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A less conservative criterion for stability analysis of linear systems with time-varying delay

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Abstract. This paper investigates the problem of stability analysis for linear systems with time-varying delay. To reduce the conservativeness of sufficient stability conditions, a novel augmented Lyapunov-Krasovskii Functional (LKF) which includes quadratic terms of double-integral phrases is introduced; as well, the technique of free-weighting matrices with new slack variables is employed; moreover, a tighter integral inequality is derived for bounding the cross-product terms in the derivative of chosen LKF. Numerical examples are presented to illustrate the superiority of the proposed method compared to some of the previously developed approaches.

1. Introduction

Variable time-delays are encountered in many practical systems, especially in the processes that involve transportation or propagation of material and data such as chemical reactors, combustion engines, and networked control systems. The presence of time-delay in the dynamic equations of the system brings important challenges in its stability analysis and stabilization. Therefore, recently, considerable attention has been attracted to the analysis and control of time-delay systems [1-8].

Most of the time-domain stability criteria for systems with time-varying delay are based primarily on the Lyapunov-Krasovskii Theorem, combined with the model transformation schemes and bounding techniques for the cross-terms in the derivative of the Lyapunov-Krasovskii Functional (LKF) [1]. In order to obtain less conservative delay-dependent conditions, more efficient augmented LKFs were constructed; besides more accurate bounding methods were developed for the cross-terms in the derivative of the energy functional [9,10]. In [11], delay-dependent stability criteria were derived for linear systems with a constant discrete delay, wherein a new simple LKF was introduced by uniformly dividing the delay interval into multiple segments and choosing different weight matrices corresponding to each segment. The idea of [11] was generalized in [12] to obtain less conservative stability conditions for linear systems with time-varying delay in which the delay interval is decomposed into equidistant subintervals; then, choosing different matrix pairs for each subinterval, a new LKF was constructed. In [13], Wirtinger inequality was used for the first time to reduce the conservatism in computing the derivative of LKFs. This more accurate integral inequality depends not only on the state, but also on the integral of the state over the delay interval.

The method of free-weighting matrices that injects additional variables to add extra-degree of freedom in the sufficient stability condition made an important progress in delay-dependent stability criteria of systems with time-varying delay [14,15]. The stability

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measures were further improved by involving novel slack matrices in the derivative of energy functionals. On the other hand, the technique of free-weighting matrices was combined with the conventional ideas, such as descriptor transformation approach and augmented vectors method, to further reduce the conservativeness of stability conditions [16,17]. But, in some cases, using functional parameters and/or free-weighting matrices leads to complex stability criteria. Therefore, in order to reduce computational burden, a simplified criterion was obtained in [18].

Most of the mentioned methods, such as [19] utilize LKFs, include double-integral terms. For the first time, triple-integral terms were added to the LKF in [20] to enhance stability criterion for the constant delay system. This type of augmented LKF was used to develop the stability measure for the linear systems with time varying-delay in [21]. It is worth noting that some types of LKFs yield infinite dimensional LMIs. That is why many authors have considered the simple form of LKFs and thus derived simpler, but more conservative, conditions [22].

In [23], delay-range-dependent stability criterion for linear systems with time-varying delay was derived, where a new estimation method, along with the convex combination and delay partitioning was employed to obtain a less conservative stability condition. More recently, the stability of a linear system with interval time-varying delay was investigated in [24], where a new LKF was presented and some novel integral inequalities were established for bounding the cross terms appearing in the derivative of the chosen LKF. Moreover, the matrix-based quadratic convex approach was used to ensure both the positive definiteness of LKF and negative definiteness of its derivative.

This paper presents an improved stability condition for systems with time-varying delay compared to some of the currently available conditions in literature. To develop this novel condition, a new type of LKF containing quadratic phrases of double-integral terms is introduced and the method of free-weighting matrices with novel slack variables is utilized. Moreover, a tighter integral inequality is derived to bound cross-terms in the derivative of LKF. Efficiency of the suggested approach is demonstrated by two illustrative examples. Note that since less conservativeness of the proposed stability condition is the consequence of increasing the number of matrix parameters in the derived LMIs, more computation time is needed in the testing procedure.

This paper is organized as follows: The problem is described in Section 2. Section 3 introduces the sufficient condition for stability analysis of the linear system with time-varying delay. In Section 4, two numerical examples are presented to demonstrate the advantages of the proposed method compared to some of the existing approaches. Section 5 concludes the paper.

Notations: In this paper, R denotes real numbers set. The symbol * stands for symmetric block in the symmetric matrices. I is identity matrix with appropriate dimensions. The notation $P > 0$ ($P \geq 0$) means that P is real symmetric and positive definite (positive semidefinite). The superscript T stands for matrix transposition. col$\{\}$ shows column vector of the elements in the bracket.

