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Interval-valued trapezoidal intuitionistic fuzzy generalized aggregation operators and application to multi-attribute group decision making

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Abstract. An Interval-Valued Trapezoidal Intuitionistic Fuzzy Number (IVTrIFN) is a special case of an Intuitionistic Fuzzy Set (IFS), which is defined on a real number set. From a geometric viewpoint, the expectation and expectant score of an IVTrIFN are defined using the notion of a barycenter, and a new method is developed to rank IVTrIFNs. Hereby, some generalized aggregation operators of IVTrIFNs are defined, including the generalized ordered weighted averaging operator and the generalized hybrid weighted averaging operator, which are employed to solve multi-attribute group decision making problems. Using the weighted average operator of IVTrIFNs, the attribute values of alternatives are integrated into the individual comprehensive ratings, which are further aggregated into the collective one by the generalized hybrid weighted averaging operator of IVTrIFNs. The ranking orders of alternatives are then generated according to the expectation and expectant score of the collective comprehensive ratings of alternatives. A numerical example is examined to demonstrate the applicability and implementation process of the decision method proposed in this paper.

1. Introduction

Atanassov [1] introduced the Intuitionistic Fuzzy Set (IFS), which is a generalization of the fuzzy set [2]. The IFS has received considerable attention since its appearance. Gau and Buehrer [3] proposed the notion of vague set, which is identified with that of IFS, as pointed out by Bustince and Burillo [4]. Atanassov and Gargov [5] further generalized the IFS in the spirit of the ordinary Interval-Valued Fuzzy Set (IVFS), and defined the notion of an Interval-Valued Intuitionistic Fuzzy Set (IVIFS). There is much research regarding IFS and IVIFS in the applications of Multi-Attribute Decision Making (MADM) [6-10] and Multi-Attribute Group Decision Making (MAGDM) [11-19].

Fuzzy numbers are a special case of fuzzy sets. As a generalization of fuzzy numbers [20], an Intuitionistic Fuzzy Number (IFN) seems to suitably describe an ill-known quantity [21-34]. Shu et al. [22] defined the concept of a Triangular IFN (TIFN) in a similar way to that of the fuzzy number [20] and developed an algorithm for intuitionistic fuzzy fault tree analysis. Li [21] pointed out and corrected some errors in the definition of the four arithmetic operations over the TIFNs in [22]. Li [23] discussed the concept of the TIFN and ranking method on the basis of the concept of a ratio of the value index to the ambiguity index, as well as applications to MADM problems in depth. Li et al. [24] proposed a value and ambiguity

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based method to rank TIFNs and applied it to solve MADM problems, in which the ratings of alternatives on attributes are expressed using TIFNs. Nan et al. [25] defined the ranking order relations of TIFNs and investigated the matrix games with payoffs of TIFNs. Chen and Li [26] constructed a dynamic MADM model based on TIFNs. Wan et al. [27] introduced the possibility mean, variance and covariance of TIFNs. Subsequently, Wan and Li [28] developed a possibility mean and variance based method for MADM with TIFNs. Further, Wan [29] proposed the possibility variance coefficient method for MADM with TIFNs. Wan et al. [30] extended the classical VIKOR method for solving MAGDM with TIFNs. Wang et al. [31] proposed new arithmetic operations and logic operators for TIFNs and applied them to fault analysis of a printed circuit board assembly system. Dong and Wan [32] investigated a new method for MAGDM with TIFNs. Wan and Dong [33] defined Choquet integral operator of TIFNs and applied to MADM with TIFNs. Wan and Dong [34] proposed the possibility method for MAGDM with TIFNs and incomplete weight information.

In a similar way to TIFNs, Wang [35] defined the Trapezoidal IFN (TrIFN) and Interval-Valued Trapezoidal IFN (IVTrIFN). Both TrIFN and IVTrIFN are extensions of TIFNs. Wang and Zhang [36] investigated the weighted arithmetic averaging operator and weighted geometric averaging operator on TrIFNs and their applications to MADM problems. Wei [37] investigated some arithmetic aggregation operators with TrIFNs and their applications to MAGDM problems. Du and Liu [38] extended the fuzzy VIKOR method with TrIFNs. Wu and Cao [39] developed some families of geometric aggregation operators with TrIFNs and applied them to MAGDM problems. Wan and Dong [40] defined the expectation and expectant score, ordered weighted aggregation operator and hybrid aggregation operator for TrIFNs and employed them for MAGDM. Ye [41] developed the expected value method for intuitionistic trapezoidal fuzzy multicriteria decision-making problems. Ye [42] proposed the MAGDM method using vector similarity measures for TrIFNs. Wan [43] developed four kinds of power average operator of TrIFNs, involving the power average operator, weighted power average operator of, power order weighted average operator of, and power hybrid average operator of TrIFNs. Wan [44] firstly defined some operational laws and the weighted arithmetical average operator of IVTrIFNs. Based on the score function and accurate function, an approach is presented to rank IVTrIFNs. The MAGDM method using IVTrIFNs is then proposed. Wan [45] constructed nonlinear fractional programming models to estimate the alternative's relative closeness. After transformation into linear programming models, the interval of the alternative's relative closeness is obtained through solving these linear programming models. Then, the fractional programming method is proposed for the MADM with IVTrIFNs. Wu and Liu [46] defined some interval-valued trapezoidal intuitionistic fuzzy geometric aggregation operators and applied to them MAGDM with IVTrIFNs. Wei et al. [47] defined the interval trapezoidal intuitionistic fuzzy ordered weighted geometric operator and the interval trapezoidal intuitionistic fuzzy hybrid geometric operator. An approach based on these operators is developed to solve the MAGDM problems with IVTrIFNs.

It is worthwhile to mention that the domains of the IFS and IVIFS are discrete sets, which are also the same as fuzzy sets. TIFN, TrIFN and IVTrIFN extend the domain of IFS from the discrete set to the continuous set. They are the extensions of fuzzy numbers [36]. Compared with the IFS, both TrIFN and IVTrIFN are defined by using trapezoidal fuzzy numbers expressing their membership and non-membership functions. Hence, TrIFN and IVTrIFN may better reflect the information of decision problems than IFS. Some practical decision problems generally involve multiple attributes with different physical dimensions and units, such as risk investment and performance evaluation of military systems, as well as partner selection of supply chain management. For example, during the process of evaluating the supplier quality of the product, Decision Makers (DMs) have some hesitancy or lack of knowledge, due to the knowledge structure and specialty, as well as the complexity of the evaluation object. The evaluation results often present three aspects: affirmation, negation, and hesitancy, between the affirmation and negation. Thus, DMs may give an evaluation in the format of IVTrIFN ([1-4]; [0.6,0.8], [0.1,0.2]), which means that the most possible value is [2,3], and the lower and upper limits are 1 and 4, respectively. Meanwhile, the maximum membership degree for the most possible value [2,3] is between 0.6 and 0.8, the minimum non-membership degree for the most possible value [2,3] is between 0.1 and 0.2, and the indeterminacy is between 0 and 0.3. An IVTrIFN, described using three characteristic functions: membership degree, non-membership degree and hesitancy degree, is just a strong instrument to represent the information of these three aspects. The ability of an IVTrIFN to capture vagueness and uncertainty is stronger than that of the TIFN and TrIFN. The IVTrIFN can subtly and effectively describe the decision information with different dimensions and units. Therefore, IVTrIFNs are of great importance in scientific research and real life applications. Although Wan [44], Wu and Liu [46], and Wei et al. [47] investigated some arithmetic aggregation operators and geometric aggregation operators for IVTrIFNs, there exists no investigation about the generalized aggrega-

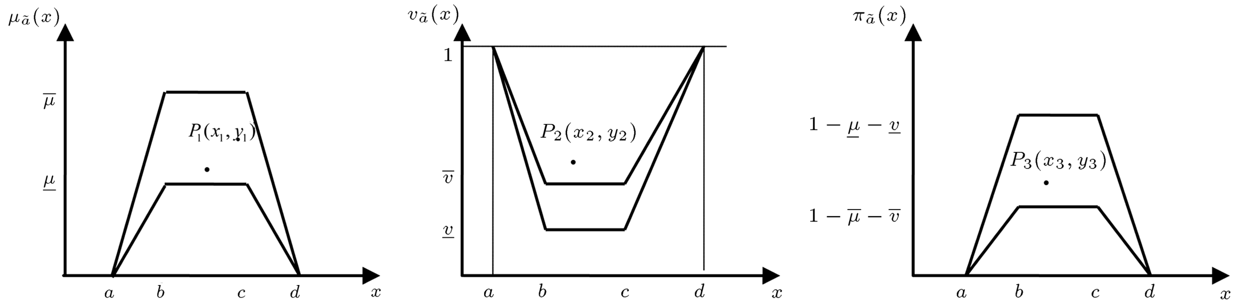

Figure 1. Membership function, non-membership function and hesitancy function for IVTrIFN.

tion operators for IVTrIFNs. The aim of this paper is to study the expectation and expectant score and new ranking method for IVTrIFNs from the geometry point of view, and develop some generalized aggregation operator of IVTrIFNs, as well as their application to MAGDM problems. The main contributions of this paper are as follows:

(i) A new ranking method for IVTrIFNs is developed from a geometric viewpoint, which is very intuitive, simple and effective;

(ii) The generalized ordered weighted average operator for IVTrIFNs and the generalized hybrid weighted averaging operator of IVTrIFNs are investigated and the desirable properties are discussed. These operators are helpful extensions of the intuitionistic fuzzy aggregation operators;

(iii) The MAGDM method proposed in this paper enriches the research content and decision approaches for MAGDM in intuitionistic fuzzy environments.

