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# Numerical approach for solving nonlinear stochastic Itô-Volterra integral equations using Fibonacci operational matrices

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**Abstract.** This article proposes an efficient method based on the Fibonacci functions for solving nonlinear stochastic Itô-Volterra integral equations. For this purpose, we obtain stochastic operational matrix of Fibonacci functions. We use the proposed basis function in combination with stochastic operational matrix. This problem is then reduced into a system of nonlinear equations which can be solved by Newton's method. Also, the existence, uniqueness, and convergence of the proposed method are discussed. Furthermore, in order to show the accuracy and reliability of the proposed method, the new approach is applied to some practical problems.



## 1. Introduction

Some mathematical objects are defined by a formula or an expression. Some other mathematical objects are defined by their properties, not explicitly by an expression. That is, the objects are defined by how they act, not by what they are, such as Brownian motion. Brownian motion is the physical phenomenon named after the English botanist Robert Brown who discovered it in 1827. Brownian motion is the zigzag motion exhibited by a small particle, such as a grain of pollen, immersed in a liquid or a gas. Albert Einstein gave the first explanation of this phenomenon in 1905. He explained Brownian motion by assuming the immersed particle was constantly buffeted by the molecules of the surrounding medium. Since then, the abstracted process has been used for modeling the stock market and in quantum mechanics. The French mathematician and father of mathematical finance Louis Bachelier initiated the mathematical equations of Brownian motion in his thesis "Thórie de la Spćulation" (1900). Later, in the mid-seventies, the Bachelier theory was improved by the American economists Fischer Black, Myron Scholes, and Robert Merton, which has had an almost indescribable influence on today's derivative pricing and international economy. Here, Brownian motion is still very important as it is in many other more recent financial models.

In many problems that involve modeling the behavior of some system, we lack sufficiently detailed information to determine how the system behaves; or the behavior of the system is so complicated that an exact description of it becomes irrelevant or impossible. In that case, a probabilistic model is often useful. A probability space $(\Omega, \mathcal{F}, P)$ consists of:

(i) A sample space, whose points label all possible outcomes of a random trial;

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(ii) A $\sigma$-algebra $\mathcal{F}$ of measurable subsets of $\Omega$, whose elements are the events about which it is possible to obtain information;

(iii) A probability measure $P : \mathcal{F} \to [0, 1]$, where $0 \leq P(A) \leq 1$ is the probability that the event $A \in \mathcal{F}$ occurs.

If $P(A) = 1$, we almost surely say that an event $A$ occurs.

Stochastic Volterra Integral Equations (SVIEs) are the natural extension of deterministic ones that were first studied by Berger and Mizel [1,2] for equation:

$$Y(t) = Y_0 + \int_0^t f(t, s, Y(s))ds + \int_0^t g(t, s, Y(s))dW(s), \quad 0 \leq t \leq T.$$

Such equations arise in many applications such as mathematical finance, engineering, biology, medical, and social sciences [1,3-5]. Because most SVIEs cannot be solved explicitly or do not have analytic solutions, it is important to provide numerical schemes [6-8]. Also, many functions or polynomials, such as modified block pulse functions [9], triangular functions [10], generalized hat basis functions [11], Taylor series [12], delta functions [13], Chebyshev wavelets [14], and Bernstein polynomials [15], were used to derive solutions of SVIEs. However, there are still very few papers discussing the numerical solutions for stochastic Volterra integral equations.

In this paper, a stochastic operational matrix for Fibonacci polynomials is derived. Then application of this stochastic operational matrix in solving stochastic Itô-Volterra integral equation is investigated. During the last decade, operational matrices have received considerable attention for making an ideal base in the procedure of approximation [16-20]. The main characteristic behind this approach is that it reduces such problems to those of solving a system of algebraic equations which greatly simplifying the problem. Furthermore, operational matrix can be computed at once for large values of $N$, and stored for use in various problems.

Let $W(t)$ be a standard Brownian motion defined on the probability space. The aim of this paper is introducing a numerical scheme to solve Itô type of stochastic Volterra integral equation of the form:

$$Y(t) = Y_0 + \lambda_1 \int_0^t a(s, Y(s))\, ds + \lambda_2 \int_0^t b(s, Y(s))\, dW(s), \quad 0 \leq t \leq T, \quad (1)$$

where, $Y_0$ is a random variable independent of $W(t)$, $\lambda_1$ and $\lambda_2$ are parameters and $Y(t)$, $a(t, Y(t))$ and $b(t, Y(t))$ for $t \in [0, T]$ are stochastic processes defined on the some probability space $(\Omega, \mathcal{F}, P)$. Also $Y(t)$ is unknown function.

The paper is divided into 8 sections. Next Section has an introductory character and provides deterministic and stochastic tools that are needed later. In Section 3, we describe the basic properties of the Fibonacci functions and functions approximation by these functions and an integration operational matrix. In Section 4, we solve nonlinear stochastic Itô-Volterra integral equations by using the stochastic integration operational matrix. Existence of the solution is discussed in Section 5. Convergence analysis of the presented method is discussed in Section 6. In Section 7, some examples illustrate the accuracy of the presented results. Finally, Section 8 gives some brief conclusions.

