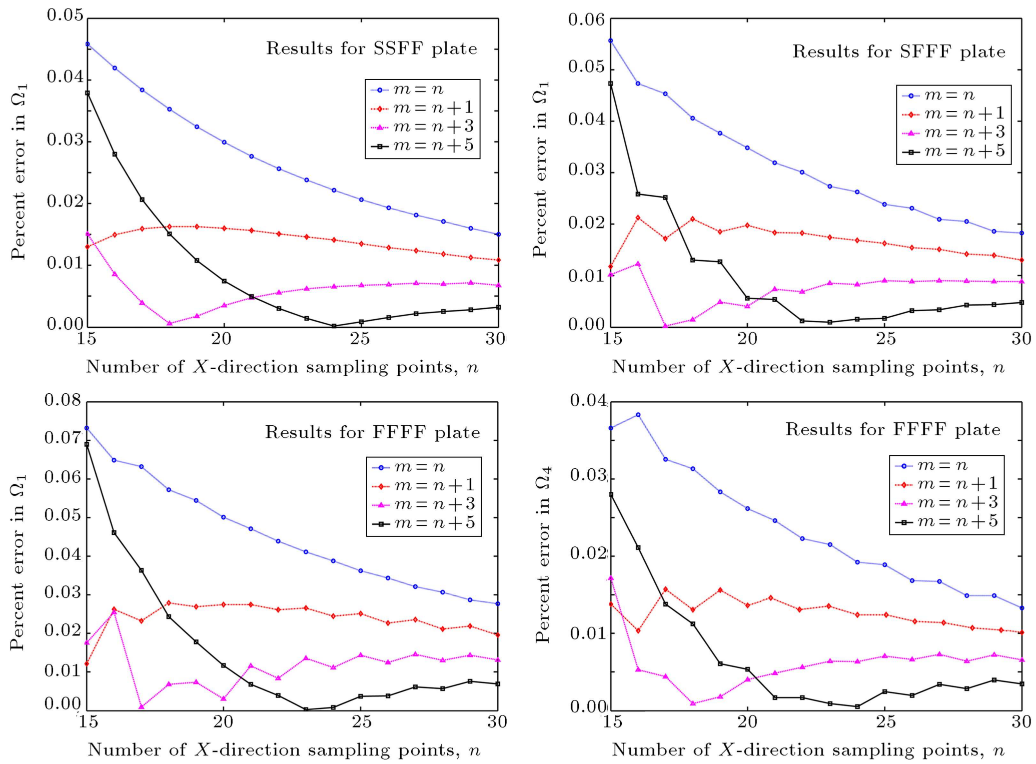

Figure 3. Convergence behavior and accuracy of natural frequencies of square plates with SSFF, SFFF, and FFFF boundary conditions.

the percent error in quadrature solutions (defined as $|\Omega_{\text{DQM}}-\Omega_{\text{Ritz [5]}}|/\Omega_{\text{Ritz [5]}}\times 100$) is shown in this table. Needless to say that better accuracy is achieved by the proposed DQM.

6. Conclusions

A simple and accurate differential quadrature formulation is developed to study the free vibration of rectangular plates. The proposed formulation reduces the original plate problem to two simple beam problems whose solution procedure is significantly simpler than the case where the conventional DQM is fully applied to the plate problem. A simple scheme is also proposed to implement the free edge and free corner boundary conditions of the plate problem. It is revealed that the proposed method can produce much better accuracy than the GDQM for plates involving free corners. Unlike the GDQM, the solutions of the proposed method for plates with free corners are not very sensitive to the sampling point distribution.

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Appendix A

Elements of matrices $[\mathbf{A}^{(4)}]_{bl}$ ***and*** $[\mathbf{A}^{(2)}]_{bl}$

The elements of matrices $[\mathbf{A}^{(4)}]_{bl}$ and $[\mathbf{A}^{(2)}]_{bl}$ depend on the boundary conditions of the plate in the X-direction and can be obtained from quadrature analog equations given in Section 4.1.2 as follows:

(I) *Simply supported end condition at* $X = X_1$:

$$[\mathbf{A}^{(4)}]_{bl}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ A_{11}^{(2)} & A_{12}^{(2)} & A_{13}^{(2)} & \cdots & A_{1n}^{(2)} \end{bmatrix}, \tag{A.1}$$

$$[\mathbf{A}^{(2)}]_{bl}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{A.2}$$