This paper is structured as follows. Section 2 introduces the definition and operation laws of IVTrIFNs and presents their new ranking method. Section 3 develops some generalized aggregation operators over IVTrIFNs. Section 4 gives the problem description of MAGDM with IVTrIFNs and proposes the corresponding group decision making method. A numerical example and comparison analysis are given in Section 5. Concluding remarks are given in Section 6.

2. Interval-valued trapezoidal intuitionistic fuzzy numbers and barycenter based ranking method

In this section, the definition, operation laws and properties for IVTrIFNs are reviewed. Then, a ranking method of IVTrIFNs is proposed based on barycenter.

2.1. Operation laws and properties for IVTrIFNs

Definition 1 [35]. Let $\tilde a$ be an IFN in the set of real numbers whose membership function and non-membership function are defined as:

$$\mu_{\tilde a} = \begin{cases} \frac{x-a}{b-a}\mu, & \text{if} \quad a \le x < b \\ \mu, & \text{if} \quad b \le x \le c \\ \frac{d-x}{d-c}\mu, & \text{if} \quad c < x \le d \\ 0, & \text{else} \end{cases}$$

and:

$$v_{\tilde a}(x) = \begin{cases} \frac{b-x+(x-a)v}{b-a}, & \text{if} \quad a \le x < b \\ v, & \text{if} \quad b \le x \le c \\ \frac{x-c+(d-x)v}{d-c}, & \text{if} \quad c < x \le b \\ 1, & \text{else} \end{cases}$$

respectively (see Figure 1), where a, b, c, and d are all real numbers, $\mu = [\underline{\mu}, \overline{\mu}]$ and $v = [\underline{v}, \overline{v}]$ are intervals, which represent the maximum membership degree interval and the minimum non-membership degree interval, respectively, such that they satisfy the following conditions:

$$0 \le \underline{\mu} \le 1, \quad 0 \le \overline{\mu} \le 1, \quad 0 \le \underline{v} \le 1,$$

$$0 \le \overline{v} \le 1 \quad \text{and} \quad \overline{\mu} + \overline{v} \le 1.$$

Then, the IFN $\tilde a$ is called IVTrIFN, denoted by $\tilde a = \big([a,b,c,d]; [\underline{\mu}, \overline{\mu}], [\underline{v}, \overline{v}]\big)$. Let the function $\pi_{\tilde a}(x) = 1 - \mu_{\tilde a}(x) - v_{\tilde a}(x)$ denote the hesitation of $\tilde a$. The smaller $\pi_{\tilde a}(x)$, the more certain $\tilde a$.

If $a \ge 0$ and one of the four values, a, b, c and d, is not equal to 0, then the IVTrIFN, $\tilde a$, is called a positive IVTrIFN, denoted by $\tilde a > 0$. Likewise, if $d \le 0$ and one of the four values, a, b, c and d, is not equal to 0, then the IVTrIFN, $\tilde a$, is called a negative IVTrIFN, denoted by $\tilde a < 0$. If $\underline{\mu} = \overline{\mu}$ and $\underline{v} = \overline{v}$, then, the IVTrIFN, $\tilde a$, is called the TrIFN. When $b = c$, a TrIFN reduces to a TIFN. Therefore, both TrIFN and TIFN are special cases of the IVTrIFN.

For example, there is an IVTrIFN $\tilde 6$ =([4, 5, 7, 8]; [0.5, 0.7], [0.1, 0.2]). Then, when $x = 5$, its membership degree being an IVTrIFN $\tilde 6$ is [0.5, 0.7], its non-membership degree not being an IVTrIFN $\tilde 6$ is [0.1, 0.2], and its hesitation being or not being an IVTrIFN $\tilde 6$ is [0.1, 0.4].

Definition 2 [44]. Let:

$$\tilde{a}_1 = \left([a_1, b_1, c_1, d_1]; [\underline{\mu}_1, \overline{\mu}_1], [\underline{v}_1, \overline{v}_1]\right),$$

and:

$$\tilde{a}_2 = \left([a_2, b_2, c_2, d_2]; [\underline{\mu}_2, \overline{\mu}_2], [\underline{v}_2, \overline{v}_2]\right),$$

be two IVTrIFNs and $\lambda \geq 0$. Then the operation laws for IVTrIFNs are defined as follows:

1. $\tilde{a}_1 + \tilde{a}_2 = \Big([a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; [\underline{\mu}_1 + \underline{\mu}_2 - \underline{\mu}_1\underline{\mu}_2 - \underline{\mu}_1\underline{\mu}_2, \overline{\mu}_1 + \overline{\mu}_2 - \overline{\mu}_1\overline{\mu}_2], [\underline{v}_1\underline{v}_2, \overline{v}_1\overline{v}_2]\Big)$;
2. $\tilde{a}_1.\tilde{a}_2 = \Big([a_1a_2, b_1b_2, c_1c_2, d_1d_2]; [\underline{\mu}_1\underline{\mu}_2, \overline{\mu}_1\overline{\mu}_2], [\underline{v}_1 + \underline{v}_2 - \underline{v}_1\underline{v}_2, \overline{v}_1 + \overline{v}_2 - \overline{v}_1\overline{v}_2]\Big)$, where $\tilde{a}_1 > 0$, $\tilde{a}_2 > 0$;
3. $\lambda\tilde{a}_1 = \Big([\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1]; [1 - (1 - \underline{\mu}_1)^\lambda, 1 - (1 - \overline{\mu}_1)^\lambda], [\underline{v}_1^\lambda, \overline{v}_1^\lambda]\Big)$;
4. $\tilde{a}_1^\lambda = \Big([a_1^\lambda, b_1^\lambda, c_1^\lambda, d_1^\lambda]; [\underline{\mu}_1^\lambda, \overline{\mu}_1^\lambda], [1 - (1 - \underline{v}_1)^\lambda, 1 - (1 - \overline{v}_1)^\lambda]\Big)$, where $\tilde{a}_1 > 0$.

From Definition 2, the following properties are proven:

1. $\tilde{a}_1 + \tilde{a}_2 = \tilde{a}_2 + \tilde{a}_1$,
   $\tilde{a}_1.\tilde{a}_2 = \tilde{a}_2.\tilde{a}_1$;
2. $\lambda(\tilde{a}_1 + \tilde{a}_2) = \lambda\tilde{a}_1 + \lambda\tilde{a}_2$,
   $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_1 = (\lambda_1 + \lambda_2)\tilde{a}_1$,
   $\tilde{a}_1^{\lambda_1}.\tilde{a}_1^{\lambda_2} = \tilde{a}_1^{\lambda_1+\lambda_2}$, where $\lambda, \lambda_1, \lambda_2 \geq 0$;
3. $(\tilde{a}_1^\lambda)^k = \tilde{a}_1^{\lambda k}$, where $\tilde{a}_1 > 0$ and $\lambda, k \geq 0$.

2.2. A new ranking method of IVTrIFNs based on barycenter

Since the three characteristic functions of IVTrIFNs are all piecewise continuous, their function images are the plain regions, as depicted in Figure 1. We view them as sheets with uniform density and calculate their barycentric coordinates. For the MAGDM in intuitionistic fuzzy environments, the ranking of intuitionistic fuzzy numbers plays an important role. In order to compare the IVTrIFNs, it is necessary to develop a method to rank the IVTrIFNs. Motivated by the score and accuracy functions of IFS [48], we define the expectation and expectant score of IVTrIFNs and thereby propose a new ranking method of IVTrIFNs.