## 2. Stochastic calculus

We start by recalling the definition of Brownian motion, which is a fundamental example of a stochastic process. The underlying probability space $(\Omega, \mathcal{F}, P)$ of Brownian motion can be constructed on the space $\Omega = C_0(R_+)$ of continuous real-valued functions on $R_+$ started at 0.

**Definition 1.** A scalar standard Brownian motion, or standard Wiener process, over $[0, T]$ is a random variable $W(t)$ that depends continuously on $t \in [0, T]$ and satisfies the following three conditions:

(i) $W(0) = 0$ (with probability 1);

(ii) For $0 \leq s < t \leq T$, the random variable given by the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, equivalently, $W(t) - W(s) \sim \sqrt{t-s}\ N(0, 1)$, where $N(0, 1)$ denotes a normally distributed random variable with zero mean and unit variance;

(iii) For $0 \leq s < t < u < v \leq T$, the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

Suppose that $p \geq 2$. Consider random variable $Y$ with distribution $f_Y$, so:

$$E[Y^p] = \int_{-\infty}^{\infty} Y^p f_Y dY < \infty.$$

Let $L^p(\Omega, H)$ be the collection of all strongly measurable, $p$-th integrable and $H$-valued random variables. It is routine to check that $L^p(\Omega, H)$ is a Banach space with:

$$\|V\|_{L^p(\Omega,H)} = [E\|V\|^p]^{\frac{1}{p}}.$$

**Definition 2 [21].** The sequence $Y_n$ converge to $Y$ in $L^2$ if for each $n, E(|Y_n|^2) < \infty$ and $E(\|Y_n - Y\|)^2 \to 0$ as $n \to \infty$.

Suppose $0 \leq S \leq T$, let $v = v(S,T)$ be the class of functions $f(t,w) : [0,1] \times \Omega \to R^n$, that satisfy:

(i) The function $(t,w) \to f(t,w)$ is $\beta \times \mathcal{F}$ measurable, where $\beta$ is the Borel algebra;

(ii) $f$ is adapted to $\mathcal{F}_t$;

(iii) $E\left[\int_S^T f^2(t,w)\ dt\right] < \infty$.

**Definition 3 [22].** Let $f \in v(S,T)$, then the Itô integral of $f$ is defined by:

$$\int_S^T f(t,w)dW(t)(w) = \lim_{n\to\infty} \int_S^T \varphi_n(t,w)dW(t)(w),$$

where $\varphi_n$ is the sequence of elementary functions such that:

$$E\left[\int_S^T (f-\varphi_n)^2 dt\right] \to 0, \qquad n \to \infty.$$

**Theorem 1 [22].** Let $f \in v(S,T)$, then:

$$E\left[\left(\int_S^T f(t,w)dW(t)(w)\right)^2\right] = E\left[\int_S^T f^2(t,w)dt\right].$$

## 3. Fibonacci polynomials and their properties

### *3.1. The Fibonacci polynomials*

The Fibonacci polynomials $\{F_n(x)\}$ are defined by the recursion $F_{n+2}(x) = x\ F_{n+1}(x) + F_n(x),\ n \geq 1$ with initial values $F_1(x) = 1$ and $F_2(x) = x$. They are given by the explicit formula:

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i},\ n \geq 0,$$

where $\lfloor \frac{n}{2} \rfloor$ denotes the greatest integer in $\frac{n}{2}$. Also for $x = k \in N$, we obtain the elements of the $k$-Fibonacci sequences [23,24]. Suppose that:

$$F(x) = [F_1(x), F_2(x), F_3(x), \ldots, F_{N+1}(x)]^T. \tag{2}$$

This equation can be written in the matrix form as follows:

$$F(x) = BX(x), \tag{3}$$

where $X(x) = [1, x, x^2, x^3, \ldots, x^N]^T$, and $B$ is the lower triangular matrix which its entrances are the coefficients appearing in the expansion of the Fibonacci polynomials in increasing powers of $x$. Note that matrix $B$ is invertible, so $x^n$ may be written as a linear combination of Fibonacci polynomials [23,24].

Suppose that a function $f(x)$ can be expressed in terms of the Fibonacci polynomials. In particular, only the first-$(N+1)$-term of Fibonacci polynomials is considered. Hence, the function $f(x)$ can be written in the matrix form:

$$f(x) \simeq AF(x), \tag{4}$$

where $A = [a_1, a_2, \ldots, a_{N+1}]$.

The integration of $F(x)$ is approximated as:

$$\int_0^t F(s)ds = \int_0^t BX(s)ds \simeq BP_X X(t),$$

where:

$$P_X = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{N} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

is operational matrix of integration of Taylor polynomials. Therefore:

$$\int_0^t F(s)ds \simeq PF(t), \tag{5}$$

where $P = BP_X B^{-1}$ is an $(N+1) \times (N+1)$ operational matrix; $B$ was introduced in Eq. (3).