2. System description and preliminaries

The linear system with time-varying delay is described as follows:

$$\dot{x}(t) = Ax(t) + A_1 x(t - \eta(t)), \qquad t \geq 0,$$

$$x(t) = \varphi(t), \qquad t \in [-\eta_2, 0), \tag{1}$$

where $x(t) \in R^n$ is the state vector; A and A_1 are system matrices with appropriate dimensions; and $\eta(t)$ denotes time-varying delay and satisfies:

$$0 \leq \eta_1 \leq \eta(t) \leq \eta_2, \tag{2}$$

where, $\eta_1 < \eta_2$ are the constant upper and lower bounds of time-varying delay.

The problem is to find a new delay-dependent condition in terms of linear matrix inequalities to investigate the asymptotic stability of the system (Eqs. (1)) with the delay satisfying Relation (2). Before proceeding further, inspired by Jensen's inequality in [23], the following lemma is extracted.

Lemma 1. Suppose $0 \leq \eta_1 \leq \eta_2$ and $\omega(t) \in R^n$; for any positive definite matrix $Z \in R^{n\times n}$, the following inequality holds:

$$(\eta_2^2 - \eta_1^2) \int_{-\eta_2}^{-\eta_1} \int_{t+\beta}^{t} \omega^T(\alpha) Z \omega(\alpha) d\alpha d\beta$$

$$\geq 2 \left(\int_{-\eta_2}^{-\eta_1} \int_{t+\beta}^{t} \omega(\alpha) d\alpha d\beta \right)^T$$

$$Z \left(\int_{-\eta_2}^{-\eta_1} \int_{t+\beta}^{t} \omega(\alpha) d\alpha d\beta \right).$$

Proof: Regarding the Schur complement, the following holds:

$$\begin{bmatrix} \omega^T(\alpha) Z \omega(\alpha) & \omega^T(\alpha) \\ \omega(\alpha) & Z^{-1} \end{bmatrix} \geq 0,$$

double integrating both sides of the above relation yield:

$$\begin{bmatrix} \int\limits_{-\eta_2}^{-\eta_1}\int\limits_{t+\beta}^{t} \omega^T(\alpha)Z\omega(\alpha)d\alpha d\beta & \int\limits_{-\eta_2}^{-\eta_1}\int\limits_{t+\beta}^{t} \omega^T(\alpha)d\alpha d\beta \\ \int\limits_{-\eta_2}^{-\eta_1}\int\limits_{t+\beta}^{t} \omega(\alpha)d\alpha d\beta & \frac{(\eta_2^2-\eta_1^2)}{2}Z^{-1} \end{bmatrix} \geq 0.$$

Applying the Schur complement completes the proof. ■

3. Main results

In this section, a new delay-dependent stability condition is derived to check the asymptotic stability of the linear system (Eqs. (1)) using appropriate LKF and Lemma 1.

Theorem 1. Given $0 < \eta_1 < \eta_2$, A and A_1, linear system in Eqs. (1) with time-varying delay satisfying Relation (2) is asymptotically stable if there are matrices $Y_0, Y_1, Y_2, M, X_0, X_1, X_2$, and symmetric matrices:

$$P = [P_{ij}]_{7\times 7} > 0 \qquad (\text{for } i,j = 1,2,\cdots,7),$$

$$Q_1 = [Q_{1_{ij}}]_{2\times 2} > 0, \qquad Q_2 = [Q_{2_{ij}}]_{2\times 2} > 0,$$

$$T_1 = [T_{1_{ij}}]_{2\times 2} > 0, \qquad T_2 = [T_{2_{ij}}]_{2\times 2} > 0,$$

$$Z_1 = [Z_{1_{ij}}]_{2\times 2} > 0, \qquad Z_2 = [Z_{2_{ij}}]_{2\times 2} > 0,$$

$$(\text{for } i,j = 1,2),$$

with appropriate dimensions satisfying the following inequality:

$$\begin{bmatrix} \Sigma & \eta_1\bar{Y}_0 & \eta_{21}\bar{Y}_1 & \eta_{21}\bar{Y}_2 \\ * & -\eta_1\begin{bmatrix} T_{1_{11}} & T_{1_{12}}+X_0 \\ * & T_{1_{22}} \end{bmatrix} & 0 & 0 \\ * & * & -\eta_{21}\begin{bmatrix} T_{2_{11}} & T_{2_{12}}+X_1 \\ * & T_{2_{22}} \end{bmatrix} & 0 \\ * & * & * & -\eta_{21}\begin{bmatrix} T_{2_{11}} & T_{2_{12}}+X_2 \\ * & T_{2_{22}} \end{bmatrix} \end{bmatrix} < 0, \tag{3}$$