Definition 3. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ be the barycentric coordinates of the images of membership, non-membership and hesitation functions for an IVTrIFN, $\tilde{a} = \left([a, b, c, d]; [\underline{\mu}, \overline{\mu}], [\underline{v}, \overline{v}]\right)$, respectively. Then, the expectation of $\tilde{a}$ is defined as follows:

$$E(\tilde{a}) = x_1y_1 + x_2y_2 + x_3y_3. \quad (1)$$

The expectant score of $\tilde{a}$ is defined as follows:

$$S(\tilde{a}) = E(\tilde{a})y_1(2 - y_1 - y_2). \quad (2)$$

As the three characteristic functions of an IVTrIFN are equal in characterizing the IVTrIFN, the total mass of the sheet focuses on the barycenter. If the three characteristic functions of the IVTrIFN are regarded as the three states of a discrete random variable in probabilistic theory, then, the horizontal and vertical coordinates of each barycenter represent, respectively, the value (or state value) and its membership (or corresponding state probability) of the IVTrIFN. Eq. (1) defines the expectation of the IVTrIFN by summing the products of horizontal and vertical coordinates of each barycenter, which is similar to the mathematical expectation of a discrete random variable. So, the expectation defined by Eq. (1) reflects accurately the value distribution of the IVTrIFN and its geometrical meaning is very clear.

Eq. (2) may be interpreted by a vote model as follows. The values, $\mu_{\tilde{a}}(x)$, $v_{\tilde{a}}(x)$ and $\pi_{\tilde{a}}(x)$, can be interpreted as proportions of the affirmation, dissension and abstention in a vote, respectively. By considering the possibility that in the abstention group some people tend to cast affirmative votes, others are dissenters and still others tend to abstain from voting, we can divide the abstention proportion, $\pi_{\tilde{a}}(x)$, into three parts: $\mu_{\tilde{a}}(x)\pi_{\tilde{a}}(x)$, $v_{\tilde{a}}(x)\pi_{\tilde{a}}(x)$ and $\pi_{\tilde{a}}(x)\pi_{\tilde{a}}(x)$, which express the proportions of the affirmation, dissension and abstention in the original part of abstention. Thus, the total affirmative proportion is $\mu_{\tilde{a}}(x)+\mu_{\tilde{a}}(x)\pi_{\tilde{a}}(x) = \mu_{\tilde{a}}(x)[2 - \mu_{\tilde{a}}(x) - v_{\tilde{a}}(x)]$. As aforementioned, the vertical coordinates, y_1 and y_2, of membership function $\mu_{\tilde{a}}(x)$ and non-membership function $v_{\tilde{a}}(x)$ represent, respectively, the corresponding state probabilities of the IVTrIFN. Then, we can use y_1 to replace $\mu_{\tilde{a}}(x)$, and y_2 to replace $v_{\tilde{a}}(x)$. So, $\mu_{\tilde{a}}(x)[2 - \mu_{\tilde{a}}(x) - v_{\tilde{a}}(x)] = y_1(2 - y_1 - y_2)$. We use $y_1(2 - y_1 - y_2)$ in Eq. (2) to characterize the score function under the situation without considering domain $[a, b, c, d]$, where y_1 and y_2 correspond to $\mu_{\tilde{a}}(x)$ and $v_{\tilde{a}}(x)$, respectively. Consequently, the expectant score defined by Eq. (2) considers both the score function and the domain $[a, b, c, d]$ of the IVTrIFN and is more reasonable and comprehensive.

Remark 1. Using calculous, the barycentric coordinates are easily obtained. We take the image of the membership function $\mu_{\tilde{a}}(x)$ (see Figure 1) as an example to illustrate the computing process.

The area of the sheet is:

$$M=\int_a^b \frac{x-a}{b-a}\overline{\mu}dx+\int_b^c \overline{\mu}dx+\int_c^d \frac{d-x}{d-c}\overline{\mu}dx$$
$$-\left(\int_a^b \frac{x-a}{b-a}\underline{\mu}dx+\int_b^c \underline{\mu}dx\right.$$
$$\left.+\int_c^d \frac{d-x}{d-c}\underline{\mu}dx\right)=\frac{1}{2}(-a-b+c+d)(\overline{\mu}-\underline{\mu}).$$

The static moment to the x-axis is:

$$M_x=\frac{1}{2}\left\{\int_a^b \left(\frac{x-a}{b-a}\overline{\mu}\right)^2 dx+\int_b^c \overline{\mu}^2 dx\right.$$
$$+\int_c^d \left(\frac{d-x}{d-c}\overline{\mu}\right)^2 dx-\left[\int_a^b \left(\frac{x-a}{b-a}\underline{\mu}\right)^2 dx\right.$$
$$\left.\left.+\int_b^c \underline{\mu}^2 dx+\int_c^d \left(\frac{d-x}{d-c}\underline{\mu}\right)^2 dx\right]\right\}$$
$$=\frac{1}{2}(-a-2b+2c+d)(\overline{\mu}^2-\underline{\mu}^2).$$

The static moment to the y-axis is:

$$M_y=\int_a^b x\left(\frac{x-a}{b-a}\overline{\mu}\right)dx+\int_b^c x\overline{\mu}dx$$
$$+\int_c^d x\left(\frac{d-x}{d-c}\overline{\mu}\right)dx-\left[\int_a^b x\left(\frac{x-a}{b-a}\underline{\mu}\right)dx\right.$$
$$\left.+\int_b^c x\underline{\mu}dx+\int_c^d x\left(\frac{d-x}{d-c}\underline{\mu}\right)dx\right]$$
$$=\frac{1}{2}(-a^2-ab-b^2+c^2+d^2+dc)(\overline{\mu}-\underline{\mu}).$$

Hence, the horizontal and vertical coordinates of the barycenter $P_1(x_1,y_1)$ are obtained as:

$$x_1=\frac{M_y}{M}=\frac{-a^2-ab-b^2+c^2+d^2+dc}{3(-a-b+c+d)},$$

and:

$$y_1=\frac{M_x}{M}=\frac{(-a-2b+2c+d)(\overline{\mu}+\underline{\mu})}{3(-a-b+c+d)},$$

respectively.

In a similar way, the coordinates of barycenters, $P_2(x_2,y_2)$ and $P_3(x_3,y_3)$, for the non-membership and hesitation functions, can be obtained as follows:

$$x_2=\frac{-a^2-ab-b^2+c^2+d^2+dc}{3(-a-b+c+d)},$$

$$y_2=\frac{(c-b)(\overline{v}+\underline{v})}{(-a-b+c+d)}+\frac{(-a+b-c+d)(1+\overline{v}+\underline{v})}{3(-a-b+c+d)},$$

$$x_3=\frac{-a^2-ab-b^2+c^2+d^2+dc}{3(-a-b+c+d)},$$

$$y_3=1-\frac{L}{3(-a-b+c+d)(\overline{\mu}-\underline{\mu}+\overline{v}-\underline{v})},$$

where:

$$L=(-a+b-c+d)(2\overline{v}\overline{\mu}-2\underline{v}\underline{\mu}+\overline{v}^2-\underline{v}^2+\overline{\mu}$$
$$-\underline{\mu}+\overline{v}-\underline{v})+3(c-b)(2\overline{v}\overline{\mu}-2\underline{v}\underline{\mu}+\overline{v}^2-\underline{v}^2)$$
$$+(-a-2b+2c+d)(\overline{\mu}^2-\underline{\mu}^2).$$

Let $\tilde{a}_1$ and $\tilde{a}_2$ be two IVTrIFNs. According to the concepts of the expectation and expectant score, a ranking method can be summarized as follows:

- If $E(\tilde{a}_1)>E(\tilde{a}_2)$, then $\tilde{a}_1$ is bigger than $\tilde{a}_2$, denoted by $\tilde{a}_1>\tilde{a}_2$;
- If $E(\tilde{a}_1)<E(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1<\tilde{a}_2$;
- If $E(\tilde{a}_1)=E(\tilde{a}_2)$, then
  - (i) if $S(\tilde{a}_1)>S(\tilde{a}_2)$, then $\tilde{a}_1>\tilde{a}_2$;
  - (ii) if $S(\tilde{a}_1)=S(\tilde{a}_2)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, denoted by $\tilde{a}_1=\tilde{a}_2$;
  - (iii) if $S(\tilde{a}_1)<S(\tilde{a}_2)$, then $\tilde{a}_1<\tilde{a}_2$.

Example. Let $\tilde{a}_1=([4,\ 5,\ 7,\ 8];[0.5,\ 0.7],[0.1,\ 0.2])$ and $\tilde{a}_2=([3,\ 4,\ 6,\ 7];\ [0.6,\ 0.8],\ [0.0,\ 0.1])$ be two IVTrIFNs. Using the above formulas, we can get their barycentric coordinates for the images of membership, non-membership and hesitancy functions as follows: $P_{11}(6.0,0.53)$, $P_{12}(6.0,0.24)$, $P_{13}(6.0,0.22)$, $P_{21}(5.0,0.47)$, $P_{22}(5.0,0.16)$ and $P_{23}(5.0,0.22)$, respectively. By Eqs. (1) and (2), we have $E(\tilde{a}_1)=6.0$, $E(\tilde{a}_2)=4.22$, $S(\tilde{a}_1)=3.91$ and $S(\tilde{a}_2)=2.71$. Since $E(\tilde{a}_1)>E_{\tilde{a}_2}$, we get $\tilde{a}_1>\tilde{a}_2$.

