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Robust economic-statistical design of multivariate exponentially weighted moving average control chart under uncertainty with interval data

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Abstract. The cost parameters in economic-statistical models of control charts are usually assumed to be deterministic in the literature. Considering uncertainty in the cost parameters of control charts is very common in application. So, several researchers used scenario-based approach for robust economic-statistical design of control charts. In this paper, we specifically concentrate on the Multivariate Exponentially Weighted Moving Average (MEWMA) control chart and consider interval uncertainty in the cost parameters of the MEWMA control chart and develop a robust economic-statistical design of the MEWMA control chart by using interval robust optimization technique. Meanwhile, the Lorenzen and Vance cost function is used, and to calculate the average run length criterion, the Markov chain approach is applied. Then, genetic algorithm for obtaining optimal solution of the proposed robust model is used and effectiveness of this model is illustrated through a numerical example. Also, a comparison with certain situation of the cost parameters is performed. Finally, a sensitivity analysis is done to investigate the effect of changing the intervals of cost parameters of the Lorenzen and Vance model on the optimal solutions. Furthermore, a sensitivity analysis on the other certain cost parameters of the Lorenzen and Vance model is done.

1. Introduction

Control charts are one of the most common tools of Statistical Process Control (SPC) for monitoring processes. Designing the control charts is determining parameters such as sample size (n), sampling interval (h) and control limits coefficient (l). Since the design of a control chart leads to various costs, authors were interested to design control charts from an economic perspective. Therefore, some authors have proposed cost functions which are a function of sample size (n), sampling interval (h) and control limits (l). Firstly, Duncan [1] presented the economic model for Shewhart [2] control charts. Also, Lorenzen and Vance [3] proposed another economic model for economic design of control charts. In the economic model of Lorenzen and Vance, in-control and out-of-control average run length criteria are used while the economic model of Duncan is based on the probabilities of Types I and II of errors. Woodall [4] expressed that economic design of control charts leads to poor statistical properties. Hence, several researchers, such as Saniga [5] and Montgomery et al. [6], proposed economic-statistical design of univariate control charts.

In most of cases, the quality of a process is

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represented by two or more quality characteristics and monitored by multivariate control charts such as T^2 Hotelling control chart. Another multivariate control chart is the Multivariate Exponentially Weighted Moving Average (MEWMA) control chart that was first introduced by Lowry et al. [7] and has advantages rather than Shewhart control chart in detecting small shifts in the mean vector of quality characteristics.

Since the cost parameters in designing the control charts are not deterministic in some real cases, using robust optimization approaches in economic-statistical design of control charts is necessary. By investigating the literature of robust economic and economic-statistical design of control charts in the next paragraphs, robust economic-statistical design of the MEWMA control chart by using interval robust optimization is not considered. In fact, the main idea of this paper is robust economic-statistical design of control charts by using interval robust optimization method. In this paper, cost parameters are not deterministic and value of each uncertainty data is taken from an uncertainty interval. Then, we develop an interval robust model by using interval robust optimization method that is applied to optimize nonlinear programming problems. In this paper, we use the Lorenzen and Vance cost model with considering Taguchi loss approach and apply the Markov chain approach to calculate the in-control and out-of-control average run length criteria.

In the literature of economic-statistical design of control charts, firstly Linderman and Love [8] presented an economic-statistical design of the MEWMA control chart and obtained the ARL by using simulation method. Then, Molnau et al. [9] applied the same proposed model of Linderman and Love [8] by using the Markov chain approach in determining the ARL. Furthermore, several researchers presented economic-statistical model for the MEWMA control chart, such as Testik and Borror [10] and Niaki et al. [11]. They applied the Lorenzen and Vance cost model for economic-statistical design of the MEWMA control chart. For more information about the economic-statistical design of control charts refer to review paper by Celano [12]. In all of these researches, the cost parameters are considered deterministic. However, in some real applications this assumption is violated. Hence, we consider uncertainty in the cost parameters of the Lorenzen and Vance function in this paper.