### *3.2. Stochastic operational matrix based on Fibonacci polynomials*

Let $F(x)$ be the vector defined in Eq. (2). The Itô integral of $F(x)$ can be computed as follows:

$$\int_0^t F(s)dW(s) = \int_0^t BX(s)dW(s)$$
$$= B\left[\int_0^t dW(s), \int_0^t sdW(s), \ldots, \int_0^t s^N dW(s)\right]^T. \tag{6}$$

We can write:

$$\begin{pmatrix} \int_0^t dW(s) \\ \int_0^t sdW(s) \\ \vdots \\ \int_0^t s^N dW(s) \end{pmatrix} = W(t)X(t)$$

$$- \begin{pmatrix} 0 \\ \int_0^t W(s)ds \\ \vdots \\ N\int_0^t s^{N-1}W(s)ds \end{pmatrix} = \Lambda(t) = (\lambda_i)_{i=0,1,\ldots,N},$$

where:

$$\lambda_i = t^i W(t) - i\int_0^t s^{i-1}W(s)ds,$$

$$i = 0, 1, \ldots, N.$$

Using composite trapezium rule, we obtain:

$$\lambda_i = t^i W(t) - \frac{it}{4}\left(2(\frac{t}{2})^{i-1}W(\frac{t}{2}) + t^{i-1}W(t)\right)$$

$$= \left[(1-\frac{i}{4})W(t) - \frac{i}{2^i}W(\frac{t}{2})\right] t^i,$$

$$i = 0, 1, \ldots, N.$$

After approximating $W(t)$ and $W(\frac{t}{2})$, for $0 \leq t \leq 1$, by $W(0.5)$ and $W(0.25)$ and substituting these approximations in Eq. (6), we obtain the equations shown in Box I. Therefore:

$$B\Lambda(t) = B\Gamma_s X(t) = B\Gamma_s B^{-1}F(t) = P_s F(t),$$

where $P_s = B\Gamma_s B^{-1}$ is $(N+1)\times(N+1)$ stochastic operational matrix. So, we have:

$$\int_0^t F(s)dW(s) \simeq P_s F(t). \tag{7}$$

## 4. Method of solution

Consider the nonlinear stochastic Itô integral Eq. (1) and let:

$$\Psi_1(t) = a\left(t, Y(t)\right), \qquad \Psi_2(t) = b\left(t, Y(t)\right). \tag{8}$$

We approximate $\Psi_1(t)$ and $\Psi_2(t)$ by Fibonacci polynomials as follows:

$$\Psi_1(t) \simeq A_1 F(t), \qquad \Psi_2(t) \simeq A_2 F(t), \tag{9}$$

such that $(N+1)$-vectors $A_1$ and $A_2$ are Fibonacci coefficients of $\Psi_1(t)$ and $\Psi_2(t)$, respectively. By using Eqs. (5), (7), and (9) we have:

$$\int_0^t \Psi_1(s)ds \simeq A_1\int_0^t F(s)ds = A_1\ P\ F(t), \tag{10}$$

and:

$$\int_0^t \Psi_2(s)dW(s) \simeq A_2\int_0^t F(s)dW(s) = A_2\ P_s\ F(t). \tag{11}$$

On the other hand, from Eqs. (1) and (8) we get:

$$\begin{cases} \Psi_1(t) = a(t, Y_0 + \lambda_1\int_0^t \Psi_1(s)ds \\ \qquad +\lambda_2\int_0^t \Psi_2(s)dW(s)), \\ \\ \Psi_2(t) = b(t, Y_0 + \lambda_1\int_0^t \Psi_1(s)ds \\ \qquad +\lambda_2\int_0^t \Psi_2(s)dW(s)). \end{cases} \tag{12}$$

After substituting the approximate Eqs. (9), (10), and (11) in Eq. (12), we get:

$$\begin{cases} A_1\ F(t) = a(t, Y_0 + \lambda_1\ A_1\ P\ F(t) \\ \qquad +\lambda_2\ A_2\ P_s\ F(t)), \\ \\ A_2\ F(t) = b(t, Y_0 + \lambda_1\ A_1\ P\ F(t) \\ \qquad +\lambda_2\ A_2\ P_s\ F(t)). \end{cases} \tag{13}$$

Now, we collocate Eq. (13) in $N+1$ Newton-cotes nodes, $t_i = \frac{2i-1}{2(N+1)}$, $i = 1, 2, \ldots, N+1$, as:

$$\begin{cases} A_1\ F(t_i) = a(t_i, Y_0 + \lambda_1\ A_1\ P\ F(t_i) \\ \qquad +\lambda_2\ A_2\ P_s\ F(t_i)), \\ \\ A_2\ F(t_i) = b(t_i, Y_0 + \lambda_1\ A_1\ P\ F(t_i) \\ \qquad +\lambda_2\ A_2\ P_s\ F(t_i)). \end{cases} \tag{14}$$

$$B\Lambda(t) = B\begin{pmatrix} W(0.5) & 0 & \cdots & 0 \\ 0 & \frac{3}{4}W(0.5) - \frac{1}{2}W(0.25) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1-\frac{N}{4})W(0.5) - \frac{N}{2^N}W(0.25) \end{pmatrix}\begin{pmatrix} 1 \\ t \\ \vdots \\ t^N \end{pmatrix}.$$

Let:

$$\Gamma_s = \begin{pmatrix} W(0.5) & 0 & \cdots & 0 \\ 0 & \frac{3}{4}W(0.5) - \frac{1}{2}W(0.25) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1-\frac{N}{4})W(0.5) - \frac{N}{2^N}W(0.25) \end{pmatrix}.$$

Box I

Finally, by solving this nonlinear system with Newton's method and determining $A_1$ and $A_2$, we obtain the approximate solution of the problem as follows:

$$Y_N(t) = Y_0 + \lambda_1\ A_1\ P\ F(t) + \lambda_2\ A_2\ P_s\ F(t).$$

## 5. Existence of the solution

Picard's iteration has been used to prove the existence and uniqueness of the solution of stochastic integral equations. In [25], the authors used Schauder's fixed point theorem to give a new existence theorem about the solution of a stochastic integral equation. The theorem can weaken some conditions gotten by applying Banach's fixed point theorem.