3. Some generalized aggregation operators for IVTrIFNs

In a group decision, different DM play different roles. To emphasize the individual influence on the decision results, some generalized aggregation operators of IVTrIFNs are developed on the basis of Definition 2 and the ranking method in Subsection 2.2, in this section. For convenience, let I be the set of all IVTrIFNs.

Definition 4 [44]. Assume that $\tilde{a}_j$ $(j = 1, 2, ..., n)$ is a collection of the IVTrIFNs. Let IVTrIFWA: $I^n \to I$. If:

$$\text{IVTrIFWA }(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{a}_j, \tag{3}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of $\tilde{a}_j$ $(j = 1, 2, ..., n)$, satisfying that $0 \le w_j \le 1$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$, then, the function, IVTrIFWA, is called the n-dimensional weighted averaging operator of the IVTrIFNs. Especially, if $w_j = 1/n$ $(j = 1, 2, ..., n)$, then the IVTrIFWA operator is reduced to the arithmetic averaging operator of the IVTrIFNs.

Theorem 1 [44]. Let $\tilde{a}_j (j = 1, 2, ..., n)$ be a collection of IVTrIFNs, then, their aggregated value, using the IVTrIFWA operator, is also an IVTrIFN, and:

$$\text{IVTrIFWA }(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \Bigg(\Bigg[\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j, \sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j\Bigg]; \Bigg[1 - \Pi_{j=1}^{n} (1 - \underline{\mu}_j)^{w_j}, 1 - \Pi_{j=1}^{n} (1 - \overline{\mu}_j)^{w_j}\Bigg], \Bigg[\Pi_{j=1}^{n} (\underline{v}_j)^{w_j}, \Pi_{j=1}^{n} (\overline{v}_j)^{w_j}\Bigg]\Bigg). \tag{4}$$

Definition 5. Assume that $\tilde{a}_j$ $(j = 1, 2, ..., n)$ is a collection of IVTrIFNs. Let IVTrIFOWA: $I^n \to I$. If:

$$\text{IVTrIFOWA }(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \sum_{j=1}^{n} \omega_j \tilde{a}_{\sigma(j)}, \tag{5}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector associated with IVTrIFOWA, satisfying that $0 \le \omega_j \le 1$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} \omega_j = 1$, $\tilde{a}_{\sigma(j)}$ is the jth largest IVTrIFN of $\tilde{a}_j$ $(j = 1, 2, ..., n)$, then, the function IVTrIFOWA is called the n-dimensional ordered weighted average operator of the IVTrIFNs. Especially, if $\omega_j = 1/n$ $(j = 1, 2, ..., n)$, then the IVTrIFOWA operator is reduced to the arithmetic averaging operator of the IVTrIFNs.

Theorem 2. Let $\tilde{a}_j$ $(j = 1, 2, ..., n)$ be a collection of IVTrIFNs, $\tilde{a}_{\sigma(j)} = \Big(\big[a_{\sigma(j)}, b_{\sigma(j)}, c_{\sigma(j)}, d_{\sigma(j)}\big]; \big[\underline{\mu}_{\sigma(j)}, \overline{\mu}_{\sigma(j)}\big], \big[\underline{v}_{\sigma(j)}, \overline{v}_{\sigma(j)}\big]\Big)$. Then, their aggregated value, using the IVTrIFOWA operator, is also an IVTrIFN, and:

$$\text{IVTrIFOWA }(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \Bigg(\Bigg[\sum_{j=1}^{n} \omega_j a_{\sigma(j)}, \sum_{j=1}^{n} \omega_j b_{\sigma(j)}, \sum_{j=1}^{n} \omega_j c_{\sigma(j)}, \sum_{j=1}^{n} \omega_j d_{\sigma(j)}\Bigg]; \big[1 - \Pi_{j=1}^{n} (1 - \underline{\mu}_{\sigma(j)})^{\omega_j}, 1 - \Pi_{j=1}^{n} (1 - \overline{\mu}_{\sigma(j)})^{\omega_j}\big], \Big[\Pi_{j=1}^{n} (\underline{v}_{\sigma(j)})^{\omega_j}, \Pi_{j=1}^{n} (\overline{v}_{\sigma(j)})^{\omega_j}\Big]\Bigg). \tag{6}$$

Definition 6. Assume that $\tilde{a}_j$ $(j = 1, 2, ..., n)$ is a collection of IVTrIFNs. Let IVTrIFGOWA: $I^n \to I$. If:

$$\text{IVTrIFGOWA }(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\sum_{j=1}^{n} \omega_j (\tilde{a}_{\sigma(j)})^{\lambda}\right)^{1/\lambda}, \tag{7}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector associated with IVTrIFGOWA, satisfying that $0 \le \omega_j \le 1$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} \omega_j = 1$, $\tilde{a}_{\sigma(j)}$ is the jth largest IVTrIFN of $\tilde{a}_j$ $(j = 1, 2, ..., n)$, $\lambda \in [0, +\infty)$ is a parameter, then, function IVTrIFGOWA is called the n-dimensional generalized ordered weighted averaging operator of the IVTrIFNs.

Theorem 3. Let $\tilde{a}_j$ $(j = 1, 2, ..., n)$ be a collection of IVTrIFNs, $\tilde{a}_{\sigma(j)} = \Big(\big[a_{\sigma(j)} b_{\sigma(j)}, c_{\sigma(j)}, d_{\sigma(j)}\big]; \big[\underline{\mu}_{\sigma(j)}, \overline{\mu}_{\sigma(j)}\big], \big[\underline{v}_{\sigma(j)}, \overline{v}_{\sigma(j)}\big]\Big)$. Then, their aggregated value, using the IVTrIFGOWA operator, is also an IVTrIFN, and:

$$\text{IVTrIFGOWA }(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \Bigg(\Bigg[\left(\sum_{j=1}^{n} \omega_j a_{\sigma(j)}^{\lambda}\right)^{1/\lambda}, \left(\sum_{j=1}^{n} \omega_j b_{\sigma(j)}^{\lambda}\right)^{1/\lambda}, \left(\sum_{j=1}^{n} \omega_j c_{\sigma(j)}^{\lambda}\right)^{1/\lambda}, \left(\sum_{j=1}^{n} \omega_j d_{\sigma(j)}^{\lambda}\right)^{1/\lambda}\Bigg]; \big[\big(1 - \Pi_{j=1}^{n} (1 - \underline{\mu}_{\sigma(j)}^{\lambda})^{\omega_j}\big)^{1/\lambda}, \big(1 - \Pi_{j=1}^{n} (1 - \overline{\mu}_{\sigma(j)}^{\lambda})^{\omega_j}\big)^{1/\lambda}\big], \big[1 - \big(1 - \Pi_{j=1}^{n} (1 - (1 - \underline{v}_{\sigma(j)})^{\lambda})^{\omega_j}\big)^{1/\lambda}, 1 - \big(1$$

$$-\Pi_{j=1}^{n}(1-(1-\overline{v}_{\sigma(j)})^{\lambda})^{\omega_j})^{1/\lambda}]\Big). \qquad (8)$$

According to Definitions 2 and 4, proofs of Theorems 2 and 3 can be easily completed by using the mathematical induction on n.

The weight vector, $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_n)^T$, associated with IVTrIFOWA (or IVTrIFGOWA), can be obtained by the fuzzy linguistic quantifier [49] as follows:

$$\omega_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \qquad (i = 1, 2, ..., n), \qquad (9)$$

where Q is the fuzzy linguistic quantifier and:

$$Q(t) = \begin{cases} 0, & t < \xi \\ (t-\xi)/(\eta-\xi), & \xi \le t < \eta \\ 1, & t \ge \eta \end{cases}$$

with $\xi, t, \eta \in [0, 1]$. For the criteria "at least half", "most", and "as many as possible", the parameter pair (ξ, η) takes the values (0,0.5), (0.3,0.8), and (0.5,1), respectively.

Combined with Definition 2 and Theorem 3, the following theorems can be obtained.

Theorem 4 (idempotency). Let $\tilde{a}_j$ $(j = 1, 2, ..., n)$ be a collection of IVTrIFNs. If all the $\tilde{a}_j$ $(j = 1, 2, ..., n)$ are equal, i.e. $\tilde{a}_j = \tilde{a}$ for all j, then IVTrIFGOWA $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}$.