where:

$$\eta_{21} = \eta_2 - \eta_1,$$

and:

$$\bar{Y}_0^T = \begin{bmatrix} \tilde{Y}_0^T & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{Y}_1^T = \begin{bmatrix} \tilde{Y}_1^T & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{Y}_2^T = \begin{bmatrix} \tilde{Y}_2^T & 0 & 0 & 0 & 0 \end{bmatrix},$$

with:

$$\tilde{Y}_0 = \begin{bmatrix} 0 & Y_0 \end{bmatrix}, \qquad \tilde{Y}_1 = \begin{bmatrix} 0 & Y_1 \end{bmatrix}, \qquad \tilde{Y}_2 = \begin{bmatrix} 0 & Y_2 \end{bmatrix},$$

also:

$$\sum = \hat{\sum} + \Omega + \Omega^T,$$

with:

$$\Omega^T = \begin{bmatrix} -MA & 0 & 0 & M & 0 & 0 & -MA_1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and:

$$\hat{\Sigma} = \left[\begin{array}{ccccccccccc} \hat{\Sigma}_{1_1} & \hat{\Sigma}_{1_2} & \hat{\Sigma}_{1_3} & \hat{\Sigma}_{1_4} & P_{12} & P_{13} & 0 & \hat{\Sigma}_{1_8} & \hat{\Sigma}_{1_9} & \hat{\Sigma}_{1_{10}} & \hat{\Sigma}_{1_{11}} \\ * & \hat{\Sigma}_{2_2} & \hat{\Sigma}_{2_3} & \hat{\Sigma}_{2_4} & \hat{\Sigma}_{2_5} & P_{23} & 0 & \hat{\Sigma}_{2_8} & \hat{\Sigma}_{2_9} & \hat{\Sigma}_{2_{10}} & \hat{\Sigma}_{2_{11}} \\ * & * & \hat{\Sigma}_{3_3} & \hat{\Sigma}_{3_4} & P_{23}^T & \hat{\Sigma}_{3_6} & 0 & \hat{\Sigma}_{3_8} & \hat{\Sigma}_{3_9} & -P_{56} & -P_{57} \\ * & * & * & \hat{\Sigma}_{4_4} & 0 & 0 & 0 & \hat{\Sigma}_{4_8} & \hat{\Sigma}_{4_9} & \hat{\Sigma}_{4_{10}} & \hat{\Sigma}_{4_{11}} \\ * & * & * & * & \hat{\Sigma}_{5_5} & 0 & 0 & P_{24} & P_{25} & P_{26} & P_{27} \\ * & * & * & * & * & -Q_{2_{22}} & 0 & P_{34} & P_{35} & P_{36} & P_{37} \\ * & * & * & * & * & * & \hat{\Sigma}_{7_7} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Sigma}_{8_8} & \hat{\Sigma}_{8_9} & \hat{\Sigma}_{8_{10}} & -P_{67} \\ * & * & * & * & * & * & * & * & \hat{\Sigma}_{9_9} & -P_{67}^T & \hat{\Sigma}_{9_{11}} \\ * & * & * & * & * & * & * & * & * & -Z_{1_{11}} & 0 \\ * & * & * & * & * & * & * & * & * & * & -Z_{2_{11}} \end{array}\right],$$

in which:

$$\hat{\Sigma}_{1_1} = P_{14}^T + P_{14} + Q_{1_{11}} + \eta_1 T_{1_{11}} + \eta_{21}T_{2_{11}} + \frac{\eta_1^4}{4}Z_{1_{11}} + \frac{(\eta_2^2-\eta_1^2)^2}{4}Z_{2_{11}} - \eta_1^2 Z_{1_{22}} - \eta_{21}^2 Z_{2_{22}} + X_0,$$

$$\hat{\Sigma}_{1_4} = P_{11} + \eta_1 P_{16} + \eta_{21}P_{17} + Q_{1_{12}} + \eta_1 T_{1_{12}} + \eta_{21}T_{2_{12}} + \frac{\eta_1^4}{4}Z_{1_{12}} + \frac{(\eta_2^2-\eta_1^2)^2}{4}Z_{2_{12}},$$

$$\hat{\Sigma}_{2_2} = -P_{24}^T + P_{25}^T - P_{24} + P_{25} - Q_{1_{11}} + Q_{2_{11}} - X_0 + X_1,$$

$$\hat{\Sigma}_{4_4} = Q_{1_{22}} + \eta_1 T_{1_{22}} + \eta_{21} T_{2_{12}} + \frac{\eta_1^4}{4} Z_{1_{22}} + \frac{\left(\eta_2^2 - \eta_1^2\right)^2}{4} Z_{2_{22}},$$

