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Research Note

Fahraeus-Lindqvist effect in an Oldroyd 8-constant fluid

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Abstract. A pressure driven axisymmetric flow in a circular tube, having a non-Newtonian Oldroyd 8-constant fluid in the core, is considered in this paper. The Oldroyd 8-constant fluid is surrounded by a Newtonian fluid in the present study. The exact solution of the governing equation is obtained in the form of an integral which is evaluated using Gaussian quadrature. The expression for the apparent viscosity is obtained. The graphical results are presented for the profiles of apparent viscosity for different values of the material parameters plotted against tube radius. It is found that for all the values of the material parameters, the apparent viscosity decreases as the tube radius decreases which is the Fahraeus-Lindqvist effect. The results for the case, when there is no Newtonian fluid present in the periphery, are also deduced.

1. Introduction

In 1931, Fahraeus and Lindqvist [1] provided some experimental observations which reveal that blood viscosity decreases by decreasing the radius of the tube; this effect is known as F-L effect. This experimental evidence was justified through the existence of a plasma layer near the wall of the tube which exists due to the movement of suspended cells towards the core. Blood is a multi-component mixture of platelets, plasma, red and white blood cells and many more. It is well known that blood has a shear dependent viscosity [2] and behaves like viscoelastic fluid under certain conditions [3,4]. Therefore, it is not easy to represent blood by a single constitutive equation. Furthermore, rheological properties of blood also depend on the geometry in which it flows. It is also an established fact that plasma behaves like a Newtonian fluid, but the red blood cells exhibit shear thinning characteristics, and thus one needs a non-Newtonian constitutive relationship between shear stress and rate of deformation to make the analysis of blood flow [5-8]. In the studies [9,10] a generalized Oldroyd-B model is used to discuss the shear dependent behavior of blood. The results obtained in these studies match well with the experimental data. A literature survey indicates that very few studies have been taken into account particularly for non-Newtonian fluids for studying the F-L effect. For the Newtonian fluids, Haynes [11] discussed the physical basis of dependence of blood viscosity on the radius of the tube. For the case of non-Newtonian fluids the readers may be referred to the articles in [12-15]. Majhi and Usha [16-18] made contributions for studying F-L effect in Newtonian, power-law and third grade fluids. For the numerical solution of nonlinear differential equations arising in case of non-Newtonian fluids using different numerical methods see [19-21].

Due to complexity and diversity many constitutive relationships exist for non-Newtonian fluids. Our aim in the present paper is to investigate the F-L effect

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for an Oldroyd 8-constant fluid. In Section 2 we present mathematical analysis of the problem. The obtained results and their discussion are given in Section 3. Section 4 is devoted to some conclusions.

2. Mathematical analysis

The constitutive equation for the Cauchy stress tensor in an Oldroyd 8-constant fluid is:

$$\boldsymbol{\tau} = -p\mathbf{I} + \mathbf{S}, \tag{1}$$

where p is the pressure, $\mathbf{I}$ is the identity tensor, and $\mathbf{S}$ satisfies:

$$\mathbf{S} + \lambda_1 \frac{D\mathbf{S}}{Dt} + \frac{\lambda_5}{2}(\mathbf{S}\mathbf{A}_1 + \mathbf{A}_1\mathbf{S}) + \frac{\lambda_5}{2}(\mathrm{tr}\mathbf{S})\mathbf{A}_1 + \frac{\lambda_6}{2}(\mathrm{tr}(\mathbf{S}\mathbf{A}_1))\mathbf{I} = \mu\left[\mathbf{A}_1 + \lambda_2\frac{D\mathbf{A}_1}{Dt} + \lambda_4\mathbf{A}_1^2 + \frac{\lambda_7}{2}(\mathrm{tr}(\mathbf{A}_1^2))\mathbf{I}\right], \tag{2}$$

in which:

