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A new complex-valued method and its applications in solving differential equations

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Abstract. In this paper, complex operational matrices of Euler functions and their interesting properties are obtained to provide a novel method for solving linear complex differential equations under mixed initial conditions. Convergence conditions of this method are studied in depth, and numerical experiments show the efficiency of this method. In addition, reasonable numerical results are obtained by selecting a small number of basis functions.

1. Introduction

The study of differential equations is a wide field in pure and applied mathematics, physics, and engineering. All these disciplines are concerned with the properties of differential equations of various types. Complex differential equations, such as those used to solve real-life problems [1], may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods. Many authors have investigated complex linear differential equations and achieved many valuable results [2-8]. We assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinnas theory of meromorphic functions and the theory of complex linear differential equations [9].

The basic motivation of this work, by considering the mentioned studies, is to develop a new numerical method, called a matrix method, based on Euler polynomials [10,11], and collocation points for the approximate solution of complex differential equation. We will consider high-order linear complex differential equations with variable coefficients:

$$\sum_{k=0}^{m} \mu_k(x+it)y^{(k)}(x+it) = g(x+it),$$

$$m \geq 1, \quad a \leq x \leq b, \quad c \leq t \leq d, \tag{1}$$

which is a generalized case of the complex differential equations given in [12,13], with the following mixed initial conditions:

$$\sum_{i=0}^{m-1} c_{ik} y^{(k)}(0) = \lambda_k, \qquad k = 0, 1, \cdots, m-1, \tag{2}$$

where $\mu_k(z)$ and $g(z)$ are analytic functions and c_{ik} and λ_k are real or complex constants. The aim of this study is to obtain a solution to problems in Eqs. (1) and (2) as the truncated Euler series, defined by:

$$y(z) = \sum_{n=0}^{N} \rho_n E_n(z),$$

$$z = x + it, \quad a \leq x \leq b, \quad c \leq t \leq d, \tag{3}$$

so that $\rho_n, n = 0, 1, 2, ..., N$ are the unknown Euler

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coefficients; N is chosen as any positive integer such that $N \geq m$ and $E_n(x), n = 0, 1, 2, ..., N$ are the Euler polynomials of the first kind defined by the generating function [14]:

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{N} E_n(x)\frac{t^n}{n!}. \tag{4}$$

The reader is referred to [15] for some advantages of Euler functions, in approximating an arbitrary unknown function, over some classical orthogonal polynomials. This paper is outlined as follows. In the next section, our own contribution, i.e. the new matrix method, is presented. In Section 3, the main results for convergence properties are proposed. We provide four numerical examples to show the feasibility and effectiveness of the proposed method in Section 4. Finally, concluding remarks are given in Section 5.

2. Method description and problem formulation

The Euler basis polynomials, $E_n(x)$, of degree n are constructed from the following relation:

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n, \quad n = 0, 1, ..., N. \tag{5}$$

Euler polynomials satisfy the well-known relation:

$$E_n'(x) = nE_{n-1}(x), \quad n \geq 1. \tag{6}$$

With the aid of Eq. (3), we convert the solution $y(z)$ to matrix form:

$$y(z) = \mathbf{E}(z)\rho, \tag{7}$$

where $\mathbf{E}(z) = [E_0(z), E_1(z), E_2(z), \ldots, E_N(z)]$ and $\rho = [\rho_0, \rho_1, \rho_2, \ldots, \rho_N]^T$. Secondly, the relation between $\mathbf{E}(z)$ and its first derivative, $\mathbf{E}'(z)$, is given by Eq. (6). So, we have:

$$\mathbf{E}'(z) = \mathbf{E}(z)\mathbf{M},$$

and, repeating the process:

$$\mathbf{E}^{(k)}(z) = \mathbf{E}(z)\mathbf{M}^k, \qquad k = 0, 1, 2, ..., \tag{8}$$

where:

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Note that $\mathbf{M^0}$ is the unit matrix defined as:

$$\mathbf{M}^0 = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Consequently, from the matrix forms in Eqs. (7) and (8), we have:

$$y^{(k)}(z) = \mathbf{E}(z)\mathbf{M}^k\rho, \qquad k = 0, 1, 2, ... \tag{9}$$

We recall that the Euler expansion is not based on orthogonal functions, but it possesses the operational matrices of differentiations (and also integration). We are now ready to construct the fundamental matrix equation corresponding to problem in Eq. (1). To compute the unknown Euler coefficients, we use the collocation points defined by:

$$z_{jj} = x_j + it_j,$$

so that:

$$x_j = a + \frac{b-a}{N}j, \qquad t_j = c + \frac{d-c}{N}j,$$

$$j = 0, 1, 2, ..., N. \tag{10}$$

Now, we put the collocation points into Eq. (9) and, thus, we can write:

$$\mathbf{Y^{(k)}} = \begin{bmatrix} y^{(k)}(z_{00}) \\ y^{(k)}(z_{11}) \\ y^{(k)}(z_{22}) \\ \vdots \\ y^{(k)}(z_{NN}) \end{bmatrix} = \mathbf{EM^k}\rho, \tag{11}$$

where:

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}(z_{00}) \\ \mathbf{E}(z_{11}) \\ \mathbf{E}(z_{22}) \\ \vdots \\ \mathbf{E}(z_{NN}) \end{bmatrix}$$

$$= \begin{bmatrix} E_0(z_{00}) & E_1(z_{00}) & E_2(z_{00}) & ,\ldots & E_N(z_{00}) \\ E_0(z_{11}) & E_1(z_{11}) & E_2(z_{11}) & ,\ldots & E_N(z_{11}) \\ E_0(z_{22}) & E_1(z_{22}) & E_2(z_{22}) & ,\ldots & E_N(z_{22}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_0(z_{NN}) & E_1(z_{NN}) & E_2(z_{NN}) & ,\ldots & E_N(z_{NN}) \end{bmatrix}, \tag{12}$$

and:

$$y^{(k)}(z_{jj}) = \mathbf{E}(z_{jj})\mathbf{M}^k\rho, \quad j = 0, 1, 2, ..., N. \tag{13}$$

On the other hand, by substituting the collocation points (10) into Eq. (1), we have:

$$\sum_{k=0}^{m} \mu_k(z_{jj}) y^{(k)}(z_{jj}) = g(z_{jj}), \quad j = 0, 1, 2, \ldots.$$

Briefly, this system can be written in the matrix form:

$$\sum_{k=0}^{m-1} \mu_k \underbrace{\mathbf{Y}^{(k)}}_{\mathbf{EM}^k \rho} = \mathbf{G},$$

and, therefore, the fundamental matrix equation is:

$$\left\{ \sum_{k=0}^{m} \mu_k \mathbf{EM}^k \right\} \rho = \mathbf{G}, \tag{14}$$

where:

$$\mu_k = \begin{bmatrix} \mu_k(z_{00}) & 0 & \ldots & 0 \\ 0 & \mu_k(z_{11}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_k(z_{NN}) \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} g(z_{00}) \\ g(z_{11}) \\ g(z_{22}) \\ \vdots \\ g(z_{NN}) \end{bmatrix}.$$

Now, let us write briefly the fundamental matrix equation (Eq. (14)), corresponding to Eq. (1) as:

$$\mathbf{W}\rho = [\mathbf{G}] \quad \text{or} \quad [\mathbf{W}; \mathbf{G}]. \tag{15}$$

This system corresponds to a linear system of $N+1$ algebraic equations in $N+1$ unknown Euler coefficients, such that:

$$\mathbf{W} = \sum_{k=0}^{m} \mu_k \mathbf{EM}^k.$$

We can obtain the corresponding matrix form to conditions (2) as follows. By means of Eq. (9), we have the matrix equation:

$$\sum_{i=0}^{m-1} c_{ik} \mathbf{E}(0) \mathbf{M}^k \rho = [\lambda_k], \qquad k = 0, 1, \ldots, m-1,$$

briefly:

$$\mathbf{U}_k \rho = [\lambda_k] \quad \text{or} \quad [\mathbf{U}_k; \lambda_k] \quad k = 0, 1, \ldots, m-1, \tag{16}$$

where:

$$\mathbf{U}_k = [u_{k0}, u_{k1}, \ldots, u_{kN}] = \sum_{i=0}^{m-1} c_{ik} \mathbf{E}(0) \mathbf{M}^k \rho.$$