**Theorem 2.** Let $Q = \{(t, Y(t)) \in R^2;\ t \in [0, T]$, and $|Y(t)| \leq r$ for fixed $r > 0\}$. Assume the following conditions:

(i) $a(t, Y(t)),\ b(t, Y(t)) : Q \to R$ are continuous and measurable on $[0, T] \times \Omega$;

(ii) We define:

$$d = \sup_{(t,Y(t))\in Q} \{||a(t, Y(t))||, ||b(t, Y(t))||\},$$

and let the real number $T, d$, and random variable $h$ be given that:

$$3E[h^2] + 3T(1 + T)d^2 \leq r^2.$$

(iii) We set $M = \{Y(t) \in \mathcal{X},\ \|Y(t)\| \leq r\}$ in which $\mathcal{X}$ denotes the space of all stochastic processes $f(t, w),\ 0 \leq t \leq T$ that are adapted to filtrate $\mathcal{F}_t$ and $\int_0^T E(|Y(t)|^2)dt < +\infty$.

Then the stochastic integral Eq. (1) has at least one solution $Y(t) \in M$.

**Proof [25].**

**Theorem 3.** Assume the following conditions:

(i) $a(t, Y(t))$ and $b(t, Y(t))$ are measurable on $[0, T]\times \Omega$;

(ii) $|a(t, Y(t)) - a(t, X(t))| \leq k_1|Y(t) - X(t)|$, $|b(t, Y(t)) - b(t, X(t))| \leq k_2|Y(t) - X(t)|$;

(iii) Let the real number $T, c = \max\{k_1, k_2\}$ be given such that:

$$0 \leq 2Tc^2(1 + T) < 1.$$

Then the stochastic integral Eq. (1) has a unique solution $Y(t) \in M$.

**Proof [25].**

## 6. Convergence analysis

The Fibonacci polynomials can be expressed in terms of some orthogonal polynomials, such as Chebychev polynomial of the second kind $u_n(t)$ [26]. It can be shown that:

$$F_{N+1}(t) = i^N u_N\left(\frac{t}{2i}\right).$$

that $i^2 = -1,\ N \geq 0$.

As we know, the expansion of $f(t)$ in the approximated form of Fibonacci polynomials can be written as:

$$f(t) \simeq p_n(t) = \sum_{i=1}^{N+1} a_i F_i(t).$$

On the other hand, it can eventually be expressed as:

$$p_n(t) = \sum_{j=0}^{N} c_j u_j(t),$$

where $c_j,\ j = 0, 1, \ldots, N$, can be expressed in terms of $a_i,\ i = 1, 2, \ldots, N + 1$. If $\mathfrak{u}_j(t) = \sqrt{(\frac{2}{\pi})}\ u_j(t)$, then $\mathfrak{u}_j(t), j = 0, 1, \ldots, N$, form an orthonormal polynomial basis in $[-1, 1]$ with respect to weight function $\omega(t) = (1 - t^2)^{\frac{1}{2}}$, that can be mapped into $[0, 1]$. Therefore, this procedure yields:

$$p_n(t) = \sum_{j=0}^{N} \sqrt{(\frac{2}{\pi})}\ c_j\mathfrak{u}_j(t).$$

Golberg and Chen [27] proved that when we are approximating a continuously differentiable function ($g \in C^r,\ r > 0$) by Chebychev polynomials, then:

$$\|g - p_n\|_\omega < \alpha N^{-r},$$

where $\alpha$ is some constant. So, above statements prove the following theorem.

**Theorem 4.** Suppose that $\mathbf{F}_n(g(t))$ is expansion of $g(t)$ in Fibonacci basis. For all function $g$ in $C[0, 1]$, the sequence $\{\mathbf{F}_n(g),\ n = 1, 2, \ldots\}$ converges uniformly to $g$.

**Proof.** Considering the above descriptions, proof is clear.

This theorem shows that for any $g \in C[0, 1]$ and for any $\varepsilon$, there exists $n$ such that:

$$\|\mathbf{F}_n(g) - g\| < \varepsilon.$$

We suppose $\|\cdot\|$ be the $L^2$ norm on $[0, 1]$. We define the error function as:

$$e_N(t) = Y(t) - Y_N(t),$$

in which $Y(t)$ and $Y_N(t)$ are the exact and approximate solution of Eq. (1), respectively. So, we have:

$$Y(t) - Y_N(t) = \lambda_1 \int_0^t (\Psi_1(s) - \hat{\Psi}_1(s))\, ds$$

$$+ \lambda_2 \int_0^t (\Psi_2(s) - \hat{\Psi}_2(s))\, dW(s),$$

where $\Psi_i(s),\ i = 1, 2$, is defined in Eq. (8). Also $\hat{\Psi}_i(s), i = 1, 2$, is approximated form of $\Psi_i(s),\ i = 1, 2$, by Fibonacci approximation.