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^T, \tag{3}$$

is the Rivlin-Ericksen tensor; $\lambda_i (i = 1-7)$ are material parameters; and D/Dt is the contravariant convective derivative defined by:

$$\frac{D\mathbf{S}}{Dt} = \frac{d\mathbf{S}}{dt} - (\nabla\mathbf{V})\mathbf{S} - \mathbf{S}(\nabla\mathbf{V})^T. \tag{4}$$

It is important to mention here that this model contains, as special cases, the Newtonian, Maxwell, second grade, Oldroyd-B, Johnson-Segalman and Oldroyd 6-constant models. For mathematical modeling of flow through circular tube we consider cylindrical coordinates (r, θ, z), and assume a velocity field $(0, 0, U_1(r))$ in the core and $(0, 0, U_2(r))$ in the periphery. The governing equations of motion through a tube of radius b for the flow of an Oldroyd 8-constant fluid, in the core and Newtonian fluid in the periphery, take the form:

$$\frac{1}{r}\frac{\partial}{\partial r}[rS_{rz}] = \frac{\partial p}{\partial z}, \tag{5}$$

in which:

$$S_{rz} = \begin{cases} \mu\left\{\frac{1+\alpha_1\left(\frac{\partial U_1}{\partial r}\right)^2}{1+\alpha_2\left(\frac{\partial U_1}{\partial r}\right)^2}\right\}\frac{\partial U_1}{\partial r}, & 0 \le r \le a \\ \mu_2\frac{\partial U_2}{\partial r}, & 0 \le r \le b \end{cases} \tag{6}$$

where:

$$\alpha_1 = \lambda_1(\lambda_4+\lambda_7) - (\lambda_3+\lambda_5)(\lambda_4+\lambda_7-\lambda_2) - \frac{\lambda_5\lambda_7}{2},$$

$$\alpha_2 = \lambda_1(\lambda_3+\lambda_6) - (\lambda_3+\lambda_5)(\lambda_3+\lambda_6-\lambda_1) - \frac{\lambda_5\lambda_6}{2}.$$

The boundary conditions applicable to the present flow are:

$$\frac{\partial U_1}{\partial r} = 0 \quad \text{at} \quad r = 0,$$

$$U_2 = 0 \quad \text{at} \quad r = b, \tag{7}$$

$$U_1 = U_2, \quad \text{and :}$$

$$\mu\left\{\frac{1+\alpha_1\left(\frac{\partial U_1}{\partial r}\right)^2}{1+\alpha_2\left(\frac{\partial U_1}{\partial r}\right)^2}\right\}\frac{\partial U_1}{\partial r} = \mu_2\frac{\partial U_2}{\partial r} \quad \text{at} \quad r = a. \tag{8}$$

Defining the dimensionless variables:

$$\bar{U}_{1,2} = \frac{U_{1,2}}{U_0}, \qquad \bar{r} = \frac{r}{b}, \qquad \bar{H} = \frac{-\frac{\partial p}{\partial z}b^2}{\mu U_0},$$

$$\bar{\alpha}_{1,2} = \frac{\alpha_{1,2}U_0^2}{b^2}, \qquad r_0 = \frac{a}{b}. \tag{9}$$

Upon substituting Eq. (9) into Eqs. (5)-(8), and dropping the asterisks for simplicity, we get the following governing problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\frac{1+\alpha_1\left(\frac{\partial U_1}{\partial r}\right)^2}{1+\alpha_2\left(\frac{\partial U_1}{\partial r}\right)^2}\right\}\frac{\partial U_1}{\partial r}\right] = -H,$$

$$0 \le r \le r_0, \tag{10}$$

$$\frac{\mu_2}{\mu}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial U_2}{\partial r}\right] = -H, \qquad r_0 \le r \le 1, \tag{11}$$

$$\frac{\partial U_1}{\partial r} = 0 \quad \text{at} \quad r = 0,$$

$$U_2 = 0 \quad \text{at} \quad r = 1, \tag{12}$$

$$U_1 = U_2 \quad \text{and}$$

$$\mu\left\{\frac{1+\alpha_1\left(\frac{\partial U_1}{\partial r}\right)^2}{1+\alpha_2\left(\frac{\partial U_1}{\partial r}\right)^2}\right\}\frac{\partial U_1}{\partial r} = \mu_2\frac{\partial U_2}{\partial r} \quad \text{at} \quad r = r_0. \tag{13}$$