One of the latest approaches to deal with the optimization under uncertainty is robust optimization. The goal of this approach is to find a solution which is robust under uncertainty of input data. This approach is first proposed in the early 1970s and has recently been extensively studied and extended. Some robust optimization methods, such as simple weighting method, considering the probability of occurrence, regret value and min-max regret model are the most significant among others. Pignatiello and Tsai [13] were the first ones who proposed the robust idea for control charts when the process parameters are not precisely known. In uncertainty case, i.e. when process parameters are not known, the costs of control charts can be handled by robust designs. In this case estimating the cost parameters is very important to obtain the optimal solution. In their paper, the values of process parameters can only be estimated with varying degrees of precision. These parameters were each investigated at three levels and were considered by several separate scenarios that these different scenarios represented different degrees to which the parameter estimates are known. Keats et al. [14] proposed robust approach for economic design of control charts with stressing on limitations and barriers of the economic design. Actually, they illustrated the importance of robust design procedures for control charts when the parameters are uncertainty. In this paper, the cost model is optimized by using robust optimization.

One of the robust optimization approaches to design control charts in the literature of the robust optimization is the scenario-based approach in which cost parameters are defined by different scenarios. Linderman and Choo [15] proposed the scenario-based approach for robust economic design of a single process. They considered three discrete robustness measures for cost parameters by using several scenarios and calculated the suitable control chart parameters under all scenarios. Vommi and Seetala [16] presented an approach to robust economic design of $\bar{X}$ control charts and obtained the best solution by using genetic algorithm. In fact, several authors applied scenario-based approach for robust economic and economic-statistical design of Shewhart control charts.

Also, another robust optimization approach is the interval robust optimization. Firstly, Soyster [17] proposed a linear optimization model to construct a solution that is feasible for all input data such that each uncertain input data can take any values from an interval. This approach, however, tends to find solutions which are over-conservative. An important step for developing a theory for robust optimization is taken independently by Ben-Tal and Nemirovski [18-20], El-Ghaoui and Lebret [21] and El-Ghaoui et al. [22]. To address the issue of over conservatism, these papers proposed less conservative models by considering uncertain linear problems with ellipsoidal uncertainties to solve the robust counterparts. Then, Bertsimas and Sim [23,24] suggested a new method which can find a robust counterpart for a linear problem. Furthermore, by introducing the parameter which is called the level of conservatism, of solution, their approach produces solution that has more flexible rather than the conservatism, but it can only be applied for robust optimization of linear programming problem. Soares et al. [25] developed interval robust optimization approach for linear and nonlinear problems either with single or multiple objectives. In this approach, an uncertainty interval is specified, and the set of scenarios is the Cartesian product of all uncertainty intervals. For each solution, the first step is to find the worst scenario and the second step is to find the optimum solution among the worst scenarios. In fact, there are the methods for robust modeling with interval data for linear and nonlinear problems. Generally, there are two methods for robust modeling of nonlinear problems with interval data described by Averbakh and Lebedev [26] and Soares et al. [25]. In this paper, we apply the combination of the two methods by Averbakh and Lebedev [26] and Soares et al. [25] for robust economic-statistical design of the MEWMA control chart by using interval data for the cost parameters.

The rest of the paper is organized as follows: In Section 2, first the MEWMA control chart is introduced. Then Lorenzen and Vance cost function is explained and economic-statistical design of control charts is described. Also, Taguchi loss function is presented. Finally, Markov chain approach is explained in this section. In Section 3, first the concept of the robust optimization approach is explained briefly and then the interval robust optimization method is stated. Moreover, robust economic-statistical model of the MEWMA control chart by using interval robust approach is developed. In Section 4, the Genetic Algorithm (GA) as the optimization method is proposed for solving the developed robust economic-statistical model. In Section 5, the performance of the proposed robust model is evaluated through a numerical example and then a comparison with certain cost parameters is performed. In Section 6, a sensitivity analysis on the interval range of uncertainty in the cost parameters of the developed model is presented. Also, effects of some cost parameters of Lorenzen and Vance cost function on the best solution of robust economic-statistical model of the MEWMA control chart are studied. Our concluding remarks are given in the final section.