Consequently, to obtain the solution to Eq. (1) by replacing m rows of the matrix in Eq. (15) by the row matrices in Eq. (16) such that rank $\mathbf{W}^* = rank[\mathbf{W}^*; \mathbf{G}^*] = N+1$, we have the new augmented matrix:

$$\mathbf{W}^* \rho = [\mathbf{G}^*] \quad \text{or} \quad [\mathbf{W}^*; \mathbf{G}^*]. \tag{17}$$

The above system can be solved by using well-known software such as Matlab. We use Matlab software while we solve our examples in the section concerning numerical examples. Thus, we can write:

$$\rho = (\mathbf{W}^*)^{-1} \mathbf{G}^*,$$

and, hence, elements $\rho_0, \rho_1 \ldots, , \rho_N$ of ρ are uniquely determined.

3. Error analysis and convergence

In this section, we perform the error estimation for Eq. (1).

Theorem 1 [16]. Function $f(x) \in L^2[0,1]$ can be expanded as a finite sum of the Euler series. Then, coefficients f_n, for all $n = 0, 1, \ldots, N$, can be calculated as follows:

$$f_n = \begin{cases} \frac{1}{N!} \int_0^1 f^{(N)}(x) dx, \qquad n = N \\ \frac{1}{n!} \Big(\int_0^1 f^{(n)}(x) dx + \sum_{k=0}^{N-n-1} \frac{2(n!)}{k+2} \\ \qquad \binom{k+n+1}{k+1} E_{k+2}(0) f_{n+k+1} \Big), \\ \quad n = N-1, N-2, \ldots, 0. \end{cases} \tag{18}$$

Theorem 2 [16]. Function $m(x,t) \in L^2[0,1] \times L^2[0,1]$ can be expanded as a finite sum of two variable truncated Euler series, $\sum_{r=0}^{N} \sum_{s=0}^{N} m_{r,s} E_r(x) E_s(t)$, and coefficients $m_{r,s}$, for all $r, s = 0, 1, \ldots N$, can be calculated from the following backward linear relation:

$$\begin{aligned} &\int_0^1 \int_0^1 \frac{\partial^{r+s} m(x,t)}{\partial x^r \partial t^s} dx dt \\ &\quad = \sum_{i=r}^{N} \sum_{j=s}^{N} \frac{4(r!s!) g_{i,j}}{(i-r+1)(j-s+1)} \\ &\quad \binom{i}{i-r} \binom{j}{j-s} E_{i-r+1}(0) E_{j-s+1}(0). \end{aligned} \tag{19}$$

Theorem 3 [17]. The following relationship:

$$E_n(x) = \sum_{k=0}^{n} \frac{-2}{k+1} \binom{n}{k} E_{k+1}(0) B_{n-k}(x),$$

holds true between classical Euler polynomials and classical Bernoulli polynomials. According to the above theorem, function approximations in Eqs. (18) and (19) expressed by Euler polynomials, can convert to corresponding approximate polynomial in terms of Bernoulli polynomials.

Theorem 4 [18]. Let $m(x,t)$ and $f(x)$ be enough smooth functions and $m_N(x,t)$ and $f_N(x)$ be the approximated polynomials in terms of Euler polynomials, respectively. Then, the error bounds would be obtained as follows:

(i) $\|f(x) - f_N(x)\|_\infty \leq C\hat{F}(2\pi)^{-N}$, $x \in [0,1]$, where $\hat{F} = \sup\{||f^{(i)}(x)||_\infty, i = 0, 1, ...\}$.

(ii) $\|m(x,t) - m_N(x,t)\|_\infty \leq \hat{C}N(2\pi)^{-N}$, where $\hat{C}$ is a positive constant.

In the following Theorem, we consider $m = 1$ and $c_{ik} = 1$ in Eqs. (1) and (2) for clarity of presentation. We note that a similar procedure can be applied for the case of other values of m. Eq. (1), with the above-mentioned assumptions, has the following form:

$$y'(z) = \mu(z)y(z) + g(z), \tag{20}$$

where:

$$y(z) = \omega(x,t) + i\tau(x,t), \quad \mu(z) = \delta(x,t) + i\zeta(x,t),$$

$$g(z) = \alpha(x,t) + i\gamma(x,t). \tag{21}$$

Also, the initial condition is $y(0) = \omega(0,0) + i\tau(0,0) = \mu_1 + i\mu_2$.

