Scientia Iranica B (2015) 22(5), 1911–1917

Sharif University of Technology
Scientia Iranica
Transactions B: Mechanical Engineering
www.scientiairanica.com

Research Note

Application of variational iteration method to large vibration analysis of slenderness beams considering mid-plane stretching

M. Daeichi and M.T. Ahmadian*

Center of Excellence in Design, Robotics and Automation (CEDRA), School of Mechanical Engineering, Sharif University of Technology, P.O. Box: 11155-9567, Iran.

Received 15 June 2014; received in revised form 11 November 2014; accepted 4 May 2015

KEYWORDS
Large vibration;
Variational iteration method;
Nonlinear natural frequency;
Galerkin technique.

Abstract. In this paper, the nonlinear vibration of transversely vibrated beams with large slenderness and immovable ends is analyzed using the Variational Iteration Method (VIM). The nonlinear partial differential equation of motion is converted to a set of coupled ordinary differential equations using the Galerkin technique. A two mode expansion for the system response is considered and a second order analytical solution of the equations is obtained using VIM. Two algebraic coupled equations are also developed to find the nonlinear frequencies of the system. Numerical simulations are performed for various initial conditions. Close agreement between the numerical and analytical results is achieved. Also, a frequency analysis is performed for a range of initial amplitudes of vibration.

1. Introduction

Due to the wide application of beam-type structures in mechanical, civil and railway engineering, investigation of the dynamic behavior of such structures has become of significant scientific and technological importance [1-2]. The large vibration of beams with large slenderness and immovable ends is a nonlinear problem, due to the mid-plane stretching effect, and linearization in the governing equations leads to remarkable errors in predicting the dynamic behavior of the system [3-5]. Surveying literature reveals that a large number of research papers are dedicated to the analysis of the nonlinear vibration of beams using analytic and semi-analytic approaches. Nayfeh implemented multiple scales, normal forms, and Shaw and Pierre methods, to investigate the free vibration of beams resting on nonlinear foundations [6]. A harmonically exited beam with a nonlinear elastic foundation was studied by Santee and Goncalves [7]. They applied the Melnikov method to obtain the frequency response, as well as bifurcations of the structure. Kargarnovin et al. obtained the response of beam subjected to a harmonic moving load while resting on a nonlinear viscoelastic foundation [8]. Malekzadeh and Vosoughi investigated a restrained edge composite beam vibrating with large amplitude and resting on a nonlinear elastic foundation [9]. Vibration of a Timoshenko beam on a random viscoelastic foundation under a harmonic moving load was analyzed by Younesian et al. using perturbation techniques [10].

Previous studies are mostly based on classical methods. However, in recent years, novel analytical approximate methods have been introduced to analyze the dynamic behavior of nonlinear oscillatory systems with simpler and more efficient procedures. The Homotopy Analysis Method (HAM) [11], the Energy Balance Method (EBM) [12], Frequency-Amplitude

*. *Corresponding author. Tel.: +98 21 66165503;*
E-mail address: ahmadian@sharif.edu (M.T. Ahmadian)

Formulation (FAF) [13] and the Variational Iteration Method (VIM) [14] are some of these methods. For example, Liao et al. introduced the homotopy analysis method and employed it for various systems, such as fractional partial differential equations, nonlinear eigenvalue problems, and wave motion in deep water [15-17]. The energy balance method has been utilized effectively to find periodic solutions for the generalized nonlinear equations of oscillators containing fraction order elastic force [18]. Jamshidi and Ganji employed EBM to investigate the oscillation of a mass attached to a stretched elastic wire [19]. Younesian et al. also obtained analytical approximate solutions for the generalized nonlinear oscillator using this method [20]. Frequency-amplitude formulation was implemented in many equations containing discontinuity [21], irrational force [22] and the nonlinear Schrodinger equation [23]. The variational iteration method is a quite efficient method applicable for nonlinear ordinary and partial differential equations. Wazwaz solved linear and nonlinear Volterra integral and integro-differential equations using this method [24]. Investigation of linear and nonlinear Schrodinger equations was carried out by him using VIM [25]. Odibat and Momani implemented this method for fractional partial differential equations in fluid mechanics [26]. Analytical solutions for nonlinear wave propagation in shallow media were obtained by Younesian et al. applying VIM [27]. Diffusion equations with local and nonlocal functional were analyzed using VIM by Guo-Cheng Wu [28]. He also employed VIM for Burgers' flow with fractional derivatives using the new Lagrange multiplier [29].