2. Economic-statistical design of MEWMA control chart

2.1. The MEWMA control chart

The univariate EWMA control chart is first introduced by Robert [27]. Suppose X_i to be ith sample of a quality characteristic with mean and variance of μ and σ^2, respectively. The EWMA statistic is as follows:

$$Z_i = r(X_i - \mu) + (1-r)Z_{i-1}, \tag{1}$$

where $0 < r \leq 1$ is the smoothing parameter and $Z_0 = 0$. So, the mean of Z_i is 0 and variance of Z_i is:

$$\sigma^2_{Z_i} = \left\{ \frac{r\left[1-(1-r)^{2i}\right]}{2-r} \right\} \sigma^2. \tag{2}$$

The advantage of the EWMA control chart is that the statistic of this chart considers the effect of the previous samples. Hence, researchers introduced a similar control chart for multivariate processes named as MEWMA control chart. In the multivariate case, consider a process with p quality characteristics. Let $\mathbf{X}$ to be a p-dimensional vector of quality characteristics that has a multivariate normal distribution $N_p \sim (\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with the mean vector μ and the covariance matrix $\boldsymbol{\Sigma}$. The statistic of MEWMA control chart proposed by Lowry et al. [9] is calculated by:

$$Q_i = \mathbf{Z}_i^T \boldsymbol{\Sigma}_z^{-1} \mathbf{Z}_i, \tag{3}$$

where $\mathbf{Z}_i$ is calculated by Eq. (1) as a vector, $\mathbf{Z}_0 = 0$, and the variance-covariance matrix of the $\mathbf{Z}$ vector is computed as:

$$\boldsymbol{\Sigma}_{\mathbf{Z}} = \left(\frac{r}{2-r} \right) \boldsymbol{\Sigma}_{\mathbf{X}}. \tag{4}$$

The MEWMA control chart signals an out-of-control state when $Q_i > l$, where l is a predefined upper control limit. The two main parameters of the MEWMA control chart (l and r) are determined such that a particular in-control average run length (ARL_0) is achieved.

2.2. Economic cost function

Most of researches of economic and economic-statistical design of control charts used either the Duncan [1] or Lorenzen and Vance [3] cost function. In this paper, we use the Lorenzen and Vance [3] cost function for economic-statistical design of the MEWMA control chart. In this economic model, the total cost in a cycle involves sampling, searching and repairing costs and the cost due to producing nonconforming items. Expected total cost per time unit is computed by dividing the expected total cost in a cycle by the expected cycle time. In this cost model, it is assumed that the process has been started from an in-control state and the time to the occurrence of an assignable cause has an exponential distribution by the mean of $1/\theta$. The Lorenzen and Vance cost function is calculated by Eq. (5) as shown in Box I, where:

C_0	Cost per hour due to nonconforming items produced by an in-control process;
C_1	Cost per hour due to nonconforming items produced by an out-of-control process;
a	Fixed cost per sample;
b	Cost per unit sampling;

F Cost per false alarm;

W Cost to locate and repair the assignable cause;

E Time to sampling and charting one item;

T_0 Expected search time when signal is a false alarm;

T_1 Expected time to detect an assignable cause;

T_2 Expected time to repair the process;

ARL_0 Average run length of an in-control process;

ARL_1 Average run length of an out-of-control process;

S Expected number of samples taken when process is in-control;

τ Expected time of occurrence of assignable cause;

γ_1 1 if production process continues during search and 0 if production process stops during search;

γ_2 1 if production process continues during repair and 0 if production process stops during repair.

Moreover, expected number of samples when process is in-control (S) and expected time of occurrence of assignable cause (τ) are computed as:

$$S=\frac{e^{(-\theta h)}}{1-e^{(\theta h)}}, \tag{6}$$

and:

$$\tau=\frac{1-(1+\theta h)e^{(-\theta h)}}{\theta\left(1-e^{(-\theta h)}\right)}. \tag{7}$$

The Average Run Length (ARL) is the expected number of successive samples taken until the sample statistic falls outside the control limits. When the process is in-control, higher value of ARL_0 is more preferable. So, for an out-of-control process the lower value of ARL_1 is more preferable. In this paper, we obtain the ARL of the MEWMA control chart by using Markov chain approach which is proposed by Runger and Prabhu [28].