Theorem 5 (boundedness). Let $\tilde{a}_j$ $(j = 1, 2, ..., n)$ be a collection of IVTrIFNs,

$$\tilde{a}^- = \Big(\big[\min_{1\le j\le n}\{a_j\}, \min_{1\le j\le n}\{b_j\}, \min_{1\le j\le n}\{c_j\} \min_{1\le j\le n}\{d_j\}\big];$$

$$\big[\min_{1\le j\le n}\{\underline{\mu}_j\}, \min_{1\le j\le n}\{\overline{\mu}_j\}\big],$$

$$\big[\max_{1\le j\le n}\{\underline{v}_j\}, \max_{1\le j\le n}\{\overline{v}_j\}\big]\Big),$$

and:

$$\tilde{a}^+ = \Big(\big[\max_{1\le j\le n}\{a_j\}, \max_{1\le j\le n}\{b_j\}, \max_{1\le j\le n}\{c_j\} \max_{1\le j\le n}\{d_j\}\big];$$

$$\big[\max_{1\le j\le n}\{\underline{\mu}_j\}, \max_{1\le j\le n}\{\overline{\mu}_j\}\big],$$

$$\big[\min_{1\le j\le n}\{\underline{v}_j\}, \min_{1\le j\le n}\{\overline{v}_j\}\big]\Big).$$

Then, $\tilde{a}^- \le$ IVTrIFGOWA $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le \tilde{a}^+$.

Theorem 6 (monotonicity). Let $\tilde{a}_j$ and $\tilde{a}'_j$ $(j = 1, 2, ..., n)$ be two collections of IVTrIFNs. If $\tilde{a}_j \le \tilde{a}'_j$ $(j = 1, 2, ..., n)$, then, IVTrIFGOWA $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le$ IVTrIFGOWA $(\tilde{a}'_1, \tilde{a}'_2, ..., \tilde{a}'_n)$.

Theorem 7. Let $\tilde{a}_j$ $(j = 1, 2, ..., n)$ be a collection of IVTrIFNs and $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weighted vector correlating with IVTrIFGOWA. Then:

1. If $\lambda = 0$, then;

$$\text{IVTrIFGOWA } (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \Pi_{i=1}^{n}\left(\tilde{a}_{\sigma(j)}\right)^{\omega_i},$$

which is called the interval-valued trapezoidal intuitionistic fuzzy ordered weighted geometrical (IVTrIFOWG) operator;

2. If $\lambda = 1$, then:

$$\text{IVTrIFGOWA } (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \sum_{i=1}^{n} \omega_i \tilde{a}_{\sigma(j)},$$

which is reduced to the IVTrIFOWA operator;

3. If $\lambda \to +\infty$ and $\omega_i \ne 0 (i = 1, 2, .., n)$, then:

$$\text{IVTrIFGOWA } (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) =$$

$$\max\{\tilde{a}_i | i = 1, 2, ..., n\},$$

which is called the Max operator of IVTrIFNs;

4. If $\omega_j = 1/n$ $(j = 1, 2, ..., n)$, then:

$$\text{IVTrIFGOWA } (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\sum_{j=1}^{n} \frac{1}{n}(\tilde{a}_{\sigma(j)})^{\lambda}\right)^{1/\lambda},$$

which is called the generalized mean operator of the IVTrIFNs;

5. If $\omega_1 = 1$, then IVTrIFGOWA $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \max\{\tilde{a}_i | i = 1, 2, ..., n\}$;
6. If $\omega_n = 1$, then VTrIFGOWA $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \min\{\tilde{a}_i | i = 1, 2, ..., n\}$;
7. If $\omega_j = 1$, then VTrIFGOWA $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}_{\sigma(j)}$, where $\tilde{a}_{\sigma(j)}$ is the jth largest of $\tilde{a}_i$ $(i = 1, 2, ..., n)$.

Definition 7. Let $\tilde{a}_j$ $(j = 1, 2, ..., n)$ be a collection of IVTrIFNs. If IVTrIFGHWA: $I^n \to I$, so that:

$$\text{IVTrIFGHWA}_{w,\omega}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\sum_{j=1}^{n} \omega_j (\tilde{a}'_{\sigma(j)})^{\lambda}\right)^{1/\lambda}, \qquad (10)$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector associated with IVTrIFGHWA, satisfying that $0 \le \omega_j \le 1$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} \omega_j = 1$, $\tilde{a}'_{\sigma(j)}$ is the jth largest number of IVTrIFNs $\tilde{a}'_i$ $(i = 1, 2, ..., n)$ with

$\tilde{a}'_i = nw_i\tilde{a}_i$, $\mathbf{w} = (w_1, w_2, ..., w_n)$ is the weighting vector of $\tilde{a}_i$ $(i = 1, 2, ..., n)$ with $0 \leq w_j \leq 1$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$, n is the balancing coefficient, then, function IVTrIFGHWA is called the n-dimensional generalized hybrid weighted averaging operator of the IVTrIFNs.

Especially, if $w_j = 1/n$ $(j = 1, 2, ..., n)$, the IVTrIFGHWA operator is reduced to an IVTrIFGOWA operator. Therefore, the $\varphi^H_{w,\omega}$ operator generalizes both φ^G_w and φ^O_w operators and reflects the important degrees of both the given IVTrIFNs and the ordered position of the IVTrIFNs.

It is easy to obtain the following theorem by Definition 2.

Theorem 8. Let $\tilde{a}_j$ $(j = 1, 2, ..., n)$ be a collection of IVTrIFNs, $\tilde{a}'_i = \left([a'_i, b'_i, c'_i, d'_i]; [\underline{\mu}'_i, \overline{\mu}'_i], [\underline{v}'_i, \overline{v}'_i]\right)$, $\tilde{a}'_{\sigma(j)} = \left([a'_{\sigma(j)}, b'_{\sigma(j)}, c'_{\sigma(j)}, d'_{\sigma(j)}]; [\underline{\mu}'_{\sigma(j)}, \overline{\mu}'_{\sigma(j)}], [\underline{v}'_{\sigma(j)}, \overline{v}'_{\sigma(j)}]\right)$. Then, the aggregated value, using the IVTrIFGHWA operator, is also an IVTrIFN, and:

$$\begin{aligned}
&\mathrm{IVTrIFGHWA}_{w,\omega}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \\
&= \Bigg(\Bigg[\left(\sum_{j=1}^{n} \omega_j {a'}^{\lambda}_{\sigma(j)}\right)^{1/\lambda}, \left(\sum_{j=1}^{n} \omega_j {b'}^{\lambda}_{\sigma(j)}\right)^{1/\lambda}, \\
&\left(\sum_{j=1}^{n} \omega_j {c'}^{\lambda}_{\sigma(j)}\right)^{1/\lambda}, \left(\sum_{j=1}^{n} \omega_j {d'}^{\lambda}_{\sigma(j)}\right)^{1/\lambda}\Bigg]; \\
&\Big[(1 - \Pi_{j=1}^{n}(1 - \underline{\mu}'^{\lambda}_{\sigma(j)})^{\omega_j})^{1/\lambda}, \\
&(1 - \Pi_{j=1}^{n}(1 - \overline{\mu}'^{\lambda}_{\sigma(j)})^{\omega_j})^{1/\lambda}\Big], \Big[1 - \big(1 - \Pi_{j=1}^{n}(1 \\
&- (1 - \underline{v}'^{\lambda}_{\sigma(j)}))^{\omega_j}\big)^{1/\lambda}, 1 - \big(1 - \Pi_{j=1}^{n}(1 \\
&- (1 - \overline{v}'_{\sigma(j)})^{\lambda})^{\omega_j}\big)^{1/\lambda}\Big]\Bigg).
\end{aligned} \tag{11}$$