In this paper, the variational iteration method is employed to investigate the nonlinear transverse free vibration of beams with large slenderness and immovable ends. The governing partial differential equation of motion is discretized using the Galerkin technique. A coupled set of nonlinear ordinary differential equations is obtained considering two mode shapes. Closed form expressions for the system response are obtained and nonlinear frequencies are evaluated from algebraic equations. Obtained analytical approximate results are compared with the numerical results of different initial conditions to examine the accuracy and validity of the solution procedures. The sensitivity of the nonlinear frequencies, with respect to initial conditions, is studied for a wide range of vibration amplitudes. Findings indicate that the solution procedure is valid, while simple to use for nonlinear systems.

2. Mathematical modeling

The nonlinear free vibration of a beam is considered in the section, considering the mid-plane stretching effect. The beam is clamped-hinged, and the corresponding governing equation of motion is derived as [1]:

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} - \left[\frac{EA}{2l}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} = 0, \tag{1}$$

where w shows the transverse deflection; ρ, E, I, A and l are density, modulus of elasticity, moment of inertia, cross section area, and length of the beam, respectively. Also, x is the axial axis and t represents the time. The boundary conditions of the clamped-hinged beam are expressed as:

$$w = 0, \qquad \frac{\partial w}{\partial x} = 0 \qquad \text{at} \qquad x = 0, \tag{2}$$

$$w = 0, \qquad \frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{at} \qquad x = l. \tag{3}$$

The following two mode discretized response is considered by employing the Galerkin method:

$$w(x,t) = u(t)\varphi_1(x) + v(t)\varphi_2(x), \tag{4}$$

where $\varphi_i(x)$, $(i = 1,2)$, are linear mode shapes of a clamped-hinged beam. Substituting Eq. (4) into Eq. (1), and using the orthogonality principle of the mode shapes, one can arrive at the following coupled ordinary differential equations:

$$\ddot{u} + \omega_1^2 u + \alpha_1 u^3 + \alpha_2 u^2 v + \alpha_3 u v^2 + \alpha_4 v^3 = 0, \tag{5}$$

$$\ddot{v} + \omega_2^2 v + \beta_1 v^3 + \beta_2 v^2 u + \beta_3 v u^2 + \beta_4 u^3 = 0, \tag{6}$$

where the linear frequencies, ω_1 and ω_2, and coefficients, $\alpha_{i,i=1,2,3,4}$ and $\beta_{i,i=1,2,3,4}$, are non-dimensional constants, which can be computed according to the physical and geometrical parameters of the beam presented in the Appendix.

3. Variational iteration method

In this section, a brief review of VIM is presented first [14] and then VIM is employed to obtain approximate solutions of the coupled system in Eqs. (5) and (6). Consider the following general differential equation:

$$Ly(t) + Ny(t) = 0, \tag{7}$$

where L and N represent the linear and nonlinear operators, respectively. Based on the variational iteration method, the iteration formulation is constructed as follows [14]:

$$y_{n+1} = y_n + \int_0^t \lambda\left(Ly_n(s) + N\tilde{y}_n(s)\right) ds, \tag{8}$$

where λ denotes a general Lagrange multiplier which can be identified via the variational theory; y_n is the nth approximate solution for Eq. (7), and $\tilde{y}_n$ is a restricted variation in which $\delta\tilde{y}_n = 0$ [14]. Eq. (8) is called a correction functional. Making the above correction functional stationary, noticing that $\delta y_n(0) = 0$, the following iteration can be written:

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda \left[\ddot{y}_n + \omega_1^2 y_n + N\tilde{y}_n\right] ds$$

$$= \delta y_n(t) + \lambda(s)\delta \dot{y}_n(s)\big|_{s=t}$$

$$- \dot{\lambda}(s)\delta y_n(s)\Big|_{s=t} + \int_0^t \left(\ddot{\lambda} + \omega_1^2 \lambda\right) \delta y_n ds = 0, \quad (9)$$

which yields the following stationary conditions:

$$\ddot{\lambda} + \omega_1^2 \lambda = 0, \quad (10a)$$

$$1 - \dot{\lambda}\Big|_{s=t} = 0, \quad (10b)$$

$$\lambda|_{s=t} = 0. \quad (10c)$$

Therefore, in this case, the Lagrange multiplier can be identified as follows:

$$\lambda = \frac{1}{\omega_1} \sin\left[\omega_1(s-t)\right]. \quad (11)$$

Applying VIM on Eqs. (5) and (6) and using Eq. (11), the following iterations are constructed:

$$u_{n+1} = u_n + \frac{1}{\omega_1}\int_0^t \sin\omega_1(s-t)\{\ddot{u}_n + \omega_1^2 u_n + \alpha_1 u_n^3 + \alpha_2 u_n^2 v_n + \alpha_3 u_n v_n^2 + \alpha_4 v_n^3\} ds, \quad (12)$$

$$v_{n+1} = v_n + \frac{1}{\omega_2}\int_0^t \sin\omega_2(s-t)\{\ddot{v}_n + \omega_2^2 v_n + \beta_1 v_n^3 + \beta_2 v_n^2 u_n + \beta_3 v_n u_n^2 + \beta_4 u_n^3\} ds. \quad (13)$$