2.3. Economic-statistical design

Design of the MEWMA control chart includes determination of four parameters including the sample size (n), the sampling interval (h), the upper control limit (l) and the smoothing parameter (r). Economic design of the MEWMA control chart leads to poor statistical properties. However, in economic-statistical design for attaining suitable statistical properties, several statistical constraints are added to the economic model. Indeed, in economic-statistical design of the MEWMA control chart, the cost function is minimized by considering constraints, such as lower bound of in-control ARL and also upper bound of out-of-control ARL. In the other words, by using economic-statistical design of control charts, statistical properties can be improved, however, the total cost increases a bit. This economic-statistical model of control charts satisfies both economic and statistical limitations simultaneously. In this paper, we use the economic-statistical model for the MEWMA control chart. To obtain the four parameters of the MEWMA control chart including n, h, l, and r, the Lorenzen and Vance cost function as an objective function is minimized. An upper bound for ARL_1 and a lower bound for ARL_0 are considered as constraints.

This model is defined as follows:

$$\begin{aligned}&\text{minimize } C(n,h,l,r),\\&\text{subject to:}\\&\text{ARL}_0\geq\text{ARL}_L,\qquad \text{ARL}_1\leq\text{ARL}_U,\\&h \text{ and } l>0,\qquad 0<r\leq 1,\\&n:\text{positive interger.}\end{aligned} \tag{8}$$

2.4. Taguchi loss function

In this paper, we used Taguchi loss function for incorporating external costs in the cost model. Taguchi

$$E(C)=\frac{\frac{C_0}{\theta}+C_1\left[-\tau+nE+h(\text{ARL}_1)+\gamma_1T_1+\gamma_2T_2\right]+\frac{\text{SF}}{\text{ARL}_0}+W}{\frac{1}{\theta}+(1-\gamma_1)\left[\frac{ST_0}{\text{ARL}_0}\right]-\tau+nE+h(\text{ARL}_1)+T_1+T_2}+\frac{\left[\frac{(a+bn)}{h}\right]\times\left[\frac{1}{\theta}-\tau+nE+h(\text{ARL}_1)+\gamma_1T_1+\gamma_2T_2\right]}{\frac{1}{\theta}+(1-\gamma_1)\left[\frac{ST_0}{\text{ARL}_0}\right]-\tau+nE+h(\text{ARL}_1)+T_1+T_2}. \tag{5}$$

Box I

loss function is used for representing economic loss due to the deviation of quality characteristic from its target. The multivariate loss function is developed for the multivariate quality characteristics by Kapur and Cho [29], and we used this loss function in this paper. The multivariate loss function is defined as:

$$L(y_1, y_2, \cdots, y_p) = \sum_{i=1}^{p}\sum_{j=1}^{i} k_{ij}(y_i - t_i)(y_j - t_j), \quad (9)$$

where k_{ij} is a constant that depends on the correlation between y_i and y_j and the rework and wasting costs. If y_i and y_j are independent, then k_{ij} is equal to zero. Also, t_i and t_j are the target values of y_i and y_j quality characteristics, and p is the number of quality characteristics. Furthermore, the expected external costs of each product for the in-control process (J_0) and the out-of-control process (J_1) are derived from the following equations, respectively:

$$J_0 = \sum_{i=1}^{p} k_{ii}\left[(\mu_{0i} - t_i)^2 + \sigma_i^2\right] + \sum_{i=2}^{p}\sum_{j=1}^{i-1} k_{ij}\left[(\mu_{0i} - t_i)(\mu_{0j} - t_j) + \sigma_{ij}\right], \quad (10)$$

$$J_1 = \sum_{i=1}^{p} k_{ii}[(\mu_{1i} - t_i)^2 + \sigma_i^2] + \sum_{i=2}^{p}\sum_{j=1}^{i-1} k_{ij}[(\mu_{1i} - t_i)(\mu_{1j} - t_j) + \sigma_{ij}], \quad (11)$$

where μ_i and σ_i^2 are the mean and variance of y_i, respectively, and σ_{ij} is the covariance of μ_i and μ_j. Suppose the in-control and out-of-control production costs are C_0 and C_1, respectively and the production rate in an hour is PR. Hence, the C_0 and C_1 parameters of the Lorenzen and Vance function can be calculated as follows, respectively:

$$C_0 = J_0 PR + C_0, \quad (12)$$

$$C_1 = J_1 PR + C_1. \quad (13)$$

2.5. The Markov chain approach

There are several approaches in the literature for computing ARL of the MEWMA control chart, such as integral equation approach, simulation approach and Markov chain approach. In this paper, we use Markov chain approach because the Markov chain approach is more precise rather than the simulation approach and is simpler than the integral equation approach. In addition, the simulation approach leads to increasing the variance of ARL criterion and is time consuming.