**Theorem 5.** Let $Y(t)$ be exact solution and $Y_N(t)$ be the Fibonacci approximate solution of Eq. (1). Also assume that:

(i) For every $T$ and $K$, there is a constant $D$ depending only on $T$ and $K$ such that for all $|Z|,\ |Y| \leq K$ and all $0 \leq t \leq T$,

$$|a(t, Z) - a(t, Y)| + |b(t, Z) - b(t, Y)| \leq D|Z - Y|.$$

(ii) Coefficients satisfy the linear growth condition:

$$|a(t, Z)| + |b(t, Z)| \leq D(1 + |Z|).$$

(iii) $E(|Z|^2) < \infty$.

Then $Y_N(t)$ converges to $Y(t)$ in $L^2$.

**Proof.**

$$e_N(t) = \lambda_1 \int_0^t (\Psi_1(s) - \hat{\Psi}_1(s))ds$$

$$+ \lambda_2 \int_0^t (\Psi_2(s) - \hat{\Psi}_2(s))dW(s)$$

$$E\|e_N(t)\|^2 \leq 2\Big(|\lambda_1|^2 E\| \int_0^t (\Psi_1(s) - \hat{\Psi}_1(s))ds\|^2$$

$$+|\lambda_2|^2 E\| \int_0^t (\Psi_2(s) - \hat{\Psi}_2(s))dB(s)\|^2\Big).$$

From Theorem 1, we have:

$$E\|e_N(t)\|^2 \leq 2\Big[|\lambda_1|^2 \int_0^t E\|\Psi_1(s) - \hat{\Psi}_1(s)\|^2 ds$$

$$+|\lambda_2|^2 \int_0^t E\|\Psi_2(s) - \hat{\Psi}_2(s)\|^2 ds\Big]$$

$$\leq 8\Big[\ |\lambda_1|^2 \int_0^t E\|\Psi_1(s) - \Psi_1^N(s)\|^2 ds$$

$$+|\lambda_1|^2 \int_0^t E\|\Psi_1^N(s) - \hat{\Psi}_1(s)\|^2 ds.$$

$$+|\lambda_2|^2 \int_0^t E\|\Psi_2(s) - \Psi_2^N(s)\|^2 ds$$

$$+|\lambda_2|^2 \int_0^t E\|\Psi_2^N(s) - \hat{\Psi}_2(s)\|^2 ds\ \Big],$$

where $\Psi_1^N(s) = a(s, Y_N(s))$ and $\Psi_2^N(s) = b(s, Y_N(s))$. By Theorem 4, there exists $N > 0$ such that for any $\varepsilon$:

$$E\|\Psi_j^N(s) - \hat{\Psi}_j(s)\|^2 \leq \varepsilon = \frac{\varepsilon_1}{16|\lambda_j|^2}, \qquad j = 1, 2.$$

So:

$$E\|e_N(t)\|^2 \leq \varepsilon_1 + 8\Big[|\lambda_1|^2 \int_0^t E\|\Psi_1(s) - \Psi_1^N(s)\|^2 ds$$

$$+|\lambda_2|^2 \int_0^t E\|\Psi_2(s) - \Psi_2^N(s)\|^2 ds\Big].$$

Using Lipschitz condition:

$$E\|e_N(t)\|^2 \leq \varepsilon_1 + 8\left(|\lambda_1|^2 + |\lambda_2|^2\right) D^2 \int_0^t E\|e_N(s)\|^2 ds. \tag{15}$$

Hence from Eq. (15) and Gronwall inequality we have:

$$E\|e_N(t)\|^2 \to 0.$$

Therefore, $Y_N(t) \to Y(t)$ in $L^2$.

## 7. Illustrative examples

To illustrate the effectiveness of the proposed method, three examples are carried out in this section. In this regard, we have presented Tables 1 to 6. All results are computed by using a program written in Matlab. Using this method, all nonlinear examples reduce to nonlinear systems of equations that we solve them by Newton's method with zero vector as the initial guess.

Let $Y(t)$ be the exact solution and $Y_N(t)$ be the Fibonacci approximate solution of Eq. (1), then we define the error in the interval [0, 1] as:

$$\|E\|_\infty = \max|e_N(t_i)|, \qquad 0 \leq t_i \leq 1,$$

where $e_N(t_i) = Y(t_i) - Y_N(t_i)$.