$$\hat{\Sigma}_{1_2} = P_{24}^T - P_{14} + P_{15},$$

$$\hat{\Sigma}_{1_3} = P_{34}^T - P_{15},$$

$$\hat{\Sigma}_{1_8} = P_{44} - P_{16} + \eta_1 Z_{1_{22}},$$

$$\hat{\Sigma}_{1_9} = P_{45} - P_{17} + \eta_{21} Z_{2_{22}},$$

$$\hat{\Sigma}_{1_{10}} = P_{46} - \eta_1 Z_{1_{12}}^T,$$

$$\hat{\Sigma}_{1_{11}} = P_{47} - \eta_{21} Z_{2_{12}}^T,$$

$$\hat{\Sigma}_{2_3} = -P_{34}^T + P_{35}^T - P_{25},$$

$$\hat{\Sigma}_{2_4} = -P_{12}^T + \eta_1 P_{26} + \eta_{21} P_{27},$$

$$\hat{\Sigma}_{2_5} = -Q_{1_{12}} + Q_{2_{12}},$$

$$\hat{\Sigma}_{2_8} = -P_{44} + P_{45}^T - P_{26},$$

$$\hat{\Sigma}_{2_9} = -P_{25} + P_{55} - P_{27},$$

$$\hat{\Sigma}_{2_{10}} = -P_{46} + P_{56},$$

$$\hat{\Sigma}_{2_{11}} = -P_{47} + P_{57},$$

$$\hat{\Sigma}_{3_3} = -P_{35}^T - P_{35} - Q_{2_{11}} - X_2,$$

$$\hat{\Sigma}_{3_4} = P_{13}^T + \eta_1 P_{36} + \eta_{21} P_{37},$$

$$\hat{\Sigma}_{3_6} = P_{33} - Q_{2_{12}},$$

$$\hat{\Sigma}_{3_8} = -P_{45}^T - P_{36},$$

$$\hat{\Sigma}_{3_9} = -P_{55} - P_{37},$$

$$\hat{\Sigma}_{4_8} = P_{14} + \eta_1 P_{46}^T + \eta_{21} P_{47}^T,$$

$$\hat{\Sigma}_{4_9} = P_{15} + \eta_1 P_{56}^T + \eta_{21} P_{57}^T,$$

$$\hat{\Sigma}_{4_{10}} = P_{16} + \eta_1 P_{66} + \eta_{21} P_{67}^T,$$

$$\hat{\Sigma}_{4_{11}} = P_{17} + \eta_1 P_{67} + \eta_{21} P_{77},$$

$$\hat{\Sigma}_{5_5} = -Q_{1_{22}} + Q_{2_{22}},$$

$$\hat{\Sigma}_{7_7} = -X_1 + X_2,$$

$$\hat{\Sigma}_{8_8} = -P_{46}^T - P_{46} - Z_{1_{22}},$$

$$\hat{\Sigma}_{8_9} = -P_{56}^T - P_{47},$$

$$\hat{\Sigma}_{8_{10}} = -P_{66} + Z_{1_{12}}^T,$$

$$\hat{\Sigma}_{9_9} = -P_{57}^T - P_{57} - Z_{2_{22}},$$

$$\hat{\Sigma}_{9_{11}} = -P_{77} + Z_{2_{12}}^T,$$

Proof: The LKF candidate is constructed as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t), \tag{4}$$

$$V_1(x_t) = \xi^T(t) P \xi(t), \tag{5}$$

$$V_2(x_t) = \int_{t-\eta_1}^{t} \tau^T(\alpha) Q_1 \tau(\alpha) d\alpha + \int_{t-\eta_2}^{t-\eta_1} \tau^T(\alpha) Q_2 \tau(\alpha) d\alpha, \tag{6}$$

$$V_3(x_t) = \int_{-\eta_1}^{0} \int_{t+\beta}^{t} \tau^T(\alpha) T_1 \tau(\alpha) d\alpha d\beta + \int_{-\eta_2}^{-\eta_1} \int_{t+\beta}^{t} \tau^T(\alpha) T_2 \tau(\alpha) d\alpha d\beta, \tag{7}$$

$$V_4(x_t) = \frac{\eta_1^2}{2} \int_{-\eta_1}^{0} \int_{\beta}^{0} \int_{t+\theta}^{t} \tau^T(\alpha) Z_1 \tau(\alpha) d\alpha d\theta d\beta + \frac{(\eta_2^2 - \eta_1^2)}{2} \int_{-\eta_2}^{-\eta_1} \int_{\beta}^{0} \int_{t+\theta}^{t} \tau^T(\alpha) Z_2 \tau(\alpha) d\alpha d\theta d\beta, \tag{8}$$