Calculating the ARLs' of the MEWMA control chart by Markov chain approach was first proposed by Runger and Prabhu [28]. Computing the in-control and out-of-control ARL of the MEWMA control chart by using the Markov chain approach is explained in the Appendix.

3. Proposed model: Robust economic-statistical design of the MEWMA control chart by using interval robust optimization method

In the interval robust optimization under uncertainty, we obtain design parameters (n, l, h, r) such that the cost function is minimized by considering the worst-case. In other words, we minimize the maximum cost due to uncertainty in the cost parameters. In this section, we briefly explain about the concept of the robust optimization as well as interval robust optimization and then we develop a model for robust economic-statistical design of the MEWMA control chart.

One of the latest methods for optimizing problem under uncertainty of parameters is robust optimization. Robust optimization is an approach which optimizes the worst case of problem under uncertainty. The main steps of the robust optimization method are as follows: In the first step, feasible solutions are obtained. In the second step, optimal solutions are selected among the feasible solutions. There are different methods in robust optimization approach. One of these methods is interval robust optimization method. In the interval robust optimization method, uncertain data are defined by lower and upper bounds as an interval. There are some methods for robust modeling with interval data for linear and nonlinear problems. The most common method for robust modeling of linear discrete problem is stated by Bertsimas and Sim [23,24]. Also, there are two methods for robust modeling of nonlinear problems with interval data described by Averbakh and Lebedev [26] and Soares et al. [25]. We use the combination of these methods for robust economic-statistical design of the MEWMA control chart.

3.1. Robust economic-statistical model of the MEWMA control chart

Uncertainty in the parameters of the cost model is a very common problem in designing control charts. In this paper, the uncertainty in the cost parameters is characterized by interval data. To model robust economic-statistical design of the MEWMA control chart with uncertain data, we use the interval robust optimization approach and suppose the vector of the design parameters for the MEWMA control chart is $x = (n\ l\ h\ r)$. In this robust approach, the main goal is minimizing the worst-case with uncertain interval data.

In economic-statistical design of the MEWMA control chart, we consider two statistical constraints including lower bound for in-control ARL (ARL_0) and upper bound for out-of-control ARL (ARL_1). In other words, the proposed model is the combination of economic-statistical design with the robust model by considering interval data for cost parameters. Note that the total cost of robust economic-statistical design may be higher than certain model of economic-statistical design of the MEWMA control chart. The robust economic-statistical model for the MEWMA control chart with using the robust optimization of nonlinear problems is explained in the next subsection.

3.2. Proposed model

Suppose that the cost parameters of the Lorenzen and Vance model (C_0, C_1, F, W) are taken from the interval data as follows:

$$C_0 \in [C_0^-, C_0^+], \qquad C_1 \in [C_1^-, C_1^+],$$

$$F \in [F^-, F^+], \qquad W \in [W^-, W^+].$$

Note that the above intervals data are not symmetric. Since the cost parameters are considered as uncertain parameters and we optimize the worst case of the costs, then only the upper bound value and the certain value of the cost parameters are considered. Hence, the lower bounds of the intervals are considered equal to the value of the parameters in certain situation. In the proposed interval robust optimization model, the cost parameters are taken from the above interval such that the problem remains feasible and the solutions obtained do not be far from the optimal solutions.