4. MAGDM method using IVTrIFNs

A MAGDM problem is one which finds a best compromise solution from all feasible alternatives assessed on multiple attributes. Assume that there is a group consisting of k DMs $\{P_1, P_2, ..., P_k\}$, which has to choose one of (or rank) m alternatives $\{A_1, A_2, ..., A_m\}$ based on n attributes $\{a_1, a_2, ..., a_n\}$. Denote an alternative set by $A = \{A_1, A_2, ..., A_m\}$ and an attribute set by $F = \{a_1, a_2, ..., a_n\}$. Suppose that the intuitionistic fuzzy rating of alternative A_i on attribute a_j, given by DM P_t, is an IVTrIFN $\tilde{a}^{(t)}_{ij} = \left([h^{(t)}_{1i}(a_j), h^{(t)}_{2i}(a_j), h^{(t)}_{3i}(a_j), h^{(t)}_{4i}(a_j)]; \mu^{(t)}_{ij}, v^{(t)}_{ij}\right)$ where $\mu^{(t)}_{ij} = [\underline{\mu}^{(t)}_{ij}, \overline{\mu}^{(t)}_{ij}] \subseteq [0, 1]$ denotes the extent to which alternative A_i belongs to trapezoidal fuzzy number $[h^{(t)}_{1i}(a_j), h^{(t)}_{2i}(a_j), h^{(t)}_{3i}(a_j), h^{(t)}_{4i}(a_j)]$ on attribute a_j by DM P_t, and $v^{(t)}_{ij} = [\underline{v}^{(t)}_{ij}, \overline{v}^{(t)}_{ij}] \subseteq [0, 1]$ denotes the extent to which alternative A_i does not belong to the trapezoidal fuzzy number $[h^{(t)}_{1i}(a_j), h^{(t)}_{2i}(a_j), h^{(t)}_{3i}(a_j), h^{(t)}_{4i}(a_j)]$ on attribute a_j by DM P_t and $v^{(t)}_{ij} = \left[\underline{v}^{(t)}_{ij}, \overline{v}^{(t)}_{ij}\right] \subseteq [0, 1]$ denotes the extent to which alternative A_i does not belong to the trapezoidal fuzzy number $[h^{(t)}_{1i}(a_j), h^{(t)}_{2i}(a_j), h^{(t)}_{3i}(a_j), h^{(t)}_{4i}(a_j)]$ on attribute a_j by DM P_t, satisfying $\overline{\mu}^{(t)}_{ij} + \overline{v}^{(t)}_{ij} \leq 1$. Hence, a MAGDM problem using IVTrIFNs can be concisely expressed in matrix format as follows:

$$\tilde{\mathbf{D}}^{(t)} = \left(\tilde{a}^{(t)}_{ij}\right)_{m \times n} \qquad (t = 1, 2, ..., k), \tag{12}$$

which are the interval-valued trapezoidal intuitionistic fuzzy decision matrixes.

An algorithm and process of MAGDM problems with IVTrIFNs may be summarized as follows:

Step 1. Identify the evaluation attributes and alternatives;

Step 2. Pool the DM's opinion to get the ratings of alternatives on attributes, i.e. to obtain the interval-valued trapezoidal intuitionistic fuzzy decision matrixes, $\tilde{\mathbf{D}}^{(t)} = \left(\tilde{a}^{(t)}_{ij}\right)_{m \times n}$ $(t = 1, 2, ..., k)$.

Step 3. Normalize matrixes $\tilde{\mathbf{D}}^{(t)} = \left(\tilde{a}^{(t)}_{ij}\right)_{m \times n}$ $(t = 1, 2, ..., k)$.

In general, attributes can be classified into two types: benefit attributes and cost attributes. In other words, attribute set F can be divided into two subsets: F_1 and F_2, which are the subsets of benefit attributes and cost attributes, respectively. Since the physical dimensions and ranges of the n attributes are different, the attribute values need to be normalized. In this paper, the attribute value:

$$\tilde{a}^{(t)}_{ij} = \left([h^{(t)}_{1i}(a_j), h^{(t)}_{2i}(a_j), h^{(t)}_{3i}(a_j), h^{(t)}_{4i}(a_j)]; \mu^{(t)}_{ij}, v^{(t)}_{ij}\right),$$

is normalized as:

$$\tilde{r}^{(t)}_{ij} = \left([r^{(t)}_{1i}(a_j), r^{(t)}_{2i}(a_j), r^{(t)}_{3i}(a_j), r^{(t)}_{4i}(a_j)]; \mu^{(t)}_{ij}, v^{(t)}_{ij}\right),$$

where:

$$r^{(t)}_{si}(a_j) = \frac{h^{(t)}_{si}(a_j) - \min\limits_{1 \leq j \leq n}\{h^{(t)}_{1i}(a_j)\}}{\max\limits_{1 \leq j \leq n}\{h^{(t)}_{4i}(a_j)\} - \min\limits_{1 \leq j \leq n}\{h^{(t)}_{1i}(a_j)\}}$$

$$(s = 1, 2, 3, 4;\ j \in F_1), \tag{13}$$

$$r_{si}^{(t)}(a_j)=\frac{\max\limits_{1\le j\le n}\{h_{4i}^{(t)}(a_j)\}-h_{5-s,i}^{(t)}(a_j)}{\max\limits_{1\le j\le n}\{h_{4i}^{(t)}(a_j)\}-\min\limits_{1\le j\le n}\{h_{1i}^{(t)}(a_j)\}}$$

$$(s=1,2,3,4;\ j\in F_2). \tag{14}$$

Note that the second term of the numerator of Eq. (14) is $h_{5-s,i}^{(t)}(a_j)$, and the subscript is $5-s$ $(s=1,2,3,4)$, which can ensure that $r_{1i}^{(t)}(a_j)\le r_{2i}^{(t)}(a_j)\le r_{3i}^{(t)}(a_j)\le r_{4i}^{(t)}(a_j)$, i.e. $[r_{1i}^{(t)}(a_j),r_{2i}^{(t)}(a_j),r_{3i}^{(t)}(a_j),r_{4i}^{(t)}(a_j)]$ is still a trapezoidal fuzzy number. Thus, the normalized value, $\tilde{r}_{ij}^{(t)}$, is still an IVTrIFN. Furthermore, all $r_{si}^{(t)}(a_j)\in[0,1]$ $(i=1,2,...,m;\ j=1,2,...,n;\ t=1,2,...,k;\ s=1,2,3,4)$, i.e. $\tilde{r}_{ij}^{(t)}$ is a normalized IVTrIFN. Then, the matrix $\tilde{\mathbf{D}}^{(t)}$ can be transformed into the normalized interval-valued trapezoidal intuitionistic fuzzy decision matrix, $\tilde{\mathbf{R}}^{(t)}=(\tilde{r}_{ij}^{(t)})_{m\times n}$ $(t=1,2,...,k)$.

Step 4. Aggregate the intuitionistic fuzzy ratings of each alternative on all attributes given by the DM. Using the IVTrIFWA operator, the individual comprehensive rating of alternative A_i for DM P_t is obtained as follows:

$$\tilde{a}_i^{(t)}=\text{IVTrIFWA}\,(\tilde{r}_{i1}^{(t)},\tilde{r}_{i2}^{(t)},...,\tilde{r}_{in}^{(t)})$$

$$=\sum_{j=1}^{n}w_j\tilde{r}_{ij}^{(t)}\qquad(i=1,2,...,m;\ t=1,2,...,k), \tag{15}$$

where $\mathbf{w}=(w_1,w_2,...,w_n)^T$ is the weight vector of attributes.

Step 5. Compute the collective comprehensive ratings of alternatives for the group. According to the method of the fuzzy linguistic quantifier [49], weight vector $\boldsymbol{\omega}=(\omega_1,\omega_2,...,\omega_k)^T$ associated with the IVTrIFGHWA operator can be obtained. A collective comprehensive rating of alternative A_i is aggregated by Eq. (11) as follows:

$$\tilde{a}_i=\text{IVTrIFGHWA}_{v,\omega}(\tilde{a}_i^{(1)},\tilde{a}_i^{(2)},...,\tilde{a}_i^{(k)})$$

$$(i=1,2,...,m), \tag{16}$$

where $\mathbf{v}=(v_1,v_2,...,v_k)^T$ is the weight vector of DMs.

Step 6. The ranking orders of alternatives are generated by the ranking method based on barycenter for IVTrIFNs $\tilde{a}_i$ $(i=1,2,...,m)$. Also the best alternative can be determined.

5. A numerical example and comparison analysis of computational results

A numerical example is analyzed and the comparison analysis is conducted in this section.

5.1. A numerical example and the analysis process

In this section, a problem concerning a manufacturing company searching for the best global supplier for a critical part used in its assembly process is designed to verify and illustrate the method proposed in this paper.

These attributes are considered in selection of three potential global suppliers, A_i $(i=1,2,3)$, as: quality of product, a_1, service performance of supplier, a_2, and supplier profile, a_3. These attributes are all benefit attributes. An expert group consists of three experts, P_t $(t=1,2,3)$, who are invited to evaluate the three potential global suppliers. Assume that the weight vector of the attributes is $\mathbf{w}=(0.26,0.32,0.42)^T$, and the weight vector of the experts is $\mathbf{v}=(0.33,0.36,0.31)^T$. The experts give the characteristics of the potential global suppliers by the IVTIFNs. Thereby, the interval-valued trapezoidal intuitionistic fuzzy decision matrices, $\tilde{\mathbf{D}}^{(t)}(t=1,2,3)$, are given in Tables 1-3. For example, $\tilde{a}_{11}^{(1)}=([1,\ 2,\ 3,\ 4];\ [0.6,0.8],\ [0.1,0.2])$ may be explained as stated in Section 1: Introduction.