**Example 1 [8].** Consider the following nonlinear stochastic Itô-Voletrra integral equation:

$$Y(t) = 0.5 + \int_0^t Y(s)(1 - Y(s))ds$$

$$+0.25 \int_0^t Y(s)dW(s), \qquad 0 \leq t \leq 1, \tag{16}$$

with the exact solution:

$$Y(t) = \frac{0.5\exp(0.96875t + 0.25W(t))}{1 + 0.5\int_0^t \exp(0.96875t + 0.25W(t))ds}. \tag{17}$$

This integral equation is a simple model for the size

**Table 1.** Mean, standard deviation, and confidence interval for error mean of Example (1) with $N = 8$ and $k = 100$.

| $t_i$ | $\bar{Y}_E$ | $S_E$ | 0.95 Confidence interval Lower bound | 0.95 Confidence interval Upper bound |
|---|---|---|---|---|
| 0.0 | 0.066787678803251 | 0.050070727528147 | 0.056973816207734 | 0.076601541398768 |
| 0.1 | 0.068740589266325 | 0.051647935696043 | 0.058617593869901 | 0.078863584662750 |
| 0.2 | 0.070747616059190 | 0.053254210986865 | 0.060309790705764 | 0.081185441412616 |
| 0.3 | 0.072801356941015 | 0.054938682939085 | 0.062033375084954 | 0.083569338797075 |
| 0.4 | 0.074900637058035 | 0.056754992712421 | 0.063776658486400 | 0.086024615629669 |
| 0.5 | 0.077121623459108 | 0.058668422068519 | 0.065622612733678 | 0.088620634184538 |
| 0.6 | 0.079469608333069 | 0.060755397319355 | 0.067561550458475 | 0.091377666207662 |
| 0.7 | 0.081954627875088 | 0.063102268436932 | 0.069586583261449 | 0.094322672488726 |
| 0.8 | 0.084584835331435 | 0.065812268823923 | 0.071685630641946 | 0.097484040020924 |
| 0.9 | 0.087465163472625 | 0.068874591335778 | 0.073965743570812 | 0.100964583374437 |

**Table 2.** Approximate infinity-norm of absolute error and CPU time(s) for Example (1).

| Methods | Maximum error $\bar{Y}_E$ | Maximum error $S_E$ | CPU time (s) |
|---|---|---|---|
| Present method | 0.0806 | 0.0508 | 3350.4 |
| Method of [15] | 0.0842 | 0.0764 | 3477.6 |

$Y$ of a population at time $t$ that is the model of exponential growth:

$$dY(t) = cY(t)dt, \tag{18}$$

where $c$ is the growth coefficient. An appropriate modification of Eq. (18) is given as a linear quadratic Verhulst equation:

$$dY(t) = Y(t)(\gamma - Y(t))dt, \tag{19}$$

where the population growth $c$ is replaced by $\gamma - Y(t)$. By randomizing the parameter $\gamma$ in Eq. (19) to $\gamma + \sigma\xi(t)$, where $\xi(t) = \frac{dW(t)}{dt}$ is a white noise of zero mean, we have the usual stochastic Verhulst equation describing more precisely the population dynamics:

$$dY(t) = Y(t)(\gamma - Y(t))dt + \sigma Y(t)dW(t), \tag{20}$$

in which $\gamma$ and $\sigma$ are positive constants [28-30]. Eq. (20) can be shown as an Iô-Voletrra integral equation:

$$Y(t) = Y_0 + \int_0^t Y(s)(\gamma - Y(s))ds + \int_0^t \sigma Y(s)dW(s), \tag{21}$$

with the exact solution:

$$Y(t) = \frac{Y_0 e^{(\gamma - \frac{1}{2}\sigma^2)t + \sigma W(t)}}{1 + Y_0 \int_0^t e^{(\gamma - \frac{1}{2}\sigma^2)s + \sigma W(s)} ds}. \tag{22}$$

**Table 4.** Approximate infinity-norm of absolute error and CPU time(s) for Example (2).

| Methods | Maximum error $\bar{Y}_E$ | Maximum error $S_E$ | CPU time (s) |
|---|---|---|---|
| Present method | 0.1363 | 0.0748 | 899.4 |
| Method of [15] | 0.1366 | 0.1113 | 1492.4 |

**Table 3.** Mean, standard deviation, and confidence interval for error mean of Example (2) with $N = 8$ and $k = 100$.

| $t_i$ | $\bar{Y}_E$ | $S_E$ | 0.95 Confidence interval Lower bound | 0.95 Confidence interval Upper bound |
|---|---|---|---|---|
| 0.0 | 0.135475215100671 | 0.099949025190492 | 0.115885206163334 | 0.155065224038007 |
| 0.1 | 0.094577113462763 | 0.067936547239583 | 0.081261550203805 | 0.107892676721722 |
| 0.2 | 0.065641356637775 | 0.046213924490262 | 0.056583427437683 | 0.074699285837866 |
| 0.3 | 0.046288849652677 | 0.032467521662404 | 0.039925215406846 | 0.052652483898509 |
| 0.4 | 0.036989038086022 | 0.025052683527720 | 0.032078712114589 | 0.041899364057455 |
| 0.5 | 0.040498134169732 | 0.019387041723278 | 0.036698273991969 | 0.044297994347494 |
| 0.6 | 0.051630444905859 | 0.015320410634470 | 0.048627644421502 | 0.054633245390215 |
| 0.7 | 0.066272860190579 | 0.010784705923229 | 0.064159057829626 | 0.068386662551532 |
| 0.8 | 0.082371739311274 | 0.007702512651059 | 0.080862046831666 | 0.083881431790881 |
| 0.9 | 0.100421704127271 | 0.006322909537695 | 0.099182413857882 | 0.101660994396659 |