First approximate solutions for $u(t)$ and $v(t)$ are considered as follows:

$$u_0 = A\cos\Omega_1 t, \quad (14)$$

$$v_0 = B\cos\Omega_2 t, \quad (15)$$

Substituting Eqs. (14) and (15) into Eqs. (12) and (13), one can arrive at:

$$u_1 = A\cos\Omega_1 t + \frac{1}{\omega_1}\int_0^t \sin\omega_1(s-t)\{A(\omega_1^2 - \Omega_1^2)\cos\Omega_1 s + \alpha_1 A^3\cos^3\Omega_1 s + \alpha_2 BA^2\cos^2\Omega_1 s\cos\Omega_2 s + \alpha_3 B^2 A\cos\Omega_1 s\cos^2\Omega_2 s + \alpha_4 B^3\cos^3\Omega_2 s\} ds, \quad (16)$$

$$v_1 = B\cos\Omega_2 t + \frac{1}{\omega_2}\int_0^t \sin\omega_2(s-t)\{B(\omega_2^2 - \Omega_2^2)\cos\Omega_2 s + \beta_1 B^3\cos^3\Omega_2 s + \beta_2 B^2 A\cos\Omega_1 s\cos^2\Omega_2 s + \beta_3 A^2 B\cos^2\Omega_1 s\cos\Omega_2 s + \beta_4 A^3\cos^3\Omega_1 s\} ds, \quad (17)$$

where Ω_1 and Ω_2 represent the nonlinear natural frequencies. Implementing sort of appropriate mathematical operations and after some simplifications, one can reach the second approximate solutions as:

$$u_1 = \left\{A + \frac{\alpha_3 AB^2}{2(\omega_1^2 - \Omega_1^2)} + \frac{\alpha_2 BA^2}{2(\omega_1^2 - \Omega_1^2)} + \frac{\alpha_3 AB^2}{4}\left(\frac{1}{\omega_1^2 - (2\Omega_2 + \Omega_1)^2} + \frac{1}{\omega_1^2 - (2\Omega_2 - \Omega_1)^2}\right)\frac{\alpha_2 BA^2}{4}\left(\frac{1}{\omega_1^2 - (2\Omega_1 + \Omega_2)^2} + \frac{1}{\omega_1^2 - (2\Omega_1 - \Omega_2)^2}\right) + \frac{\alpha_1 A^3}{4}\left(\frac{3}{\omega_1^2 - \Omega_1^2} + \frac{1}{\omega_1^2 - 9\Omega_1^2}\right)\frac{\alpha_4 B^3}{4}\left(\frac{3}{\omega_1^2 - \Omega_2^2} + \frac{1}{\omega_1^2 - 9\Omega_2^2}\right)\right\}\cos\omega_1 t$$
$$- \frac{\alpha_3 AB^2}{4}\left\{\frac{2\cos\Omega_1 t}{\omega_1^2 - \Omega_1^2} + \frac{\cos(2\Omega_2 + \Omega_1)t}{\omega_1^2 - (2\Omega_2 + \Omega_1)^2} + \frac{\cos(2\Omega_2 - \Omega_1)t}{\omega_1^2 - (2\Omega_2 - \Omega_1)^2}\right\}$$
$$- \frac{\alpha_1 A^3}{4}\left\{\frac{3\cos\Omega_1 t}{\omega_1^2 - \Omega_1^2} + \frac{\cos 3\Omega_1 t}{\omega_1^2 - 9\Omega_1^2}\right\} - \frac{\alpha_4 B^3}{4}\left\{\frac{3\cos\Omega_2 t}{\omega_1^2 - \Omega_2^2} + \frac{\cos 3\Omega_2 t}{\omega_1^2 - 9\Omega_2^2}\right\}$$
$$- \frac{\alpha_2 BA^2}{4}\left\{\frac{2\cos\Omega_2 t}{\omega_1^2 - \Omega_2^2} + \frac{\cos(\Omega_2 + 2\Omega_1)t}{\omega_1^2 - (2\Omega_1 + \Omega_2)^2} + \frac{\cos(2\Omega_1 - \Omega_2)t}{\omega_1^2 - (2\Omega_1 - \Omega_2)^2}\right\}, \quad (18)$$