wherein, augmented vectors $\tau(\alpha)$ and $\xi(t)$ are defined as:

$$\tau(\alpha) = \mathrm{col}\{x(\alpha), \dot{x}(\alpha)\},$$

and:

$$\xi(t) = \mathrm{col}\Bigg\{x(t), x(t-\eta_1), x(t-\eta_2), \int_{t-\eta_1}^{t} x(\alpha)d\alpha, \int_{t-\eta_2}^{t-\eta_1} x(\alpha)d\alpha, \int_{-\eta_1}^{0}\int_{t+\beta}^{t} x(\alpha)d\alpha d\beta, \int_{-\eta_2}^{-\eta_1}\int_{t+\beta}^{t} x(\alpha)d\alpha d\beta\Bigg\}.$$

First, time derivative of $V(x_t)$ is computed along the trajectories of Eqs. (1) as the following:

$$\begin{aligned}\dot{V}(x_t) =& 2\xi^T(t)P(t)\xi + \tau^T(t)Q_1\tau(t) \\ &- \tau^T(t-\eta_1)Q_1\tau(t-\eta_1) \\ &+ \tau^T(t-\eta_1)Q_2\tau(t-\eta_1) \\ &- \tau^T(t-\eta_2)Q_2\tau(t-\eta_2) \\ &+ \tau^T(t)(\eta_1 T_1 + \eta_{21}T_2)\tau(t) \\ &- \int_{t-\eta_1}^{t} \tau^T(\alpha)T_1\tau(\alpha)d\alpha \\ &- \int_{t-\eta(t)}^{t-\eta_1} \tau^T(\alpha)T_2\tau(\alpha)d\alpha \\ &- \int_{t-\eta_2}^{t-\eta(t)} \tau^T(\alpha)T_2\tau(\alpha)d\alpha \\ &+ \tau^T(t)\left(\frac{\eta_1^4}{4}Z_1 + \frac{(\eta_2^2-\eta_1^2)^2}{4}Z_2\right)\tau(t) \\ &- \frac{\eta_1^2}{2}\int_{-\eta_1}^{0}\int_{t+\beta}^{t} \tau^T(\alpha)Z_1\tau(\alpha)d\alpha d\beta \\ &- \frac{(\eta_2^2-\eta_1^2)}{2}\int_{-\eta_2}^{-\eta_1}\int_{t+\beta}^{t} \tau^T(t)Z_2\tau(t)d\alpha d\beta.\end{aligned}$$

Note that for any of the matrices Y_0, Y_1, Y_2, M, X_0, X_1, and X_2 with appropriate dimensions, the following equalities hold:

$$\varepsilon_1(t) = 2\zeta^T(t)Y_0\left(x(t) - x(t-\eta_1) - \int_{t-\eta_1}^{t} \dot{x}(\alpha)d\alpha\right) = 0, \tag{9}$$

$$\varepsilon_2(t) = 2\zeta^T(t)Y_1\left(x(t-\eta_1) - x(t-\eta(t)) - \int_{t-\eta(t)}^{t-\eta_1} \dot{x}(\alpha)d\alpha\right) = 0, \tag{10}$$

$$\varepsilon_3(t) = 2\zeta^T(t)Y_2\left(x(t-\eta(t)) - x(t-\eta_2) - \int_{t-\eta_2}^{t-\eta(t)} \dot{x}(\alpha)d\alpha\right) = 0, \tag{11}$$

$$\varepsilon_4(t) = 2\zeta^T(t)M\left(\dot{x}(t) - Ax(t) - A_1x(t-\eta(t))\right) = 0, \tag{12}$$

$$\varepsilon_5(t) = x^T(t)X_0x(t) - x^T(t-\eta_1)X_0x(t-\eta_1) - 2\int_{t-\eta_1}^{t} \dot{x}^T(\alpha)X_0x(\alpha)d\alpha = 0, \tag{13}$$

$$\varepsilon_6(t) = x^T(t-\eta_1)X_1x(t-\eta_1) - x^T(t-\eta(t))X_1x(t-\eta(t)) - 2\int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(\alpha)X_1x(\alpha)d\alpha = 0, \tag{14}$$

$$\varepsilon_7(t) = x^T(t-\eta(t))X_2x(t-\eta(t)) - x^T(t-\eta_2)X_2x(t-\eta_2) - 2\int_{t-\eta_2}^{t-\eta(t)} \dot{x}^T(\alpha)X_2x(\alpha)d\alpha = 0, \tag{15}$$

in which:

$$\zeta(t) = \mathrm{col}\{x(t), x(t-\eta_1), x(t-\eta_2), \dot{x}(t), \dot{x}(t-\eta_1), \dot{x}(t-\eta_2), x(t-\eta(t))\}.$$