**Table 5.** Mean, standard deviation, and confidence interval for error mean of Example (3) with $N = 8$ and $k = 100$.

| $t_i$ | $\bar{Y}_E$ | $S_E$ | 0.95 Confidence interval | |
|---|---|---|---|---|
| | | | *Lower bound* | *Upper bound* |
| 0.0 | 0.009355665647819 | 0.006755459612363 | 0.008031595563796 | 0.010679735731842 |
| 0.1 | 0.031217002113644 | 0.024322569914853 | 0.026449778410333 | 0.035984225816955 |
| 0.2 | 0.040897395484443 | 0.027392057364539 | 0.035528552240994 | 0.046266238727893 |
| 0.3 | 0.048795606340377 | 0.038178679020026 | 0.041312585252452 | 0.056278627428302 |
| 0.4 | 0.055188393237043 | 0.041477690239870 | 0.047058765950029 | 0.063318020524058 |
| 0.5 | 0.061091301361998 | 0.045404732373637 | 0.052191973816765 | 0.069990628907231 |
| 0.6 | 0.066550166176714 | 0.055272371925028 | 0.055716781279409 | 0.077383551074020 |
| 0.7 | 0.074798297097570 | 0.057252918866680 | 0.063576724999701 | 0.086019869195439 |
| 0.8 | 0.084017239030808 | 0.069273995253571 | 0.070439535961108 | 0.097594942100508 |
| 0.9 | 0.097281470236254 | 0.074331818594678 | 0.082712433791697 | 0.111850506680811 |

**Table 6.** Approximate infinity-norm of absolute error and CPU time(s) for Example (3).

| Methods | Maximum error | | CPU time (s) |
|---|---|---|---|
| | $\bar{Y}_E$ | $S_E$ | |
| Present method | 0.0085 | 0.0791 | 3761.1 |
| Method of [15] | 0.1002 | 0.0791 | 6509.9 |

By considering $Y_0 = 0.5$, $\gamma = 1$, and $\sigma = 0.25$ in Eq. (21), we get Eq. (16) that can be solved by the proposed method. The numerical results for this example are shown in Tables 1 and 2. In these tables, $\bar{Y}_E$ is the error mean and $S_E$ is the standard deviation of errors in $k$ iteration. In Table 2, we compare the maximum absolute error and measured CPU time(s) for the present method and Bernstein functions method [15] with $N = 8$ and $k = 20$.

**Example 2 [8].** Consider the following nonlinear stochastic Itô-Voletrra integral equation:

$$Y(t) = 1 + \int_0^t Y(s)(\frac{1}{32} - Y^2(s))ds$$
$$+0.25 \int_0^t Y(s)dW(s), \qquad 0 \leq t \leq 1, \tag{23}$$

with the exact solution:

$$Y(t) = \frac{\exp(0.25W(t))}{\sqrt{1 + 2\int_0^t \exp(0.5W(s))ds}}. \tag{24}$$

The numerical results for this example are shown in Tables 3 and 4. In these tables, $\bar{Y}_E$ is the error mean and $S_E$ is the standard deviation of errors in $k$ iteration. In Table 4, we compare the maximum absolute error and measured CPU time(s) for the present method and Bernstein functions method [15] with $N = 8$ and $k = 20$.

**Example 3 [8].** Consider the following nonlinear stochastic Itô-Voletrra integral equation:

$$Y(t) = \frac{1}{8} - 0.015625 \int_0^t Y(s)(1 - Y^2(s))ds$$
$$+0.125 \int_0^t (1 - Y^2(s))dW(s), \qquad 0 \leq t \leq 1, \tag{25}$$

with the exact solution:

$$Y(t) = \frac{\frac{9}{8}e^{0.25W(t)} - \frac{7}{8}}{\frac{9}{8}e^{0.125W(t)} + \frac{7}{8}}. \tag{26}$$

The numerical results for this example are shown in Tables 5 and 6. In these tables, $\bar{Y}_E$ is the error mean and $S_E$ is the standard deviation of errors in $k$ iteration. In Table 6, we compare the maximum absolute error and measured CPU time(s) for the present method and Bernstein functions method [15], with $N = 8$ and $k = 20$.

## 8. Conclusion

Because it is almost impossible to find the exact solution of Eq. (1), it would be convenient to determine its numerical solution based on stochastic numerical analysis. This paper suggested a numerical method to solve the nonlinear stochastic Itô-Volterra integral equations by using Fibonacci polynomials and their operational matrices. Moreover, the stochastic operational matrix of Itô-integration for Fibonacci functions was derived and the convergence and error analyses of the proposed method were established. Efficiency of this method and a good reasonable degree of accuracy is confirmed by three numerical examples. Furthermore, the results of the present method have been compared with analytical solution and the solution of Bernstein method [15].