Consider the following assumptions:

(H_1) $\mathbf{\Omega}(x,t) = [\omega(x,t), \tau(x,t)]^T$,

$$\mathbf{\Gamma}(x,t) = [\overline{\alpha}(x,t), \overline{\gamma}(x,t)]^T,$$

$$\mathbf{\Delta}(x,t) = \begin{bmatrix} \delta(x,t) & \zeta(x,t) \\ -\zeta(x,t) & \delta(x,t) \end{bmatrix},$$

$$\overline{\alpha}(x,t) = \int_0^x \int_0^t \alpha_t(x,t)dxdt + \mu_1, \quad \text{and}$$

$$\overline{\gamma}(x,t) = \int_0^x \int_0^t \gamma_t(x,t)dxdt + \mu_2.$$

(H_2) $||\mathbf{\Delta}(x,t)||_\infty \leq \overline{\Delta} < 1$, where $\overline{\Delta}$ is a real constant.

(H_3) $$||\mathbf{\Omega}(x,0) - \mathbf{\Omega}_N(x,0)||_\infty \leq \frac{1}{N}||\mathbf{\Omega}(x,t) - \mathbf{\Omega}_N(x,t)||_\infty,$$

where $\mathbf{\Omega}(x,t)$ and $\mathbf{\Omega}_N(x,t)$ are approximated by $\mathbf{\Omega}_N(x,t)$ and $\mathbf{\Omega}_N(x,0)$, respectively, by the aid of Euler polynomials. It should be noted that the condition (H_3) is based upon Theorem 4.

Theorem 5. Let the assumptions $(H_1) - (H_3)$ be satisfied. If $[a,b] = [c,d] = [0,1]$, and we use a collocation scheme for providing the numerical solution of Eq. (20), then:

$$\lim_{n\to\infty} \mathbf{\Omega}_N(x,t) = \mathbf{\Omega}(x,t).$$

Proof. Since $y(z)$ is an analytic function, then Eq. (20) could be rewritten as follows:

$$\omega_x(x,t) + i\tau_x(x,t) = (\delta(x,t) + i\zeta(x,t))(\omega(x,t) + i\tau(x,t)) + \alpha(x,t) + i\gamma(x,t). \tag{22}$$

In other words:

$$\begin{cases} \omega_x(x,t) - \big(\delta(x,t)\omega(x,t) - \zeta(x,t)\tau(x,t)\big) \\ \qquad = \alpha(x,t) \\ \tau_x(x,t) - \big(\delta(x,t)\tau(x,t) + \zeta(x,t)\omega(x,t)\big) \\ \qquad = \gamma(x,t) \end{cases} \tag{23}$$

We differentiate the above equations with respect to t, and then integrate with respect to x and t in the rectangular $[0,x] \times [0,t]$. Therefore, we have:

$$\begin{cases} \omega_{xt}(x,t) - \frac{\partial}{\partial t}\big(\delta(x,t)\omega(x,t) - \zeta(x,t)\tau(x,t)\big) \\ \qquad = \alpha_t(x,t) \\ \tau_{xt}(x,t) - \frac{\partial}{\partial t}\big(\delta(x,t)\tau(x,t) + \zeta(x,t)\omega(x,t)\big) \\ \qquad = \gamma_t(x,t) \end{cases} \tag{24}$$

and:

$$\begin{cases} \int_0^x \int_0^t \omega_{xt}(x,t)dxdt - \int_0^x \int_0^t \frac{\partial}{\partial t}\big(\delta(x,t)\omega(x,t) \\ \qquad -\zeta(x,t)\tau(x,t)\big)dxdt = \int_0^x \int_0^t \alpha_t(x,t)dxdt \\ \int_0^x \int_0^t \tau_{xt}(x,t)dxdt - \int_0^x \int_0^t \frac{\partial}{\partial t}\big(\delta(x,t)\tau(x,t) \\ \qquad +\zeta(x,t)\omega(x,t)\big)dxdt = \int_0^x \int_0^t \gamma_t(x,t)dxdt \end{cases} \tag{25}$$