(II) *Clamped end condition at* $X = X_1$:

$$[\mathbf{A}^{(4)}]_{bl}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} & \cdots & A_{1n}^{(1)} \end{bmatrix}, \tag{A.3}$$

$$[\mathbf{A}^{(2)}]_{bl}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{A.4}$$

(III) *Free end condition at* $X = X_1$:

$$[\mathbf{A}^{(4)}]_{bl}=\begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} & \cdots & A_{1n}^{(2)} \\ A_{11}^{(3)} & A_{12}^{(3)} & \cdots & A_{1n}^{(3)} \end{bmatrix}, \tag{A.5}$$

$$[\mathbf{A}^{(2)}]_{bl}=\frac{1}{2}\begin{bmatrix} \mu & 0 & 0 & \cdots & 0 \\ (2-\mu)A_{11}^{(1)} & (2-\mu)A_{12}^{(1)} & (2-\mu)A_{13}^{(1)} & \cdots & (2-\mu)A_{1n}^{(1)} \end{bmatrix}. \tag{A.6}$$

Appendix B

Elements of matrices $[\mathbf{A}^{(4)}]_{br}$ and $[\mathbf{A}^{(2)}]_{br}$

The elements of matrices $[\mathbf{A}^{(4)}]_{br}$ and $[\mathbf{A}^{(2)}]_{br}$ can also be determined from quadrature analog equations given in Section 4.1.2 as follows:

(I) *Simply supported end condition at $X = X_n$:*

$$[\mathbf{A}^{(4)}]_{br} = \begin{bmatrix} A_{n1}^{(2)} & A_{n2}^{(2)} & \cdots & A_{n(n-1)}^{(2)} & A_{nn}^{(2)} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{B.1}$$

$$[\mathbf{A}^{(2)}]_{br} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{B.2}$$

(II) *Clamped end condition at $X = X_n$:*

$$[\mathbf{A}^{(4)}]_{br} = \begin{bmatrix} A_{n1}^{(1)} & A_{n2}^{(1)} & \cdots & A_{n(n-1)}^{(1)} & A_{nn}^{(1)} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{B.3}$$

$$[\mathbf{A}^{(2)}]_{br} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{B.4}$$

(III) *Free end condition at $X = X_n$:*

$$[\mathbf{A}^{(4)}]_{br} = \begin{bmatrix} A_{n1}^{(3)} & A_{n2}^{(3)} & \cdots & A_{nn}^{(3)} \\ A_{n1}^{(2)} & A_{n2}^{(2)} & \cdots & A_{nn}^{(2)} \end{bmatrix}, \tag{B.5}$$

$$[\mathbf{A}^{(2)}]_{br} = \frac{1}{2} \begin{bmatrix} (2-\mu)A_{n1}^{(1)} & (2-\mu)A_{n2}^{(1)} & \cdots & (2-\mu)A_{n(n-1)}^{(1)} & (2-\mu)A_{nn}^{(1)} \\ 0 & 0 & \cdots & 0 & \mu \end{bmatrix}. \tag{B.6}$$

Appendix C

Elements of matrix $[\tilde{\mathbf{K}}_{bl}]$

The elements of matrix $[\tilde{\mathbf{K}}_{bl}]$ depend on the boundary conditions of the plate in Y-direction and can be obtained from quadrature analog equations given in Section 4.2.2 as follows:

(I) *Simply supported end condition at $Y = Y_1$:*

$$\left[\tilde{\mathbf{K}}_{bl}\right] = \begin{bmatrix} \left[\hat{\mathbf{I}}\right] & [\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ B_{11}^{(2)}\left[\hat{\mathbf{I}}\right] & B_{12}^{(2)}\left[\hat{\mathbf{I}}\right] & B_{13}^{(2)}\left[\hat{\mathbf{I}}\right] & \cdots & B_{1m}^{(2)}\left[\hat{\mathbf{I}}\right] \end{bmatrix}. \tag{C.1}$$

(II) *Clamped end condition at $Y = Y_1$:*

$$\left[\tilde{\mathbf{K}}_{bl}\right] = \begin{bmatrix} \left[\hat{\mathbf{I}}\right] & [0] & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ B_{11}^{(1)}\left[\hat{\mathbf{I}}\right] & B_{12}^{(1)}\left[\hat{\mathbf{I}}\right] & B_{13}^{(1)}\left[\hat{\mathbf{I}}\right] & \cdots & B_{1m}^{(1)}\left[\hat{\mathbf{I}}\right] \end{bmatrix}. \tag{C.2}$$