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## References

1. Berger, M. and Mizel, V. "Volterra equations with Itô integrals", *I. J. Int. Equ.*, **2**, pp. 187-245 (1980).
2. Berger, M. and Mizel, V. "Volterra equations with Itô integrals", *II. J. Int. Equ.*, **2**, pp. 319-337 (1980).
3. Appley, J.A.D., Devin, S. and Reynolds, D.W. "Almost sure convergence of solutions of linear stochastic Volterra equations to non-equilibrium limits", *J. Integ. Equ. Appl.*, **19**(4), pp. 405-437 (2007).
4. Pardoux, E. and Protter, P. "Stochastic Volterra equations with anticipating coefficients", *Ann. Probab.*, **18**, pp. 1635-1655 (1990).
5. Shiota, Y. "A linear stochastic integral equation containing the extended Itô integral", *Math. Rep.*, **9**, pp. 43-65 (1986).
6. Khodabin, M., Maleknejad, K., Rostami, M. and Nouri, M. "Numerical solution of stochastic differential equations by second order Runge-Kutta methods", *Math. Comput. Model.*, **53**, pp. 1910-1920 (2011).
7. Khodabin, M., Maleknejad, K., Rostami, M. and Nouri, M. "Interpolation solution in generalized stochastic exponential population growth model", *Appl. Math. Model.*, **36**(3), pp. 1023-1033 (2012).
8. Kloeden, P.E. and Platen, E. "Numerical solution of stochastic differential equations", *Appl. Math.*, Springer-Verlag, Berlin (1995).
9. Maleknejad, K., Khodabin, M. and Hosseini Shekarabi, F. "Modified block pulse functions for numerical solution of stochastic Volterra integral equations", *J. Appl. Math.*, **2014**, pp. 1-10 (2014).
10. Khodabin, M., Maleknejad, K. and Hosseini Shekarabi, F. "Application of triangular functions to numerical solution of stochastic Volterra integral equations", *IAENG Int. J. Appl. Math.*, **43**(1), pp. 1-9 (2013).
11. Heydari, M.H., Hooshmandasl, M.R., Cattani, C. and Maalek Ghaini, F.M. "An efficient computational method for solving nonlinear stochastic Itô integral equations: Application for stochastic problems in physics", *J. Comput. Phys.*, **283**, pp. 148-168 (2015).
12. Khodabin, M., Maleknejad, K. and Damercheli, T. "Approximate solution of the stochastic Volterra integral equations via expansion method", *Int. J. Indus. Math.*, **6**(1), pp. 41-48 (2014).
13. Mirzaee, F. and Hadadiyan, E. "A collocation technique for solving nonlinear stochastic Itô-Volterra integral equations", *Appl. Math. Comput.*, **247**, pp. 1011-1020 (2014).
14. Mohammadi, F. "A wavelet-based computational method for solving stochastic Itô-Volterra integral equations", *J. Comput. Phys.*, **298**, pp. 254-265 (2015).
15. Asgari, M., Hashemizadeh, E., Khodabin, M. and Maleknejad, K. "Numerical solution of nonlinear stochastic integral equation by stochastic operational matrix based on Bernstein polynomials", *Bull. Math. Soc. Sci. Math. Roumanie*, **1**, pp. 3-12 (2014).
16. Bhrawy, A.H. and Abdelkawy, M.A. "A fully spectral collocation approximation formulti-dimensional fractional Schrödinger equations", *J. Comput. Phys.*, **294**, pp. 462-483 (2015).
17. Bhrawy, A.H., Doha, E.H., Ezz-Eldien, S.S. and Abdelkawy, M.A. "A numerical technique based on the shifted Legendre polynomials for solving the time-fractional coupled KdV equations", *Calcolo* (2015) (In press).
18. Bhrawy, A.H. and Zaky, M.A. "A method based on the Jacobi tau approximation for solving multi-term time-space fractional partial differential equations", *J. Comput. Phys.*, **281**, pp. 876-895 (2015).
19. Bhrawy, A.H. and Zaky, M.A. "Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation", *Nonlinear Dyn.*, **80**, pp. 101-116 (2015).
20. Doha, E.H., Bhrawy, A.H. and Ezz-Eldien, S.S. "A new Jacobi operational matrix: An application for solving fractional differential equations", *Appl. Math. Model.*, **36**, pp. 4931-4943 (2012).
21. Klebaner, F., *Introduction to Stochastic Calculus with Applications*, Second Edition, Imperial college Press (2005).
22. Oksendal, B., *Stochastic Differential Equations, an Introduction with Applications*, Fifth Edition, Springer-Verlag, New York (1998).
23. Falcon, S. and Plaza, A. "On k-Fibonacci sequences and polynomials and their derivatives", *Chaos Soliton Fract.*, **39**, pp. 1005-1019 (2009).
24. Mirzaee, F. and Hoseini, S.F. "Solving systems of linear Fredholm integro-differential equations with Fibonacci polynomials", *Ain Shams Engin. J.*, **5**(1), pp. 271-283 (2014).
25. Chen, X., Qi, Y. and Yang, C. "New existence theorems about the solutions of some stochastic integral equations", arXiv preprint arXiv:1211.1249 (2012).
26. Rainville, E.D., *Special Functions*, New York (1960).
27. Golberg, M.A. and Chen, C.S., *Discrete Projection Methods for Integral Equations*, Southampton: Comput. Mechanics Pub. (1997).
28. Gard, T.C., *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York (1988).
29. Gard, T.C. and Kannan, D. "On a stochastic differential equations modeling of prey-predator evolution", *J. Appl. Prob.*, **13**, pp. 413-429 (1976).
30. Schoener, T.W. "Population growth regulated by intraspecific competition for energy or time: some simple representations", *Theor. Pop. Biol.*, **4**, pp. 56-84 (1973).

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