We solve this problem using the proposed method in this paper. The solving process is summarized as follows:

Table 1. Interval-valued trapezoidal intuitionistic fuzzy decision matrix $\tilde{\mathbf{D}}^{(t)}$ given by expert P_t.

	a_1	a_2	a_3
A_1	([1, 2, 3, 4];[0.6,0.8],[0.1,0.2])	([4, 5, 6, 8];[0.3,0.5],[0.2,0.4])	([1, 4, 5, 6];[0.1,0.3],[0.3,0.5])
A_2	([5, 6, 7, 8];[0.3,0.5],[0.2,0.4])	([2, 4, 6, 7];[0.4,0.6],[0.1,0.3])	([2, 4, 6, 7];[0.1,0.3],[0.4,0.6])
A_3	([2, 3, 4, 5];[0.1,0.3],[0.4,0.6])	([2, 4, 5, 7];[0.5,0.7],[0.2,0.3])	([3, 4, 6, 8];[0.4,0.6],[0.0,0.2])

Table 2. Interval-valued trapezoidal intuitionistic fuzzy decision matrix, $\tilde{\mathbf{D}}^{(2)}$, given by expert P_2.

	a_1	a_2	a_3
A_1	([3, 5, 6, 8];[0.1,0.3],[0.4,0.6])	([2, 3, 4, 5];[0.6,0.8],[0.0,0.1])	([2, 4, 5, 7];[0.4,0.7],[0.1,0.3])
A_2	([2, 3, 4, 6];[0.4,0.6], [0.2,0.3])	([1, 3, 5, 8];[0.3,0.5],[0.2,0.4])	([1, 2, 4, 6];[0.1,0.2],[0.3,0.6])
A_3	([1, 2, 3, 4];[0.1,0.2],[0.4,0.7])	([3, 4, 5, 8];[0.2,0.4],[0.2,0.5])	([3, 4, 6, 7];[0.3,0.4],[0.4,0.5])

Table 3. Interval-valued trapezoidal intuitionistic fuzzy decision matrix, $\tilde{\mathbf{D}}^{(3)}$, given by expert P_3.

	a_1	a_2	a_3
A_1	([2, 4, 5, 8];[0.1,0.2],[0.3,0.6])	([2, 3, 4, 5];[0.3,0.4],[0.2,0.4])	([1, 3, 6, 7];[0.2,0.4],[0.4,0.6])
A_2	([4, 5, 6, 7];[0.2,0.3],[0.3,0.5])	([1, 3, 5, 6];[0.4,0.6],[0.0,0.2])	([4, 6, 7, 9];[0.5,0.7],[0.1,0.3])
A_3	([1, 2, 3, 4];[0.4,0.7],[0.1,0.3])	([2, 3, 4, 5];[0.2,0.4],[0.3,0.5])	([3, 4, 6, 7];[0.1,0.4],[0.2,0.5])

Table 4. Normalized decision matrix, $\tilde{\mathbf{R}}^{(1)}$, given by expert P_1.

	a_1	a_2	a_3
A_1	([0,0.14,0.29,0.43];[0.6,0.8],[0.1,0.2])	([0.43,0.57,0.71,1.0];[0.3,0.5],[0.2,0.4])	([0,0.43,0.57,0.71];[0.1,0.3],[0.3,0.5])
A_2	([0.5,0.67,0.83,1.0];[0.3,0.5],[0.2,0.4])	([0,0.33,0.67,0.83];[0.4,0.6],[0.1,0.3])	([0,0.33,0.67,0.83];[0.1,0.3],[0.4,0.6])
A_3	([0,0.17,0.33,0.50];[0.1,0.3],[0.4,0.6])	([0,0.33,0.50,0.83];[0.5,0.7],[0.2,0.3])	([0.17,0.33,0.67,1.0];[0.4,0.6],[0.0,0.2])

Table 5. Normalized decision matrix, $\tilde{\mathbf{R}}^{(2)}$, given by expert P_2.

	a_1	a_2	a_3
A_1	([0.17,0.50,0.67,1.0];[0.1,0.3],[0.4,0.6])	([0,0.17,0.33,0.50];[0.6,0.8],[0.0,0.1])	([0,0.33,0.50,0.83];[0.4,0.7],[0.1,0.3])
A_2	([0.14,0.29,0.43,0.71];[0.4,0.6], [0.2,0.3])	([0,0.29,0.57,1.0];[0.3,0.5],[0.2,0.4])	([0,0.14,0.43,0.71];[0.1,0.2],[0.3,0.6])
A_3	([0,0.14,0.29,0.43];[0.1,0.2],[0.4,0.7])	([0.29,0.43,0.57,1.0];[0.2,0.4],[0.2,0.5])	([0.29,0.43,0.71,0.86];[0.3,0.4],[0.4,0.5])

Table 6. Normalized decision matrix, $\tilde{\mathbf{R}}^{(3)}$, given by expert P_3.

	a_1	a_2	a_3
A_1	([0.14,0.43,0.57,1.0];[0.1,0.2],[0.3,0.6])	([0.14,0.29,0.43,0.57];[0.3,0.4],[0.2,0.4])	([0,0.29,0.71,0.86];[0.2,0.4],[0.4,0.6])
A_2	([0.38,0.50,0.63,0.75];[0.2,0.3],[0.3,0.5])	([0,0.25,0.50,0.63];[0.4,0.6],[0.0,0.2])	([0.38,0.63,0.75,1.0];[0.5,0.7],[0.1,0.3])
A_3	([0,0.17,0.33,0.50];[0.4,0.7],[0.1,0.3])	([0.17,0.33,0.50,0.67];[0.2,0.4],[0.3,0.5])	([0.33,0.50,0.83,1.0];[0.1,0.4],[0.2,0.5])

Step 1. Step 1 is omitted.

Step 2. Step 2 is omitted.

Step 3. According to Eqs. (13) and (14), the normalized interval-valued trapezoidal intuitionistic fuzzy decision matrices are obtained as in Tables 4-6.

Step 4. Combining the weight vector of attributes $\mathbf{w} = (0.26, 0.32, 0.42)^T$ with the IVTrIFWA operator, the individual comprehensive rating of alternative A_1 for expert P_1 is calculated by Eq. (15) as follows:

$$\tilde{a}_1^{(1)} = \text{IVTrIFWA } (\tilde{r}_{11}^{(1)}, \tilde{r}_{11}^{(1)}, \tilde{r}_{13}^{(1)}) = \Pi_{j=1}^3 \left(\tilde{r}_{1j}^{(1)}\right)^{W_j}$$
$$= \left([0.1376,\ 0.3994,\ 0.542,\ 0.73]\,;\right.$$
$$\left.[0.3274,\ 0.5462]\,, [0.1980,\ 0.3669]\right).$$

In a similar way, the individual comprehensive ratings of each alternative A_i ($i = 1,2,3$) for experts P_t ($i = 1,2,3$) are obtained as:

$$\tilde{a}_2^{(1)} = \left([0.13, 0.4184, 0.7116, 0.8742]; [0.2595, 0.4638],\right.$$
$$\left.[0.2144, 0.4326]\right),$$

$$\tilde{a}_3^{(1)} = \left([0.0714, 0.2884, 0.5272, 0.8156];\right.$$
$$\left.[0.3711, 0.5780], [0, 0.3030]\right),$$

$$\tilde{a}_1^{(2)} = \left([0.0442, 0.323, 0.4898, 0.7686];\right.$$
$$\left.[0.4144, 0.6716], [0, 0.2528]\right),$$

$$\tilde{a}_2^{(2)} = \left([0.0364, 0.227, 0.4748, 0.8028];\right.$$
$$\left.[0.2526, 0.4252], [02371, 0.4401]\right),$$

$$\tilde{a}_3^{(2)} = \left([0.2146, 0.3546, 0.556, 0.793];\right.$$
$$\left.[0.2201, 0.3534], [0.3204, 0.5457]\right),$$

$$\tilde{a}_1^{(3)} = \left([0.0812, 0.3264, 0.584, 0.8036];\right.$$
$$\left.[0.2096, 0.3534], [0.2973, 0.5270]\right),$$

$$\tilde{a}_2^{(3)} = \left([0.2584, 0.4746, 0.6388, 0.8166];\right.$$
$$\left.[0.4011, 0.5900], [0, 0.3009]\right),$$

and:

$$\tilde{a}_3^{(3)} = \big([0.193, 0.3598, 0.5944, 0.7644]; [0.2200, 0.4989], [0.1902, 0.4378]\big),$$

respectively.