Eq. (10) is a cubic in $\partial U_1/\partial r$ and has one real and two complex conjugate roots. We are interested in a real solution, therefore:

$$\frac{\partial U_1}{\partial r} = F(r) = \frac{\frac{-\alpha_2 Hr}{6\alpha_1} - \left\{12\alpha_1 - \alpha_2^2H^2r^2\right\}}{3(2^{2/3})\alpha_1 A^{1/3}} + \frac{A^{1/3}}{6(2^{1/3})\alpha_1}, \tag{14}$$

$$A = B + \sqrt{4(12\alpha_1 - \alpha_2^2H^2r^2)^3 + B^2}, \tag{15}$$

$$B = 36Hr\alpha_1(\alpha_2 - 3\alpha_1) - 2\alpha_2^3H^3r^3, \tag{16}$$

and therefore:

$$U_1 = \int_r^{r_0} F(r)dr + \frac{\mu H}{4\mu_2}(1 - r_0^2) \qquad 0 \leq r \leq r_0, \quad (17)$$

$$U_2 = \frac{\mu H}{4\mu_2}(1 - r^2) \qquad r_0 \leq r \leq 1. \quad (18)$$

The normalized total flux is given by:

$$Q = \frac{\mu H}{16\mu_2}(1 - r_0^4) - \int_0^{r_0} r^2 F(r)dr. \quad (19)$$

On comparing the total flux with the flux of the Poiseuille flow, we get the following expression for the apparent viscosity:

$$\frac{\mu}{\mu_a} = \frac{\mu}{\mu_2}(1 - r_0^4) - \frac{8}{H}\int_0^{r_0} r^2 F(r)dr. \quad (20)$$

The integral in above equation can easily be evaluated numerically for different values of the appearing parameters. In the limiting case, when $r_0 = 1$, one can obtain the results for a single fluid, and from Eqs. (17) and (20) one obtains:

$$U_1 = \int_r^1 F(r)dr \qquad 0 \leq r \leq 1, \quad (21)$$

$$\frac{\mu}{\mu_a} = -\frac{8}{H}\int_0^1 r^2 F(r)dr. \quad (22)$$

3. Numerical results and discussion

The apparent viscosity for the F-L effect in an Oldroyd 8-constant fluid in the core surrounded by a Newtonian fluid flowing through a tube of radius b is obtained and given in Eq. (20). The Gaussian quadrature formula is used to evaluate the integral in Eq. (20). The results, thus obtained for the apparent viscosity, are shown in Figures 1-4 to see the influence of material parameters appearing in the governing equation. In these figures, the apparent viscosity is plotted against tube radius b. In Figures 1 and 2, the thickness of the plasma layer, i.e. δ, is assumed to be 3 μm where $\delta = b - a$. It is evident from Figure 1 that for all values of the material parameter, apparent viscosity decreases with decreasing tube radius, i.e. F-L effect occurs. Further, it can be seen through this figure that apparent viscosity increases with increasing the parameter α_1. This is because by increasing α_1 the relaxation time increases and fluid needs more time to relax when shear stress is taken off. The obtained theoretical results are very close to the experimental values given by Gaetghens [22]. For some particular value of material parameter, we may get a close match (see Figure 5). The effects of parameter α_2 on the apparent viscosity are displayed in Figure 2. The results illustrate that the apparent viscosity decreases by increasing the parameter α_2. The reason for this is that the retardation time increases with increase in α_2, and hence fluid require more time to move when shear stress is applied. In Figures 3 and 4, the results are plotted by assuming the thickness of the plasma layer to be 0.05 μm, and once again it is seen that F-L effect occurs; but for small radius tubes, it is observed that decrease in apparent viscosity is marginal when compared with the results of the tubes with relatively large radius tubes; this fact is evident in Figures 3 and 4. Figure 5 is displayed to show the comparison of present solution with the experimental results of