First, we consider the parameters C_0, C_1, F and W as decision variables in the following optimization model to find the worst case of the cost function under the presence of the uncertain parameters:

$$\max \frac{C_0}{\theta} + C_1(-\tau + nE + h(\text{ARL}_1) + \gamma_1 T_1 + \gamma_2 T_2) + \frac{\text{SF}}{\text{ARL}_0} + W,$$

subject to:

$$C_0^- \leq C_0 \leq C_0^+, \qquad C_1^- \leq C_1 \leq C_1^+,$$

$$F^- \leq F \leq F^+, \qquad W^- \leq W \leq W^+. \tag{14}$$

Then, the dual problem of above optimization model is written as:

$$\min -C_0^- y_1 + C_0^+ y_2 - C_1^- y_3 + C_1^+ y_4 - F^- y_5 + F^+ y_6 - W^- y_7 + W^+ y_8,$$

subject to:

$$-y_1 + y_2 \geq \frac{1}{\theta},$$

$$-y_3 + y_4 \geq -\tau + nE + h(\text{ARL}_1) + \gamma_1 T_1 + \gamma_2 T_2,$$

$$-y_5 + y_6 \geq \frac{S}{\text{ARL}_0}, \qquad -y_7 + y_8 \geq 1,$$

$$y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 > 0. \tag{15}$$

Finally, the proposed interval robust model for robust economic-statistical design of the MEWMA control chart by using the dual problem of optimization model as Eq. (15), is obtained by Eq. (16) which is shown in Box II.

4. Optimization method

One of the algorithms to solve economic-statistical design of control charts is the Genetic Algorithm (GA). The GA algorithm is a global search and optimization tool in biological system [30]. This algorithm is different from the other optimization tools because it considers many points in a search space simultaneously, works directly with a set of parameters characterized as strings of chromosomes instead of parameters themselves. In addition, it uses the probabilistic rules for the search of solutions. Since the cost function in the economic-statistical model is nonlinear, to solve the model in Eq. (16), the GA algorithm is more suitable than the classical optimization tools. In addition, the GA has been applied in many economic and economic-statistical designs of control charts, such as Chou et al. [31], Chen and Yeh [32], Kaya [33] and Niaki et al. [11]. These are the reasons that we used the GA for solving the robust optimization problem.

The steps of the GA algorithm used in this paper are explained as follows.

4.1. Generation

Each setting of the MEWMA parameters including n, h, l and r composes a chromosome including four genes and each gene represents a decision variable. A sample of a chromosome is shown in Figure 1. In this step, 30 chromosomes are generated randomly and the objective function of each chromosome is computed. Also, out-of-control average run length (ARL_1), in-control average run length (ARL_0) for each chromosome are calculated. The generated chromosomes are considered

First gene	Second gene	Third gene	Fourth gene
n	l	h	r

Figure 1. Representation of one chromosome as an example.

$$\min \frac{-C_0^- y_1 + C_0^+ y_2 - C_1^- y_3 + C_1^+ y_4 - F^- y_5 + F^+ y_6 - W^- y_7 + W^+ y_8}{\frac{1}{\theta} + (1-\gamma_1)\left[\frac{sT_0}{\mathrm{ARL}_0}\right] - \tau + nE + h(\mathrm{ARL}_1) + T_1 + T_2}$$

$$+ \frac{\left[\frac{(a_1+a_2 n)}{h}\right] \times \left[\frac{1}{\theta} - \tau + nE + h(\mathrm{ARL}_1) + \gamma_1 T_1 + \gamma_2 T_2\right]}{\frac{1}{\theta} + (1-\gamma_1)\left[\frac{sT_0}{\mathrm{ARL}_0}\right] - \tau + nE + h(\mathrm{ARL}_1) + T_1 + T_2},$$

Subject to :

$$-y_1 + y_2 \geq \frac{1}{\theta}, \qquad -y_3 + y_4 \geq -\tau + nE + h(\mathrm{ARL}_1) + \gamma_1 T_1 + \gamma_2 T_2, \qquad -y_5 + y_6 \geq \frac{S}{\mathrm{ARL}_0},$$

$$-y_7 + y_8 \geq 1, \qquad \mathrm{ARL}_0 \geq \mathrm{ARL}_L, \qquad \mathrm{ARL}_1 \leq \mathrm{ARL}_U, \qquad h \text{ and } l > 0, \qquad 0 < r \leq 1,$$

$$n : \text{positive integer}, \qquad y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 > 0. \tag{16}$$

Box II

as initial population. Then, using cross-over and mutation operators, new populations will be produced.