By imposing the initial conditions, we have:

$$\omega(x,t) - \int_0^x \Big(\big(\delta(x,t)\omega(x,t) - \zeta(x,t)\tau(x,t)\big) - \big(\delta(x,0)\omega(x,0) - \zeta(x,0)\tau(x,0)\big) \Big) dx = \overline{\alpha}(x,t), \tag{26}$$

and:

$$\tau(x,t) - \int_0^x \Big(\big(\delta(x,t)\tau(x,t) + \zeta(x,t)\omega(x,t)\big) - \big(\delta(x,0)\tau(x,0) + \zeta(x,0)\omega(x,0)\big) \Big) dx = \overline{\gamma}(x,t). \tag{27}$$

Eq. (27) could be restated in the following matrix vector form:

$$\boldsymbol{\Omega}(x,t)=\int_0^x \boldsymbol{\Delta}(x,t)\boldsymbol{\Omega}(x,t)dx-\int_0^x \boldsymbol{\Delta}(x,0)\boldsymbol{\Omega}(x,0)dx+\boldsymbol{\Gamma}(x,t). \tag{28}$$

Now, we use an Euler collocation method for providing the numerical solution of Eq. (28). Then, we have:

$$\boldsymbol{\Omega}_N(x,t)=\int_0^x \boldsymbol{\Delta}(x,t)\boldsymbol{\Omega}_N(x,t)dx-\int_0^x \boldsymbol{\Delta}(x,0)\boldsymbol{\Omega}_N(x,0)dx+\boldsymbol{\Gamma}(x,t)+\mathbf{R}_{N+1}(x,t), \tag{29}$$

where $\mathbf{R}_{N+1}(x,t)$ is the residual function that is zero at the collocation nodes. By subtracting Eq. (29) from Eq. (28) we have:

$$\begin{aligned}&\Big\|\boldsymbol{\Omega}(x,t)-\boldsymbol{\Omega}_N(x,t)\Big\|_\infty\\&=\Big\|\int_0^x(\boldsymbol{\Delta}(x,t)\boldsymbol{\Omega}(x,t)-\boldsymbol{\Delta}(x,0)\boldsymbol{\Omega}(x,0)\\&\quad-\boldsymbol{\Delta}(x,t)\boldsymbol{\Omega}_N(x,t)+\boldsymbol{\Delta}(x,0)\boldsymbol{\Omega}_N(x,0))\mathbf{d}x\\&\quad-\mathbf{R}_{N+1}(x,t)\Big\|_\infty\\&\le\Big\|\int_0^x(\boldsymbol{\Delta}(x,t)\boldsymbol{\Omega}(x,t)-\boldsymbol{\Delta}(x,t)\boldsymbol{\Omega}_N(x,t))\mathbf{d}x\Big\|_\infty\\&\quad+\Big\|\int_0^x(\boldsymbol{\Delta}(x,0)\boldsymbol{\Omega}(x,0)-\boldsymbol{\Delta}(x,0)\boldsymbol{\Omega}_N(x,0))\mathbf{d}x\Big\|_\infty\\&\quad+\Big\|\mathbf{R}_{N+1}(x,t)\Big\|_\infty\le\overline{\Delta}\int_0^x(\Big\|\boldsymbol{\Omega}(x,t)-\boldsymbol{\Omega}_N(x,t)\Big\|_\infty\\&\quad+\Big\|\boldsymbol{\Omega}(x,0)-\boldsymbol{\Omega}_N(x,0)\Big\|_\infty)\mathbf{d}x+\Big\|\mathbf{R}_{N+1}(x,t)\Big\|_\infty\\&\le\overline{\Delta}\left(\Big\|\boldsymbol{\Omega}(x,t)-\boldsymbol{\Omega}_N(x,t)\Big\|_\infty\left(1+\frac{1}{N}\right)\right)\\&\quad+\Big\|\mathbf{R}_{N+1}(x,t)\Big\|_\infty.\end{aligned} \tag{30}$$

Then, the error bound would be obtained as follows:

$$\Big\|\boldsymbol{\Omega}(x,t)-\boldsymbol{\Omega}_N(x,t)\Big\|_\infty\le\frac{\Big\|\mathbf{R}_{N+1}(x,t)\Big\|_\infty}{1-\overline{\Delta}\left(1+\frac{1}{N}\right)}. \tag{31}$$

Since:

$$\lim_{n\to\infty}\Big\|\mathbf{R}_{N+1}(x,t)\Big\|_\infty=0,$$

we have:

$$\lim_{n\to\infty}\Big\|\boldsymbol{\Omega}(x,t)-\boldsymbol{\Omega}_N(x,t)\Big\|_\infty=0.\ \square$$

4. Numerical examples

In this section, we solve some complex differential equations using the present approach.