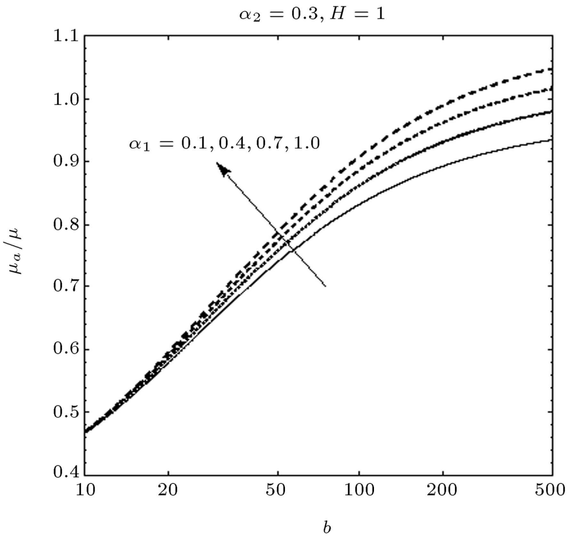

Figure 1. Influence of the material parameter, α_1, on the apparent viscosity plotted against tube radius, b, when $\delta = 3$ μm, $\mu = 0.03$ cps and $\mu_2 = 0.012$ cps.

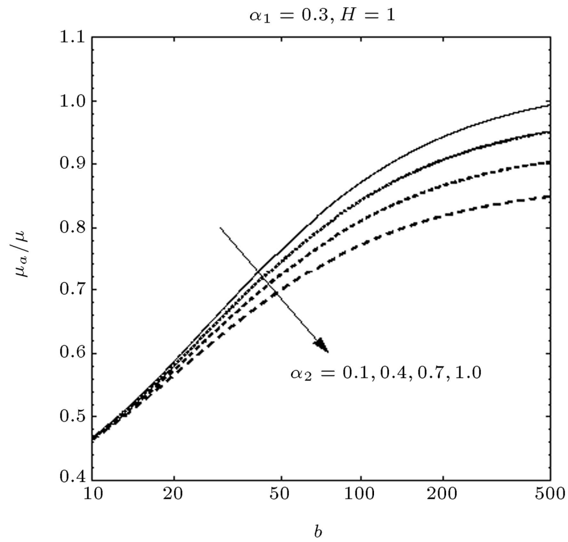

Figure 2. Influence of the material parameter, α_2, on the apparent viscosity plotted against tube radius, b, when $\delta = 3$ μm, $\mu = 0.03$ cps, $\mu_2 = 0.012$ cps.

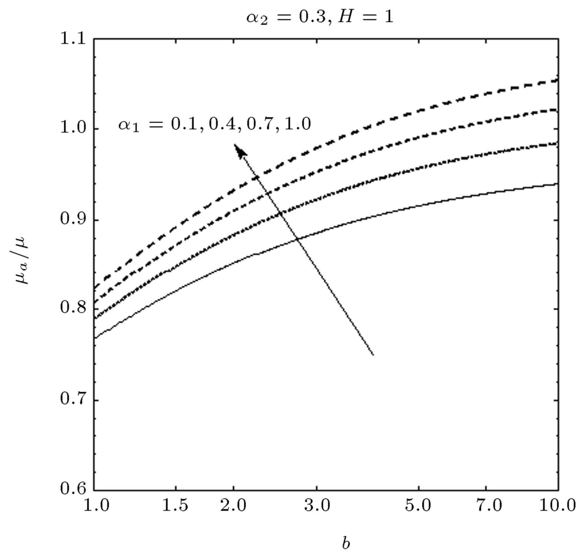

Figure 3. Influence of the material parameter, α_1, on the apparent viscosity plotted against tube radius, b, when $\delta = 0.05$ μm, $\mu = 0.03$ cps and $\mu_2 = 0.012$ cps.