and:

$$v_1 = \Bigg\{B + \frac{\beta_3 BA^2}{2(\omega_2^2-\Omega_2^2)} + \frac{\beta_2 AB^2}{2(\omega_2^2-\Omega_2^2)} + \frac{\beta_3 BA^2}{4}\left(\frac{1}{\omega_2^2-(2\Omega_1+\Omega_2)^2}+\frac{1}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right)\frac{\beta_2 AB^2}{4}\left(\frac{1}{\omega_2^2-(2\Omega_2+\Omega_1)^2}+\frac{1}{\omega_2^2-(2\Omega_2-\Omega_1)^2}\right) + \frac{\beta_1 B^3}{4}\left(\frac{3}{\omega_2^2-\Omega_2^2}+\frac{1}{\omega_2^2-9\Omega_2^2}\right) \frac{\beta_4 A^3}{4}\left(\frac{3}{\omega_2^2-\Omega_1^2}+\frac{1}{\omega_2^2-9\Omega_1^2}\right)\Bigg\}\cos\omega_2 t - \frac{\beta_3 BA^2}{4}\left\{\frac{2\cos\Omega_2 t}{\omega_2^2-\Omega_2^2}+\frac{\cos(2\Omega_1+\Omega_2)t}{\omega_2^2-(2\Omega_2+\Omega_1)^2} + \frac{\cos(2\Omega_1-\Omega_2)t}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right\} - \frac{\beta_1 B^3}{4}\left\{\frac{3\cos\Omega_2 t}{\omega_2^2-\Omega_2^2}+\frac{\cos 3\Omega_2 t}{\omega_2^2-9\Omega_2^2}\right\} - \frac{\beta_4 A^3}{4}\left\{\frac{3\cos\Omega_1 t}{\omega_2^2-\Omega_1^2}+\frac{\cos 3\Omega_1 t}{\omega_2^2-9\Omega_1^2}\right\} - \frac{\beta_2 AB^2}{4}\left\{\frac{2\cos\Omega_1 t}{\omega_2^2-\Omega_1^2}+\frac{\cos(2\Omega_2+\Omega_1)t}{\omega_2^2-(2\Omega_2+\Omega_1)^2} + \frac{\cos(2\Omega_2-\Omega_1)t}{\omega_2^2-(2\Omega_2-\Omega_1)^2}\right\}. \quad (19)$$

Elimination of secular terms in the above equations leads to the following algebraic coupled system:

$$\Bigg\{A + \frac{\alpha_3 AB^2}{2(\omega_1^2-\Omega_1^2)} + \frac{\alpha_2 BA^2}{2(\omega_1^2-\Omega_1^2)} + \frac{\alpha_3 AB^2}{4}\left(\frac{1}{\omega_1^2-(2\Omega_2+\Omega_1)^2}+\frac{1}{\omega_1^2-(2\Omega_2-\Omega_1)^2}\right) \frac{\alpha_2 BA^2}{4}\left(\frac{1}{\omega_1^2-(2\Omega_1+\Omega_2)^2}+\frac{1}{\omega_1^2-(2\Omega_1-\Omega_2)^2}\right) + \frac{\alpha_1 A^3}{4}\left(\frac{3}{\omega_1^2-\Omega_1^2}+\frac{1}{\omega_1^2-9\Omega_1^2}\right) \frac{\alpha_4 B^3}{4}\left(\frac{3}{\omega_1^2-\Omega_2^2}+\frac{1}{\omega_1^2-9\Omega_2^2}\right)\Bigg\} = 0, \quad (20)$$

and:

$$\Bigg\{B + \frac{\beta_3 BA^2}{2(\omega_2^2-\Omega_2^2)} + \frac{\beta_2 AB^2}{2(\omega_2^2-\Omega_2^2)} + \frac{\beta_3 BA^2}{4}\left(\frac{1}{\omega_2^2-(2\Omega_1+\Omega_2)^2}+\frac{1}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right) \frac{\beta_2 AB^2}{4}\left(\frac{1}{\omega_2^2-(2\Omega_2+\Omega_1)^2}+\frac{1}{\omega_2^2-(2\Omega_2-\Omega_1)^2}\right) + \frac{\beta_1 B^3}{4}\left(\frac{3}{\omega_2^2-\Omega_2^2}+\frac{1}{\omega_2^2-9\Omega_2^2}\right) \frac{\beta_4 A^3}{4}\left(\frac{3}{\omega_2^2-\Omega_1^2}+\frac{1}{\omega_2^2-9\Omega_1^2}\right)\Bigg\} = 0. \quad (21)$$

So, unknown natural frequencies can be obtained from the above coupled system, with respect to initial conditions, linear frequencies and constant coefficients.