Example 1 [12]. Consider the following linear complex differential equation:

$$\begin{cases}y^{(4)}(z)=2zy''(z)-zy(z)+z^5-29z^3+2z^2\\ \qquad\qquad +19z+24,\quad |z|\le 1,\\ y(0)=-1,\quad y'(0)=2\end{cases} \tag{32}$$

with the exact solution, $y(z)=z^4-5z^2+2z-1$. Using the procedure given in Section 2 for $N=4$, the Euler coefficient matrix is obtained as:

$$\rho=[-2\quad -1\quad -2\quad 2\quad 1]^T.$$

Thereby, the approximate solution of the problem for $N=4$ becomes:

$$y_4(z)=z^4-5z^2+2z-1,$$

which is the exact solution.

Example 2 [9]. Let us consider the linear complex differential equation of second-order:

$$\begin{cases}y''(z)=-zy(z)+e^z+ze^z,\quad |z|\le 1,\\ y(0)=1,\quad y'(0)=1\end{cases} \tag{33}$$

which has the exact solution $y(z)=e^z$. Table 1 and Figure 1 list the results obtained by the Euler collocation method in terms of absolute errors at $N=3,5,8$ and 11. It can be seen that the errors decrease when integer N is increased.

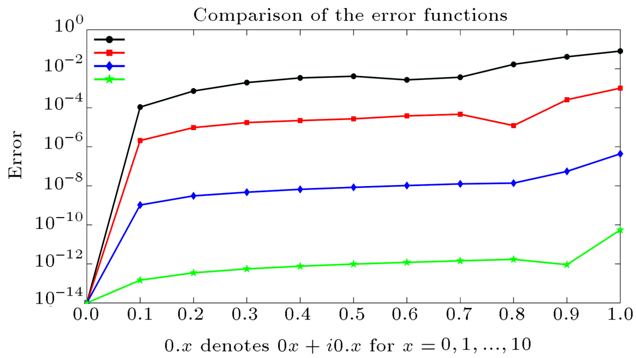

Figure 1. Comparison of the absolute errors of Example 2.

Table 1. The errors obtained by present method for various values of N of Example 2.

z_i	For $N = 3$	For $N = 5$	For $N = 8$	For $N = 11$
$0.0 + 0.0i$	0	0	0	0
$0.1 + 0.1i$	1.0807e-04	2.0957e-06	1.0372e-09	1.4653e-13
$0.2 + 0.2i$	7.2337e-04	9.6427e-06	3.0491e-09	3.4944e-13
$0.3 + 0.3i$	1.9444e-03	1.7474e-05	4.6975e-09	5.5253e-13
$0.4 + 0.4i$	3.3821e-03	2.2075e-05	6.5798e-09	7.5858e-13
$0.5 + 0.5i$	4.1150e-03	2.7155e-05	8.3848e-09	9.7501e-13
$0.6 + .06i$	2.6972e-03	3.8409e-05	1.0343e-08	1.1976e-12
$0.7 + 0.7i$	3.6562e-03	4.6562e-05	1.2535e-08	1.4378e-12
$0.8 + 0.8i$	1.6744e-02	1.2243e-05	1.3752e-08	1.7089e-12
$0.9 + 0.9i$	4.0882e-02	2.5754e-04	5.4855e-08	9.0841e-13
$1.0 + 1.0i$	8.0438e-02	1.0071e-03	4.3604e-07	5.4736e-11

Table 2. Comparison of the absolute errors for various values of N of Example 3.

x_i	Sezar method [19]	Present method		
	$N = 9$	$N = 9$	$N = 10$	$N = 12$
$0.0 + 0.0i$	0	0	0	0
$0.1 + 0.1i$	$8.000e-10$	$5.998e-11$	$3.072e-12$	$1.238e-14$
$0.2 + 0.2i$	$7.810e-09$	$1.604e-10$	$7.684e-12$	$2.832e-14$
$0.3 + 0.3i$	$2.376e-08$	$2.501e-10$	$1.209e-11$	$6.148e-14$
$0.4 + 0.4i$	$5.459e-08$	$3.472e-10$	$1.661e-11$	$1.177e-13$
$0.5 + 0.5i$	$1.077e-07$	$4.411e-10$	$2.124e-11$	$2.023e-13$
$0.6 + 0.6i$	$1.852e-07$	$5.457e-10$	$2.597e-11$	$3.208e-13$
$0.7 + 0.7i$	$2.948e-07$	$6.391e-10$	$3.116e-11$	$4.821e-13$
$0.8 + 0.8i$	$4.405e-07$	$8.026e-10$	$3.511e-11$	$6.992e-13$
$0.9 + 0.9i$	$6.264e-07$	$7.669e-10$	$8.081e-11$	$1.016e-12$
$1.0 + 1.0i$	$8.500e-07$	$2.294e-08$	$1.257e-09$	$3.731e-12$