Regarding Eqs. (9)-(15) and utilizing Lemma 1, the upper bound of $\dot{V}(x_t)$ is written as follows:

$$\begin{aligned}\dot{V}(x_t) \leq & 2\xi^T(t)P\xi(t) + \tau^T(t)Q_1\tau(t) \\ & - \tau^T(t-\eta_1)Q_1\tau(t-\eta_1) \\ & + \tau^T(t-\eta_1)Q_2\tau(t-\eta_1) \\ & - \tau^T(t-\eta_2)Q_2\tau(t-\eta_2) \\ & + \tau^T(t)(\eta_1 T_1 + (\eta_2-\eta_1)T_2)\tau(t) \\ & - \int\limits_{t-\eta_1}^{t} \tau^T(\alpha)T_1\tau(\alpha)d\alpha \\ & - \int\limits_{t-\eta(t)}^{t-\eta_1} \tau^T(\alpha)T_2\tau(\alpha)d\alpha \\ & - \int\limits_{t-\eta_2}^{t-\eta(t)} \tau^T(\alpha)T_2\tau(\alpha)d\alpha \\ & + \tau^T(t)\left(\frac{\eta_1^4}{4}Z_1 + \frac{(\eta_2^2-\eta_1^2)^2}{4}Z_2\right)\tau(t) \\ & - \begin{bmatrix} \int\limits_{-\eta_1}^{0}\int\limits_{t+\beta}^{t} x(\alpha)d\alpha d\beta \\ \eta_1 x(t) - \int\limits_{t-\eta_1}^{t} x(\alpha)d\alpha \end{bmatrix}^T \\ & \cdot \begin{bmatrix} Z_{1_{11}} & Z_{1_{12}} \\ * & Z_{1_{22}} \end{bmatrix} \\ & \cdot \begin{bmatrix} \int\limits_{-\eta_1}^{0}\int\limits_{t+\beta}^{t} x(\alpha)d\alpha d\beta \\ \eta_1 x(t) - \int\limits_{t-\eta_1}^{t} x(\alpha)d\alpha \end{bmatrix} \\ & - \begin{bmatrix} \int\limits_{-\eta_2}^{-\eta_1}\int\limits_{t+\beta}^{t} x(\alpha)d\alpha d\beta \\ (\eta_2-\eta_1)x(t) - \int\limits_{t-\eta_2}^{t-\eta_1} x(\alpha)d\alpha \end{bmatrix}^T \\ & \cdot \begin{bmatrix} Z_{2_{11}} & Z_{2_{12}} \\ * & Z_{2_{22}} \end{bmatrix} \\ & \cdot \begin{bmatrix} \int\limits_{-\eta_2}^{-\eta_1}\int\limits_{t+\beta}^{t} x(\alpha)d\alpha d\beta \\ (\eta_2-\eta_1)x(t) - \int\limits_{t-\eta_2}^{t-\eta_1} x(\alpha)d\alpha \end{bmatrix} + \sum_{i=1}^{i=7}\varepsilon_i(t). \end{aligned} \tag{16}$$

On the other hand, the following inequalities are true:

$$\begin{aligned} & -2\zeta^T(t)\tilde{Y}_0 \int\limits_{t-\eta_1}^{t} \tau(\alpha)d\alpha \\ & \leq \eta_1\zeta^T(t)\tilde{Y}_0 \begin{bmatrix} T_{1_{11}} & T_{1_{12}}+X_0 \\ * & T_{1_{22}} \end{bmatrix}^{-1} \tilde{Y}_0^T\zeta(t) \\ & + \int\limits_{t-\eta_1}^{t} \tau^T(\alpha) \begin{bmatrix} T_{1_{11}} & T_{1_{12}}+X_0 \\ * & T_{1_{22}} \end{bmatrix} \tau(\alpha)d\alpha, \\ & \tilde{Y}_0 = \begin{bmatrix} 0 & Y_0 \end{bmatrix}, \end{aligned} \tag{17}$$

$$\begin{aligned} & -2\zeta^T(t)\tilde{Y}_1 \int\limits_{t-\eta(t)}^{t-\eta_1} \tau(\alpha)d\alpha \\ & \leq \eta_{21}\zeta^T(t)\tilde{Y}_1 \begin{bmatrix} T_{2_{11}} & T_{2_{12}}+X_1 \\ * & T_{2_{22}} \end{bmatrix}^{-1} \tilde{Y}_1^T\zeta(t) \\ & + \int\limits_{t-\eta(t)}^{t-\eta_1} \tau^T(\alpha) \begin{bmatrix} T_{2_{11}} & T_{2_{12}}+X_1 \\ * & T_{2_{22}} \end{bmatrix} \tau(\alpha)d\alpha, \\ & \tilde{Y}_1 = \begin{bmatrix} 0 & Y_1 \end{bmatrix}, \end{aligned} \tag{18}$$