4. Results and discussions

In this section, obtained analytical results for the time response of the system are shown and compared with the numerical solutions for different initial conditions of two mode shapes. Also, a frequency analysis is presented here for a range of initial amplitudes of vibration. It should be noted that the non-dimensional constant of the system is considered for a practical system as the following numerical values, and computed by the formulations in the Appendix [5].

$$\begin{cases} \omega_1 = 1, \omega_2 = 3.24064, \\ \alpha_1 = 0.278769, \alpha_2 = -0.311074, \\ \alpha_3 = 1.11585, \alpha_4 = -0.386361, \\ \beta_1 = 3.87030, \beta_2 = -1.159083, \\ \beta_3 = 1.11585, \beta_4 = -0.1036913, \end{cases} \quad (22)$$

Figures 1 to 4 show the time histories of two mode shapes for both analytical and numerical solutions for some values of initial amplitudes of vibration. It is obvious that the analytical approximate solutions have a reliable agreement with the numerical results for both

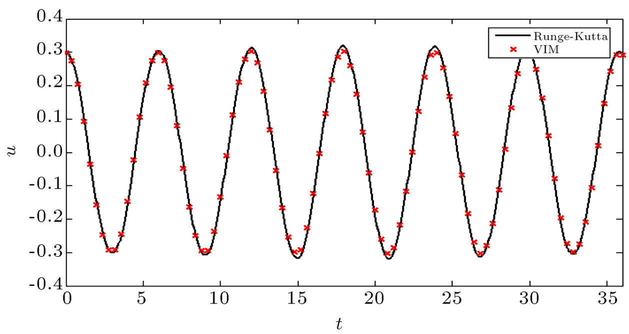

Figure 1. Time history of the dynamic response for the first mode ($A = 0.3$, $B = 0.4$).

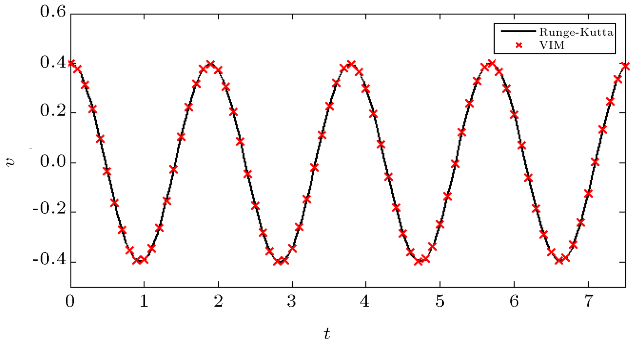

Figure 2. Time history of the dynamic response for the second mode ($A = 0.3$, $B = 0.4$).

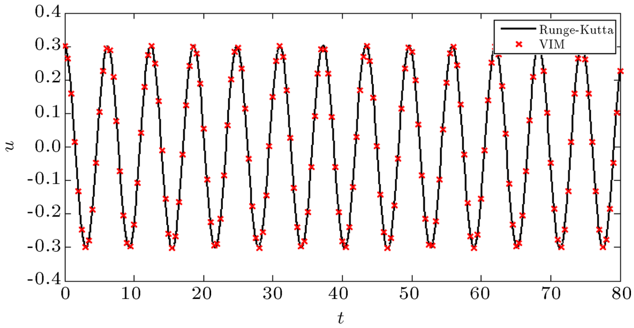

Figure 3. Time history of the dynamic response for the first mode ($A = 0.3$, $B = 0.1$).

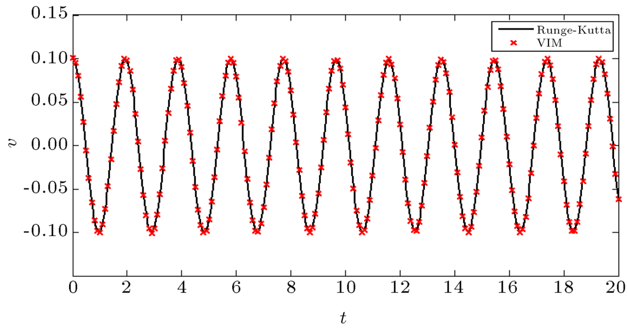

Figure 4. Time history of the dynamic response for the second mode ($A = 0.3$, $B = 0.1$).