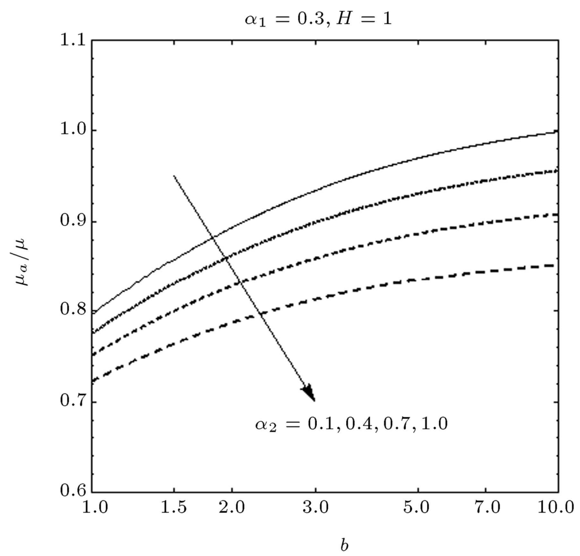

Figure 4. Influence of the material parameter, α_2, on the apparent viscosity plotted against tube radius, b, when $\delta = 0.05$ μm, $\mu = 0.03$ cps and $\mu_2 = 0.012$ cps.

Gaetghens [22]. It is evident from the results that the predicted theoretical results and experimental results are in good agreement.

4. Concluding remarks

In this paper, it was found that the experimentally observed F-L effect occurs for an Oldroyd 8-constant fluid in the core surrounded by a Newtonian fluid flowing through a circular tube. It was further observed that no inverse F-L effect exists for the present study. The exact solution of the governing flow problem was obtained and was discussed under the influence of fluid

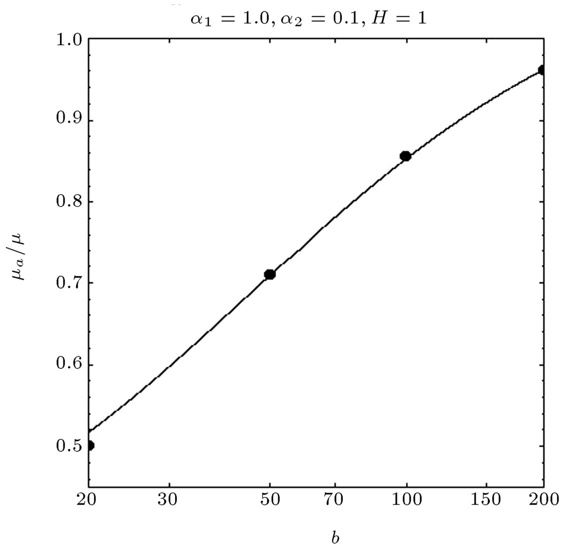

Figure 5. Comparison of the present solution (solid line) with the experimental results of Gaetghens [22] (bullets) when $\delta = 3$ μm, $\mu = 0.03$ cps and $\mu_2 = 0.012$ cps.

parameters. The results for other constitutive relationship could be obtained as special cases. The obtained theoretical results were also compared with the experimental results and were found in good agreement.

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References

1. Fahraeus, R. and Lindqvist, T. "The viscosity of the blood in narrow capillary tubes", *Am. J. Physiol.*, **96**, pp. 562-568 (1931).
2. Fung, Y.C., *Biomechanics*, Berlin: Springer Verlag (1981).
3. Thurston, G.B. "Viscoelasticity of human blood", *J. Biophys.*, **12**, pp. 1205-1217 (1972).
4. Chien, S., King, R.G., Skalak, R., Usami, S. and Copley, A.L. "Viscoelastic properties of human blood and red cell suspension", *Biorheology*, **12**, pp. 341-346 (1975).
5. Anand, M. and Rajagopal, K.R. "A shear-thinning viscoelastic fluid model for describing the flow of blood", *Int. J. Cardiovasc. Med. Sci.*, **4**, pp. 59-68 (2004).
6. Anand, M., Rajagopal, K. and Rajagopal, K.R. "A model for the formation and lysis of blood clots", *Pathophysiol. Haemost. Thromb.*, **34**, pp. 109-120 (2005).