4.2. Cross-over operator

For cross-over operation, a pair of chromosomes is selected to produce together new children. In each pair, two similar genes are randomly replaced with each other by the probability of $P_c = 0.8$. As an example, the first and the third genes of each chromosome are fixed, and the second and the fourth genes are replaced with each other.

4.3. Mutation operator

The mutation step in each loop is performed with probability of $P_m = 0.3$. In this operation, a gene in chromosomes is randomly mutated by regenerating from the length of the feasible range of n, l, h and r that lengths of the genes are 20, 20, 2 and 1, respectively.

4.4. Evaluation

In step of evaluation, the objective values of the chromosomes (C) and constraints (ARL_0, ARL_1) that are the outputs of the previous step are compared and sorted. At the end of this step, the thirty chromosomes with the lower cost values are exported to the first step to repeat the procedure.

4.5. Stopping rule

The stopping rule applied in the proposed GA is the number of iterations which is fifty. When the algorithm stops, the chromosome with the minimum cost value in the last iteration is selected as the best optimal solution and its elements are considered as the best values of the parameters.

Because, the values of ARL_0 and ARL_1 should be large and small in the proposed robust model, respectively, we use the penalty function approach for the GA algorithm in order to evaluate whether the solutions exceed from both constraints of the model. To do that, we first calculate the value of capacity variable by using violation measure of constraints as follows:

$$\text{capacity} = \max\{0, (\mathrm{ARL}_L - \mathrm{ARL}_0)\} + \max\{0, (\mathrm{ARL}_1 - \mathrm{ARL}_U)\}. \tag{17}$$

Then the value of objective function will be calculated by using the following equation:

$$\text{Objective function} = \text{Objective function} \times e^{\text{capacity}}. \tag{18}$$

In the other words, if the value of ARL_0 is smaller than its corresponding lower bound (ARL_L) or the value of ARL_1 is more than its corresponding upper bound (ARL_U), the value of capacity variable is the sum of the difference of ARL_0 from ARL_L and the difference of ARL_1 from ARL_U; thus this penalty function is applied and the solutions that violate at least one of the constraints are removed.

5. Numerical example

In this section, a numerical example is presented to illustrate performance of the proposed interval robust optimization approach for robust economic-statistical design of the MEWMA control chart. This example is extracted from the paper by Niaki et al. [11]. The radius d_1 and the weight d_2 of the automobile pistons are the two important correlated quality characteristics that must be monitored by the MEWMA control chart. The correlation matrix is as follows:

$$\mathbf{\Sigma} = \begin{bmatrix} 0.00254 & 0.00073 \\ 0.00073 & 0.00079 \end{bmatrix}.$$

In this example, the fixed cost and the variable cost of sampling are 0.5 and 0.1, respectively. Also, it takes approximately 0.05 hours to take and analyze each observation. The time between occurrences of the successive assignable causes follows an exponential distribution with an average of 100 hours. It takes 2 hours to detect an assignable cause. The cost of investigating a false alarm is 50, and the cost of corrective actions is 25. The hourly cost of operating in the in-control state is 10 and in the out-of-control state is 100. Numerical data of this example is briefly presented as follows:

$$\theta = 0.01, \qquad E = 0.05, \qquad T_0 = 0,$$

$$T_1 = 2, \qquad T_2 = 2, \qquad \gamma_1 = \gamma_2 = 1,$$

$$C_0 = 10, \qquad C_1 = 100, \qquad F = 50,$$

$$W = 25, \qquad a = 0.5, \qquad b = 0.1,$$

$$\text{PR} = 100, \qquad \boldsymbol{\mu} = \begin{bmatrix} 0.0505 & 0.0282 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} 1500 & -1000 \\ -1000 & 8000 \end{bmatrix}.$$

We compute the ARL of the MEWMA control chart by using Markov chain method. For practical use, as Runger and Prabhu [28] suggested, the number of states when the process is in-control, m, is considered equal to 25. Also, the parameters m_1 and m_2, in the out-of-control state (defined in the Appendix), are considered equal to 5. In fact, the parameters of m, m_1 and m_2 are the input parameters in Markov chain algorithm for in-control and out-of-control ARL computation.