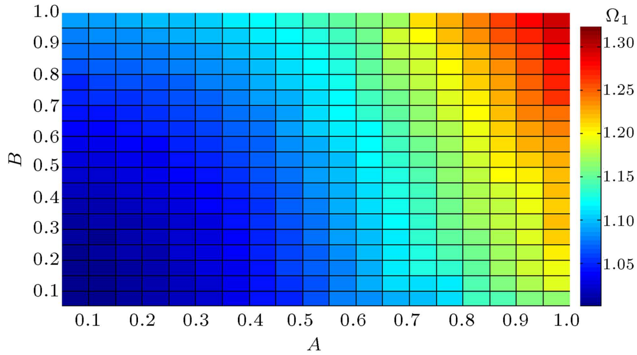

Figure 5. First nonlinear natural frequency contour map.

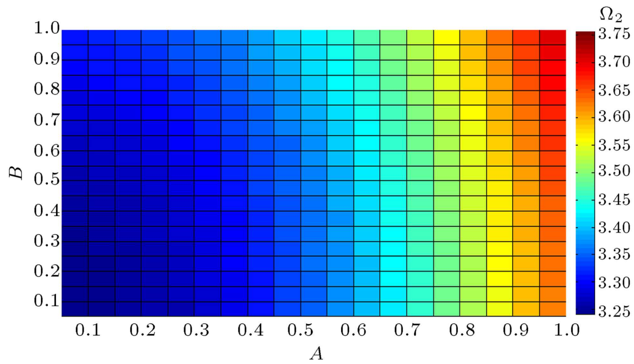

Figure 6. Second nonlinear natural frequency contour map.

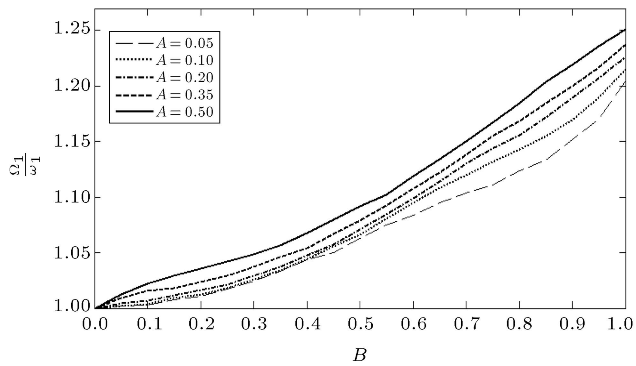

Figure 7. Frequency ratio of the first mode versus initial amplitudes.

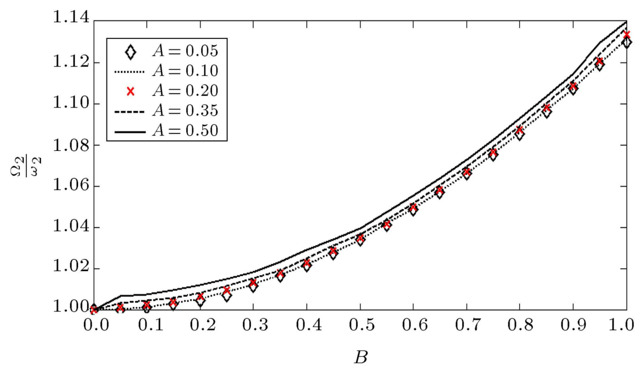

Figure 8. Frequency ratio of the second mode versus initial conditions.

modes. Consequently, our solution procedures obtain appropriate approximate expressions in analytical form for analysis of the proposed system.

Variations of the first and second nonlinear natural frequencies are illustrated in the counter maps shown in Figures 5 and 6, and the nonlinear sensitivity of the frequencies to the initial conditions is represented by the color gradient. Also, the frequency ratio, which is nonlinear to linear versus the initial amplitudes, is represented in Figures 7 and 8. It is seen that for small initial amplitudes, the ratio is close to one. However, by increasing this value, the ratio increases nonlinearly and the growth is higher for the first mode frequency as the corresponding initial condition is further effective in the nonlinear frequency.