$$\begin{aligned} & -2\zeta^T(t)\tilde{Y}_2 \int\limits_{t-\eta_2}^{t-\eta(t)} \tau(\alpha)d\alpha \\ & \leq \eta_{21}\zeta^T(t)\tilde{Y}_2 \begin{bmatrix} T_{2_{11}} & T_{2_{12}}+X_2 \\ * & T_{2_{22}} \end{bmatrix}^{-1} \tilde{Y}_2^T\zeta(t) \\ & + \int\limits_{t-\eta_2}^{t-\eta(t)} \tau^T(\alpha) \begin{bmatrix} T_{2_{11}} & T_{2_{12}}+X_2 \\ * & T_{2_{22}} \end{bmatrix} \tau(\alpha)d\alpha, \\ & \tilde{Y}_2 = \begin{bmatrix} 0 & Y_2 \end{bmatrix}. \end{aligned} \tag{19}$$

Now, the upper bound of $\dot{V}(x_t)$ is restated as follows by substituting Relations (17)-(19) in Relation (16):

$$\begin{aligned} \dot{V}(x_t) \leq \vartheta^T(t)\Bigg\{ & \Sigma + \eta_1\bar{Y}_0 \begin{bmatrix} T_{1_{11}} & T_{1_{12}}+X_0 \\ * & T_{1_{22}} \end{bmatrix}^{-1} \bar{Y}_0^T \\ & + \eta_{21}\bar{Y}_1 \begin{bmatrix} T_{2_{11}} & T_{2_{12}}+X_1 \\ * & T_{2_{22}} \end{bmatrix}^{-1} \bar{Y}_1^T \\ & + \eta_{21}\bar{Y}_2 \begin{bmatrix} T_{2_{11}} & T_{2_{12}}+X_2 \\ * & T_{2_{22}} \end{bmatrix}^{-1} \bar{Y}_2^T \Bigg\}\vartheta(t), \end{aligned} \tag{20}$$

where, $\bar{Y}_0$, $\bar{Y}_1$, and $\bar{Y}_2$ were defined previously in

Relation (3) and:

$$\vartheta(t) = \text{col}\Bigg\{x(t), x(t-\eta_1), x(t-\eta_2), \dot{x}(t), \dot{x}(t-\eta_1), \dot{x}(t-\eta_2), x(t-\eta(t), \int_{t-\eta_1}^{t} x(\alpha)d\alpha \int_{t-\eta_2}^{t-\eta_1} x(\alpha)d\alpha, \int_{-\eta_1}^{0}\int_{t+\beta}^{t} x(\alpha)d\alpha d\beta, \int_{-\eta_2}^{-\eta_1}\int_{t+\beta}^{t} x(\alpha)d\alpha d\beta\Bigg\}.$$

Regarding Relation (20), the Lyapunov-Krasovskii Theorem [1] guarantees the asymptotic stability of the system in Eqs. (1), provided that:

$$\Sigma + \eta_1 \bar{Y}_0 \begin{bmatrix} T_{1_{11}} & T_{1_{12}} + X_0 \\ * & T_{1_{22}} \end{bmatrix}^{-1} \bar{Y}_0^T + \eta_{21} \bar{Y}_1 \begin{bmatrix} T_{2_{11}} & T_{2_{12}} + X_1 \\ * & T_{2_{22}} \end{bmatrix}^{-1} \bar{Y}_1^T + \eta_{21} \bar{Y}_2 \begin{bmatrix} T_{2_{11}} & T_{2_{12}} + X_2 \\ * & T_{2_{22}} \end{bmatrix}^{-1} \bar{Y}_2^T < 0,$$

which can be transformed easily to Relation (3) by the Schur Complement.■

Remark. The novelty in the derivation of stability criterion in Relation (3) is threefold. First, new LKF was constructed by including the double-integral terms $\int_{-\eta_1}^{0}\int_{t+\beta}^{t} x(\alpha)d\alpha d\beta$ and $\int_{-\eta_2}^{-\eta_1}\int_{t+\beta}^{t} x(\alpha)d\alpha d\beta$ in the augmented vectors $\xi(t)$ which differently from [21], create innovative quadratic terms in the energy functional. Second, novel inequality was introduced in Lemma 1 to bound the cross terms appearing in the derivative of V due to the existence of newly appended terms. Finally, inventive relations were presented in Eqs. (13)-(15) to inject new free weights in the derivative of the chosen LKF in order to create more degree of freedom in the resulting sufficient condition. Combination of the mentioned tricks decreases the conservativeness of the stability condition.