(III) *Free end condition at $Y = Y_1$:*

$$\left[\tilde{\mathbf{K}}_{bl}\right] = \begin{bmatrix} B_{11}^{(2)}\left[\hat{\mathbf{I}}\right] & B_{12}^{(2)}\left[\hat{\mathbf{I}}\right] & \cdots & B_{1m}^{(2)}\left[\hat{\mathbf{I}}\right] \\ B_{11}^{(3)}\left[\hat{\mathbf{I}}\right] & B_{12}^{(3)}\left[\hat{\mathbf{I}}\right] & \cdots & B_{1m}^{(3)}\left[\hat{\mathbf{I}}\right] \end{bmatrix} + \frac{1}{\lambda^2} \begin{bmatrix} \mu\left[\tilde{\mathbf{A}}^{(2)}\right] & [\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ (2-\mu)B_{11}^{(1)}\left[\tilde{\mathbf{A}}^{(2)}\right] & (2-\mu)B_{12}^{(1)}\left[\tilde{\mathbf{A}}^{(2)}\right] & (2-\mu)B_{13}^{(1)}\left[\tilde{\mathbf{A}}^{(2)}\right] & \cdots & (2-\mu)B_{1m}^{(1)}\left[\tilde{\mathbf{A}}^{(2)}\right] \end{bmatrix}, \tag{C.3}$$

where $[\tilde{\mathbf{A}}^{(2)}]$ has already been defined in Eqs. (43)-(46), $[\hat{\mathbf{I}}]$ and $[\mathbf{0}]$ are identity and zero matrices of order $n_f \times n_f$ $(n_f = n)$.

Appendix D

Elements of matrix $[\tilde{\mathbf{K}}_{br}]$

The elements of matrix $[\tilde{\mathbf{K}}_{br}]$ can also be determined from quadrature analog equations given in Section 3.1 as follows:

(I) *Simply supported end condition at $Y = Y_m$:*

$$\left[\tilde{\mathbf{K}}_{br}\right] = \begin{bmatrix} B_{m1}^{(2)}\left[\hat{\mathbf{I}}\right] & B_{m2}^{(2)}\left[\hat{\mathbf{I}}\right] & \cdots & B_{m(m-1)}^{(2)}\left[\hat{\mathbf{I}}\right] & B_{mm}^{(2)}\left[\hat{\mathbf{I}}\right] \\ [\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{0}] & \left[\hat{\mathbf{I}}\right] \end{bmatrix}. \tag{D.1}$$

(II) *Clamped end condition at $Y = Y_m$:*

$$\left[\tilde{\mathbf{K}}_{br}\right] = \begin{bmatrix} B_{m1}^{(1)}\left[\hat{\mathbf{I}}\right] & B_{m2}^{(1)}\left[\hat{\mathbf{I}}\right] & \cdots & B_{m(m-1)}^{(1)}\left[\hat{\mathbf{I}}\right] & B_{mm}^{(1)}\left[\hat{\mathbf{I}}\right] \\ [\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{0}] & \left[\hat{\mathbf{I}}\right] \end{bmatrix}. \tag{D.2}$$

(III) Free end condition at $Y = Y_m$

$$\left[\tilde{\mathbf{K}}_{br}\right]=\begin{bmatrix} B_{m1}^{(3)}\left[\hat{\mathbf{I}}\right] & B_{m2}^{(3)}\left[\hat{\mathbf{I}}\right] & \cdots & B_{mm}^{(3)}\left[\hat{\mathbf{I}}\right] \\ B_{m1}^{(2)}\left[\hat{\mathbf{I}}\right] & B_{m2}^{(2)}\left[\hat{\mathbf{I}}\right] & \cdots & B_{mm}^{(2)}\left[\hat{\mathbf{I}}\right] \end{bmatrix}$$

$$+\frac{1}{\lambda^2}\begin{bmatrix} (2-\mu)B_{m1}^{(1)}\left[\tilde{\mathbf{A}}^{(2)}\right] & (2-\mu)B_{m2}^{(1)}\left[\tilde{\mathbf{A}}^{(2)}\right] & \cdots & (2-\mu)B_{m(m-1)}^{(1)}\left[\tilde{\mathbf{A}}^{(2)}\right] & (2-\mu)B_{mm}^{(1)}\left[\tilde{\mathbf{A}}^{(2)}\right] \\ [\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{0}] & \mu\left[\tilde{\mathbf{A}}^{(2)}\right] \end{bmatrix}, \tag{D.3}$$

where the matrices $[\tilde{\mathbf{A}}^{(2)}]$, $[\hat{\mathbf{I}}]$, and $[\mathbf{0}]$ have already been defined in Appendix C.

Biographies

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