5. Conclusion

In this paper, the nonlinear dynamic behavior of beams with large slenderness and immovable ends is analyzed using the variational iteration method. The beam is considered to be clamped-clamped and free to vibrate transversely. The Galerkin method is applied to the governing equation of motion to obtain a set of ordinary differential equations, and the approximate analytical solutions of the corresponding equations were achieved using VIM. The accuracy of the results for time responses is examined by comparing them with numerical findings. The nonlinear frequencies of the beam are analyzed for different initial conditions. Effects of initial conditions on the natural frequencies and the frequency ratio are demonstrated. It is shown that the solution procedure is accurate enough, and, in comparison with numerical findings, no significant difference is observed. The procedure is simple and applicable for analysis and modeling of nonlinear systems.

References

1. Lau, S.L. and Cheung, Y.K. "Amplitude incremental variational principle for nonlinear vibration of elastic systems", *ASME Journal of Applied Mechanics*, **48**, pp. 959-964 (1981).
2. Younesian, D., Saadatnia, Z. and Askari, H. "Analytical solutions for free oscillations of beams on nonlinear elastic foundation using the variational iteration method", *Journal of Theoretical and Applied Mechanics*, **50**, pp. 639-652 (2012).
3. Scridhar, S., Nayfeh, A.H. and Mook, D.T. "Nonlinear resonances in a class of multi-degree-of-freedom systems", *Journal of the Acoustical Society of America*, **58**, pp. 113-123 (1975).
4. Cheung, Y.K. and Lau, S.L. "Incremental time-space finite strip method for nonlinear structural vibrations", *Earthquake Engineering and Structural Dynamics*, **10**, pp. 239-253 (1982).
5. Lau, S.L., Cheung, Y.K. and Wu, S.Y. "Incremental harmonic balance method with multiple time scales for aperiodic vibration of nonlinear systems", *ASME Journal of Applied Mechanics*, **50**, pp. 871-876 (1983).
6. Nayfeh, A.H., *Nonlinear Interactions*, First Ed., John Wiley, New York (2000).
7. Santee, D.M. and Goncalves, P.B. "Oscillations of a beam on a non-linear elastic foundation under periodic loads", *Shock and Vibration*, **13**, pp. 273-284 (2006).
8. Kargarnovin, M.H., Younesian, D., Thompson, D.J. and Jones, C.J.C. "Response of beams on nonlinear viscoelastic foundations to harmonic moving loads", *Computers and Structures*, **83**, pp. 1865-1877 (2005).
9. Malekzadeh, P. and Vosoughi, A.R. "DQM large amplitude vibration of composite beams on nonlinear elastic foundations with restrained edges", *Communications in Nonlinear Science and Numerical Simulation*, **14**, pp. 906-915 (2009).
10. Younesian, D., Kargarnovin, M.H., Thompson, D.J. and Jones, C.J.C. "Parametrically excited vibration of a Timoshenko beam on random viscoelastic foundation subjected to a harmonic moving load", *Nonlinear Dynamics*, **45**, pp. 75-93 (2005).
11. Liao, S. "Homotopy analysis method: A new analytic method for nonlinear problems", *Applied Mathematics and Mechanics*, **19**(10), pp. 957-962 (1998).
12. He, J.H. "Preliminary report on the energy balance for nonlinear oscillations", *Mechanics Research Communications*, **29**, pp. 107-111 (2002).
13. Fan, J. "He's frequency-amplitude formulation for the Duffing harmonic oscillator", *Computer and Mathematics with Applications*, **58**, pp. 2473-2476 (2009).
14. He, J.H. "Variational iteration method - a kind of nonlinear analytical technique: some examples", *International Journal of Nonlinear Mechanics*, **34**, pp. 699-708 (1999).
15. Liao, S.J. and You, X.C. "Analysis of nonlinear fractional partial differential equations with the homotopy analysis method", *Communication in Nonlinear Science and Numerical Simulations*, **14**(4), pp. 1152-1156 (2009).
16. Liao, S.J. and Cheung, K.F. "Homotopy analysis of nonlinear progressive waves in deep water", *Journal of Engineering Mathematics*, **45**, pp. 105-116 (2003).
17. Liao, S. "Series solution of nonlinear eigenvalue problems by means of the homotopy analysis method", *Nonlinear Analysis: Real World Applications*, **10**(4), pp. 2455-2470 (2009).
18. Younesian, D., Askari, H., Saadatnia, Z. and Yildirim, A. "Periodic solutions for the generalized nonlinear oscillators containing fraction order elastic force", *International Journal of Nonlinear Science and Numerical Simulations*, **11**(12), pp. 1027-1032 (2010).
19. Jamshidi, N. and Ganji, D.D. "Application of energy balance method and variational iteration method to an oscillation of a mass attached to a stretched elastic wire", *Current Applied Physics*, **10**, pp. 484-486 (2010).
20. Younesian, D., Askari, H., Saadatnia, Z. and KalamiYazdi, M. "Analytical approximate solutions for the generalized nonlinear oscillator", *Applicable Analysis*, **91**(5), pp. 965-977 (2012).
21. Zhang, H.L. "Application of He's amplitude-frequency formulation to a nonlinear oscillator with discontinuity", *Computer and Mathematics with Applications*, **58**, pp. 2197-2198 (2009).
22. Zhao, L. "He's frequency-amplitude formulation for nonlinear oscillators with an irrational force", *Computer and Mathematics with Applications*, **58**, pp. 2477-2479 (2009).
23. Zhang, Y.N., Xu, F. and Deng, L.L. "Exact solution for nonlinear Schrödinger equation by He's frequency formulation", *Computer and Mathematics with Applications*, **58**, pp. 2449-2451 (2009).