Table 3. Comparison of the absolute errors for various values of N of Example 4.

z_i	Taylor collocation method[20]		Bessel method [21]		Present method	
	$N = 5$	$N = 9$	$N = 5$	$N = 9$	$N = 5$	$N = 9$
$0.0 + 0.0i$	0	0	8.5541e-13	6.4992e-13	1.7000e-16	2.2765e-16
$0.1 + 0.1i$	2.7889e-03	1.5421e-05	8.5931e-05	2.3811e-08	5.7447e-07	2.0899e-11
$0.2 + 0.2i$	3.1418e-03	1.2489e-04	2.8831e-04	6.8263e-08	2.6893e-06	5.5961e-11
$0.6 + 0.6i$	8.4872e-02	3.3937e-03	1.1734e-03	2.5256e-07	1.1016e-05	1.9648e-10
$1.0 + 1.0i$	2.4519e-01	1.5811e-02	2.2240e-03	5.1293e-07	3.9616e-04	9.2889e-09

Example 3. We consider the linear complex differential equation with variable coefficients to demonstrate that Euler polynomials have the power to approximate the solution to the desired accuracy. The equation we consider is [19]:

$$\begin{cases} y''(z) = -zy'(z) - zy(z) + e^z + 2ze^z, \quad |z| \leq 1. \\ y(0) = 1, \quad y'(0) = 1 \end{cases} \tag{34}$$

with the exact solution, $y(z) = e^z$. We compare the absolute errors of the current method with the Sezar method [19] in Table 2, which shows that the results obtained by the present method are better than those in [19].

Example 4. Our last example is the linear complex differential equation of second-order [20]:

$$\begin{cases} y''(z) = -zy'(z) - 2zy(z) + z\cos(z) + 2z\sin(z) \\ \qquad\quad -\sin(z), \quad |z| \leq 1. \\ y(0) = 0, \quad y'(0) = 1 \end{cases} \tag{35}$$

The exact solution of the problem is $f(z) = \sin(z)$. By following the procedure of our method, we obtain the approximate solutions of the problem for $N = 5$ and 9. The absolute errors of the present method, Taylor collocation method [20] and Bessel method [21] are compared in Table 3. We plot these numerical results

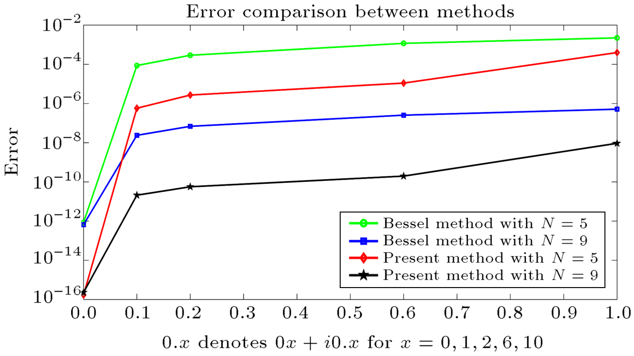

Figure 2. Comparison of the absolute errors of Example 4.

in Figure 2. It is also observed that the new method is an effective method with high accuracy.

5. Conclusion

Complex differential equations have an important role to play in physics and engineering. They are very difficult to handle analytically, so, we always obtain approximate solutions. In this study, a new method for obtaining a numerical solution for linear complex differential equations is discussed. The derivation of this method is essentially based on Euler functions. The error analysis and convergence of the method are also introduced. Thus, the proposed method is suggested as efficient. Comparison of the results obtained by the present method and those other methods shows that the present method is very convenient. Moreover, this method can be extended to complex integro-differential equations, but some modifications are needed.

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