Corollary: For $\eta_1 = 0$, the stability condition of Theorem 1 is rewritten as follows where all the notations were defined previously.

$$\begin{bmatrix} \Sigma & \eta_2 \bar{Y}_1 & \eta_2 \bar{Y}_2 \\ * & -\eta_2 \begin{bmatrix} T_{2_{11}} & T_{2_{12}} + X_1 \\ * & T_{2_{22}} \end{bmatrix} & 0 \\ * & * & -\eta_2 \begin{bmatrix} T_{2_{11}} & T_{2_{12}} + X_2 \\ * & T_{2_{22}} \end{bmatrix} \end{bmatrix} < 0. \tag{21}$$

Proof. The upper bound of $\dot{V}$ in Relation (20) is rewritten as follows for $\eta_1 = 0$ which is a result of omitting some terms in the chosen LKF in Eq. (4):

$$\dot{V} \leq \vartheta^T(t) \Bigg[\Sigma + \eta_2 \bar{Y}_1 \begin{bmatrix} T_{2_{11}} & T_{2_{12}} + X_1 \\ * & T_{2_{22}} \end{bmatrix}^{-1} \bar{Y}_1^T + \eta_2 \bar{Y}_2 \begin{bmatrix} T_{2_{11}} & T_{2_{12}} + X_2 \\ * & T_{2_{22}} \end{bmatrix}^{-1} \bar{Y}_2^T\Bigg] \vartheta(t).$$

So, regarding the Schur complement, it can be easily verified that if Relation (21) holds, then $\dot{V} < 0$.■

4. Illustrative examples

Two numerical examples are presented to compare the proposed method with some existing ones. The LMI Toolbox of Matlab is utilized to solve the LMI feasibility problems [25]. The maximum value of delay that retains the stability of system is called as Maximum Allowable Delay Bound (MADB). MADB, which is the common performance index to evaluate the conservativeness of stability tests in the literature, is computed for the proposed and rival stability criteria.

Example 1. Consider the following system from [21]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t), \tag{22}$$

which is controlled through a communication network with:

$$u(t) = \begin{bmatrix} -3.75 & -11.5 \end{bmatrix} x(t-\eta(t)),$$

wherein $\eta(t)$ denotes the time-varying transmission delay in the communication link. So, the closed-loop

Table 1. MADBs computed by different methods for Example 1 (with $\eta_1 = 0$).

	Method						
	[27]	**[7]**	**[16]**	**[28]**	**[26]**	**[21]**	**Proposed method**
MADB	0.9412	1.0081	1.0081	1.0081	1.0432	1.0629	1.0762

Table 2. Admissible upper bound η_2 for different values of η_1.

	Method						
	[31]	**[30]**	**[29]** $N=2$	**[29]** $N=4$	**[23]** $N=2$	**[23]** $N=4$	**Proposed method**
$\eta_1 = 1$	1.9008	1.8737	2.004	2.0273	2.0089	2.0448	2.0852
$\eta_1 = 2$	2.5663	2.5048	2.5650	2.5915	2.5829	2.6051	2.6767
$\eta_1 = 3$	3.3408	3.2591	3.2866	3.3010	3.2983	3.3098	3.3778
$\eta_1 = 4$	4.169	4.0744	4.0818	4.0855	4.0848	4.0877	4.123

system can be described by the following time-delay system as in [21]:

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ 0 & -0.1\end{bmatrix}x(t)+\begin{bmatrix}0 & 0\\ -0.375 & -1.15\end{bmatrix}x(t-\eta(t)). \tag{23}$$

In Table 1, the maximum allowable value of η_2 obtained from the proposed method along with the results from rival methods is listed for comparison. Table 1 clearly shows that the proposed approach outperforms the methods presented in [7,16,21,26-28].

Example 2. Considering the system in Eqs. (1) with the following matrices [23]:

$$A=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix}, \qquad A_1=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}.$$

Table 2 shows the maximum allowable delay upper bound for different values of delay lower bound η_1, obtained from the proposed approach and methods of [23,29-31]. As seen, the maximum bound of delay obtained by the proposed method is larger than the existing ones which verifies the superiority of the suggested delay-dependent condition.

Results of several recent papers are reported in Tables 1 and 2 to reveal that the improvement of stability conditions with regard to the nearby rival methods is gradual.

5. Conclusion

In this paper, a new approach has been proposed to analyze the asymptotic stability of the linear time invariant systems with time-varying delay. By constructing new augmented Lyapunov-Krasovskii functional and using free-weighting matrices, a novel delay-dependent stability condition has been derived in terms of LMIs. Numerical examples have been given to demonstrate that the proposed criterion is less conservative compared to some of the existing approaches in the literature.

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