24. Wazwaz, A.M. "The variational iteration method for solving linear and nonlinear Volterra integral and integro-differential equations", *International Journal of Computer Mathematics*, **87**, pp. 1131-1141 (2010).
25. Wazwaz, A.M. "A study on linear and nonlinear Schrodinger equations by the variational iteration method", *Chaos, Solitons and Fractals*, **37**, pp. 1136-1142 (2008).
26. Odibat, Z. and Momani, S. "The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics", *Computers and Mathematics with Applications*, **58**, pp. 2199-2208 (2009).
27. Younesian, D., Askari, H., Saadatnia, Z. and Yildirim, A. "Analytical solution for nonlinear wave propagation in shallow media using the variational iteration method", *Waves in Random and Complex Media*, **22**(2), pp. 133-142 (2012).
28. Guo-Cheng Wu "Applications of the variational iteration method to fractional diffusion equations: Local versus nonlocal ones", *International Review of Chemical Engineering*, **4**(5), pp. 505-510 (2012).
29. Guo-Cheng Wu and Dumitru Baleanu "Variational iteration method for the Burgers' flow with fractional derivatives - new Lagrange multipliers", *Applied Mathematical Modelling*, **37**, pp. 6183-6190 (2013).

Appendix

Constant non-dimensional coefficients introduced in Section 2 can be obtained from the following formulations. The linear frequencies are computed as:

$$\omega_1 = 1, \qquad \omega_2 = \left(\frac{\lambda_2}{\lambda_1}\right)^2,$$

$$\lambda_i^2 = \omega_{L_i} L_i^2 \left(\frac{\rho A}{EI}\right)^{1/2}, \qquad i = 1, 2, \tag{A.1}$$

where ω_{L_i} is the ith linear frequency of the beam. Other coefficients are:

$$\begin{cases} \alpha_1 = \frac{1}{\lambda_1^4}\gamma_{1111}, & \alpha_2 = \frac{1}{\lambda_1^4}(3\gamma_{1121}), \\ \alpha_3 = \frac{1}{\lambda_1^4}(\gamma_{1122} + 2\gamma_{1221}), & \alpha_4 = \frac{1}{\lambda_1^4}\gamma_{2221} \end{cases} \tag{A.2}$$

and:

$$\begin{cases} \beta_1 = \frac{1}{\lambda_1^4}\gamma_{2222}, & \beta_2 = \frac{1}{\lambda_1^4}(3\gamma_{2212}) \\ \beta_3 = \frac{1}{\lambda_1^4}(\gamma_{2211} + 2\gamma_{2112}), & \beta_4 = \frac{1}{\lambda_1^4}\gamma_{1112} \end{cases} \tag{A.3}$$

where:

$$\gamma_{nmpq} = \frac{1}{2}\int_0^1 \frac{\partial \varphi_n}{\partial x_1}\frac{\partial \varphi_m}{\partial x_1} dx_1 \int_0^1 \frac{\partial \varphi_p}{\partial x_1}\frac{\partial \varphi_q}{\partial x_1} dx_1,$$

$$(x_1 = x/L, n, m, p, q = 1, 2). \tag{A.4}$$

Biographies

Meysam Daeichi obtained his MS degree from the Department of Mechanical Engineering at Sharif University of Technology, Tehran, Iran, in 2013. His research interests include nonlinear dynamics and vibration.

Mohamamd Taghi Ahmadian received a PhD degree in Mechanical Engineering from the University of Kansas, USA, in 1986, and is currently Professor in the Department of Mechanical Engineering at Sharif University of Technology, Tehran, Iran. He has published more than 100 journal papers and over 250 conference papers.