7. Hron, J., Malek, J. and Turek, S. "A numerical investigation of flows of shear-thinning fluids with applications to blood rheology", *Int. J. Numer. Meth. Fluids*, **32**, pp. 829-863 (2000).

8. Massoudi, M. and Phuoc, T.X. "Pulstile flow of blood using a modified second-grade fluid model", *Comput. Math. Appl.*, **56**, pp. 199-211 (2008).

9. Yeleswarapu, K.K., Kamaneva, M.V., Rajagopal, K.R. and Antaki, J.F. "The flow blood in tubes: Theory and experiment", *Mech. Res. Commun.*, **25**, pp. 257-262 (1998).

10. Yeleswarapu, K.K. "Evaluation of continuum models for characterizing the constitutive behavior of blood", Ph.D. Dissertation, University of Pittsburgh, Pittsburgh (1996).

11. Haynes, R.H. "Physical basis of the dependence of blood viscosity on tube radius", *Am. J. Physiol.*, **198**, pp. 1193-1200 (1960).

12. Wares, J.H., Sorrell, F.Y. and Felder, R.M. "A model of steady blood flow", *Biorheology*, **11**, pp. 97-109 (1974).

13. Chathurani, P. and Upadhya, V.S. "A two fluid model for blood flow through small diameter tubes", *Biorheology*, **16**, pp. 109-118 (1979).

14. Shukla, J.B., Parihar, R.S. and Rao, B.R.P. "Effects of stenosis on non-Newtonian flow of blood in an artery", *Bull. Math. Biol.*, **42**, pp. 283-294 (1980).

15. Gupta, B.B., Nigam, K.M. and Jaffarin, M.Y. "A three layer semi-empirical model for flow of blood and other particulate suspensions through narrow tubes", *J. Biomech. Eng.*, **104**, pp. 129-135 (1981).

16. Majhi, S.N. and Usha, L. "Fahraeus-Lindqvist effect and generalized Poiseuille flow with or without wall slip", *Proc. 13th Nat. Conf. Fluid Mech. Fluid Power*, REC, Tiruchirappalli, pp. 125-128 (1984).

17. Majhi, S.N. and Usha, L. "A mathematical note on the Fahraeus-Lindqvist effect in power law fluid", *Bull. Math. Biol.*, **47**, pp. 765-769 (1985).

18. Majhi, S.N. and Usha, L. "Modelling the Fahraeus-Lindqvist effect through fluids of differential type", *Int. J. Eng. Sci.*, **26**, pp. 503-508 (1988).

19. Keimanesh, M., Rashidi, M.M., Chamkha, A.J. and Jafri, R. "Study of a third grade non-Newtonian fluid flow between two parallel plates using the multi-step differential transform method", *Comput. Math. Appli.*, **62**, pp. 2871-2891 (2011).

20. Rashidi, M.M., Beg, O.A. and Rastegari, M.T. "A study of non-Newtonian flow and heat transfer over a non-isotheral wedge using homotopy analysis method", *Chem. Eng. Commun.*, **199**, pp. 231-256 (2012).

21. Beg, O.A., Rashidi, M.M., Beg, T.A. and Asadi, M. "Homotopy analysis of transient magneto-bio-fluid-dynamics of micropolar squeeze film in a porous medium: a model for magneto-bio- rheoligical lubrication", *J. Mech. Medi. Bio.*, **12** (2012).

22. Gaetghens, R. "Flow of blood through narrow capillaries: rheological mechanisms determining capillary hematocrit and apparent viscosity", *Biorheology*, **17**, pp. 183-189 (1980).

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