Step 5. According to Eq. (9), we choose the "almost" criteria and determine that the weight vector associated with the IVTrIFGHWA operator is $\boldsymbol{\omega} = (0.067, 0.667, 0.267)^T$. The collective comprehensive rating of alternative A_1 for the group is obtained by Eq. (16) as:

$$\tilde{a}_1 = \text{IVTrIFGHWA}_{v,\omega}(\tilde{a}_1^{(1)}, \tilde{a}_1^{(2)}, \tilde{a}_1^{(3)}) = \big([0.0611, 0.3402, 0.5338, 0.8016]; [0.3751, 0.6182], [0, 0.2571]\big),$$

where $\mathbf{v} = (0.33, 0.36, 0.31)^T$ is the weight vector of the experts, and $\lambda = 1$.

Analogously, the collective comprehensive ratings of alternatives A_2 and A_3 are obtained, respectively, as:

$$\tilde{a}_2 = \big([0.1124, 0.3713, 0.6466, 0.8596]; [0.2697, 0.4657], [0, 0.3948]\big),$$

and:

$$\tilde{a}_3 = \big([0.1541, 0.3251, 0.5483, 0.7471]; [0.2553, 0.4976], [0.2434, 0.4243]\big).$$

Step 6. By Definition 3, we calculate the barycenters of three characteristic functions of $\tilde{a}_1$ for the alternative A_1 as $P_{11} = (0.4363, 0.3997)$, $P_{12} = (0.4363, 0.2986)$ and $P_{13} = (0.4363, 0.3017)$, respectively.

Analogously, the barycenters of characteristic functions of $\tilde{a}_2$ and $\tilde{a}_3$ are obtained as follows:

$$P_{21} = (0.5040, 0.3111),$$

$$P_{22} = (0.5040, 0.3209),$$

$$P_{23} = (0.5040, 0.3680),$$

$$P_{31} = (0.4641, 0.3196),$$

$$P_{32} = (0.4641, 0.4344),$$

$$P_{33} = (0.4641, 0.2460).$$

Then, by Eqs. (1) and (2), the expectation and expectant scores for $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ are obtained as $E(\tilde{a}_1) = 0.4363$, $S(\tilde{a}_1) = 0.2270$, $E(\tilde{a}_2) = 0.5040$, $S(\tilde{a}_2) = 0.2145$, $E(\tilde{a}_3) = 0.4641$ and $S(\tilde{a}_3) = 0.1848$, respectively.

According to Subsection 2.2, the ranking order of the suppliers for the group is $A_2 \succ A_3 \succ A_1$, and the best supplier is A_2.

Similarly, for different parameter values, $\lambda \in [0, +\infty)$, we can obtain the ranking order of the suppliers. The computation results and ranking orders of suppliers are listed in Table 7.

Table 7 shows that the ranking orders of alternatives may be different for different parameter values of λ. If $\lambda \leq 50$, the rank order is $A_2 \succ A_3 \succ A_1$, and the best supplier is A_2, while, for $\lambda \to +\infty$, the rank order is $A_3 \succ A_2 \succ A_1$ and the best supplier is A_3. This observation implies that DMs can select different parameter values according to their preferences in making decisions, which greatly enhances the flexibility of the decision process.

5.2. Comparison analysis of the obtained results

In this subsection, we compare the results obtained by the proposed method in this paper and the method of Wei [37]. Since [37] utilized TrIFNs to express the attribute values, we first take the interval middle-points of the maximum membership degree interval and the

Table 7. The computation results for different parameter values and ranking orders of suppliers.

λ	$E(\tilde{a}_1)$	$S(\tilde{a}_1)$	$E(\tilde{a}_2)$	$S(\tilde{a}_2)$	$E(\tilde{a}_3)$	$S(\tilde{a}_3)$	Ranking order	Best supplier
0	0.4339	0.1998	0.4935	0.1938	0.4562	0.1819	$A_2 \succ A_3 \succ A_1$	A_2
0.1	0.4386	0.2254	0.4998	0.2124	0.4619	0.1824	$A_2 \succ A_3 \succ A_1$	A_2
0.2	0.4366	0.2246	0.4983	0.2117	0.4603	0.1819	$A_2 \succ A_3 \succ A_1$	A_2
0.5	0.4358	0.2251	0.4998	0.2125	0.4612	0.1827	$A_2 \succ A_3 \succ A_1$	A_2
1	0.4363	0.2270	0.5040	0.2145	0.4641	0.1848	$A_2 \succ A_3 \succ A_1$	A_2
5	0.4469	0.2437	0.5311	0.2308	0.4799	0.2010	$A_2 \succ A_3 \succ A_1$	A_2
10	0.4584	0.2556	0.5546	0.2513	0.4910	0.2150	$A_2 \succ A_3 \succ A_1$	A_2
20	0.4692	0.2642	0.5745	0.2786	0.5092	0.2322	$A_2 \succ A_3 \succ A_1$	A_2
50	0.4808	0.1882	0.5899	0.2039	0.5360	0.1722	$A_2 \succ A_3 \succ A_1$	A_2
$+\infty$	0.4610	0.1947	0.5645	0.2827	0.5647	0.1797	$A_3 \succ A_2 \succ A_1$	A_3

minimum non-membership degree interval in Tables 4-6 to transform the IVTrIFNs into TrIFNs. Then, using method [37], the collective comprehensive ratings of alternatives, $A_i (i = 1, 2, 3)$, are obtained as:

$$\tilde{r}_1 = \big([0.0786, 0.2491, 0.3387, 0.7602]; 0.4557, 0.2365\big),$$

$$\tilde{r}_2 = \big([0.1562, 0.3455, 0.4013, 0.8453]; 0.4059, 0.2773\big),$$

and:

$$\tilde{r}_3 = \big([0.1384, 0.3231, 0.3280, 0.7312]; 0.4114, 0.2677\big),$$

respectively.

The distances between the collective comprehensive ratings of alternatives and the trapezoidal intuitionistic fuzzy positive ideal solution, $\tilde{r}^+$, [37] are calculated as: $d(\tilde{r}_1, \tilde{r}^+) = 0.7826$, $d(\tilde{r}_2, \tilde{r}^+) = 0.7534$ and $d(\tilde{r}_3, \tilde{r}^+) = 0.7826$, respectively.

Therefore, the suppliers are ranked as $A_2 \succ A_3 = A_1$, and the best supplier is A_2.

Obviously, the ranking results obtained by the method proposed in this paper are remarkably different from those obtained by the method [37]. Compared with the method in [37], the proposed method in this paper has the following advantages:

(i) The abilities of representing uncertainty are very different for IVTrIFN and TrIFN. The former uses the intervals to express the maximum membership degree and the minimum non-membership degree, while the latter uses crisp real numbers to express the maximum membership degree and the minimum non-membership degree. Consequently, the latter is only a special case of the former.

(ii) The method in [37] can only obtain a single ranking result, while the proposed method in this paper can generate different ranking results through choosing different parameter values. Thus, the latter is more flexible and agile than the former.

(iii) If there exist two different alternatives, with equal distances relative to the fuzzy positive ideal solution, then, the method in [37] cannot distinguish these two alternatives. For example, because $d(\tilde{r}_1, \tilde{r}^+) = d(\tilde{r}_3, \tilde{r}^+) = 0.7826$, A_3 is the same as A_1 when using the method in [37]. However, if there exist two different alternatives with equal expectations, the method of this paper can further distinguish these two alternatives, according to expectant scores. For example, though $E(\tilde{a}_1) = E(\tilde{a}_2) = 0.4460$, and $S(\tilde{a}_1) = 0.3175 > S(\tilde{a}_2) = 0.2967$, A_1 is superior to A_2 when using the method proposed in this paper. From this point of view, the distinguishing power of this paper's method is stronger than that of [37].

6. Conclusion

As stated previously, the IVTrIFN is a useful generalization of the IFS. From a geometric viewpoint, the expectation and expectant scores of IVTrIFNs are defined by using the notion of barycenter. A new method based on barycenter is presented to rank IVTrIFNs. Then, the IVTrIFOWA, IVTrIFGOWA, and IVTrIFHWA operators for IVTrIFNs are developed and employed to solve a MAGDM problem with IVTrIFNs. Though a supplier, a selected example is used to illustrate the applicability and implementation process of the decision method proposed in this paper. It is expected to be applicable to decision problems in many areas, such as risk investment and performance evaluation of military systems, engineering management, and partner selection of supply chain management.

However, constructing IVTrIFNs (i.e., extracting the membership and non-membership functions, whose values depend on both different intervals and trapezoidal fuzzy number) is a key problem when applying the proposed methodology to practical decision problems. Generating methods of IVTrIFNs will be investigated in the near future.

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