When the mean of process shifts, the value of non-centrality parameter is obtained as follows:

$$\delta = \left(\boldsymbol{\mu}^T \mathbf{\Sigma}^{-1} \boldsymbol{\mu}\right)^{0.5} = 1.32.$$

Also, we use Taguchi loss function to incorporate the external costs to the Lorenzen and Vance cost function. The external costs, J_0 and J_1, are obtained by using Eqs. (10) and (11), respectively, as follows:

$$J_0 = K_{11}\sigma_1^2 + K_{22}\sigma_2^2 + K_{12}\sigma_{12} = 9.4765,$$

$$J_1 = K_{11} \times (\mu_{11}^2 + \sigma_1^2) + K_{22} \times (\mu_{21}^2 + \sigma_2^2) + K_{12} \times (\mu_{21} \times \mu_{11} + \sigma_{12}) = 18.2578.$$

Using Eqs. (12) and (13), we have:

$$C_0 = (100)(9.4765) + 10 = 957.65,$$

$$C_1 = (100)(18.2578) + 100 = 1925.78.$$

Finally, we apply the GA algorithm to find the best solution of the proposed interval robust economic-statistical model. In this example, robust economic-statistical design of the MEWMA control chart in Eq. (16) is used. In this model, the lower bound of ARL_0 (ARL_L) is assumed to be equal to 200 and the upper bound of ARL_1 (ARL_U) is assumed to be equal to 2. The upper and lower bounds of decision parameters are defined as:

$$1 \leq n \leq 20, \qquad 1 \leq l \leq 20,$$

$$0 < h \leq 2, \qquad 0 < r \leq 1.$$

In this example, the cost parameters of the Lorenzen and Vance cost model (C_0, C_1, F, W) are uncertain and are taken from interval data as follows (case (a)):

$$C_0 \in [957.65, 1025], \qquad C_1 \in [1925.78, 2000],$$

$$F \in [50, 55], \qquad W \in [25, 30].$$

This robust economic-statistical model is solved by the GA algorithm and optimal values of design parameters are reported in Table 1. In addition, the results of economic-statistical design of the MEWMA control charts, when the values of cost parameters are deterministic, are extracted from Niaki et al. [11] and reported in Table 1.

The results in Table 1 show when the values of cost parameters of the Lorenzen and Vance cost

Table 1. Optimal values of robust economic-statistical design of the MEWMA control chart (case (a)).

Model	Cost	ARL5_0	ARL_1	n	l	h	r
Robust ES (case (a)) (proposed model) (uncertain data)	1073.1	695.68	1.639	8	9.777	0.372	0.674
Economic-statistical Niaki et al. [14] (certain data)	1007.3	246.17	1.356	11	11	0.484	0.802

Table 2. Optimal values of robust economic-statistical design of the MEWMA control chart by expanded interval data rather than case (a).

Model	Cost	ARL_0	ARL_1	n	l	h	r
Robust ES (b) (proposed model)	1177.93	636.65	1.032	10	5.15	0.85	0.775
Robust ES (c) (proposed model)	1305.6	530.42	1.36	9	11.44	0.635	0.216

function are taken from uncertain interval, the total cost increases 6.52% rather than the situation when the cost parameters of Lorenzen and Vance cost function are described with certain data. Also, in uncertain situation, the in-control ARL and out-of-control ARL increase with respect to the certain situations.

Table 2 reports the results of robust economic-statistical design of the MEWMA control charts when the values of cost parameters are taken from the longer intervals data (case (b)). In case (b), the value of the cost parameters of the Lorenzen and Vance cost model (C_0, C_1, F, W) are taken from the interval data with larger ranges, with respect to case (a) as follows:

$$C_0 \in [957.65, 1125], \qquad C_1 \in [1925.78, 2150],$$

$$F \in [50, 60], \qquad W \in [25, 35].$$

In case (c), the value of the cost parameters of the Lorenzen and Vance model (C_0, C_1, F, W) are taken from the interval data with larger ranges, with respect to case (b), as follows:

$$C_0 \in [957.65, 1250], \qquad C_1 \in [1925.78, 2300],$$

$$F \in [50, 65], \qquad W \in [25, 40].$$

Table 2 shows the total cost increases when the interval data of the cost parameters are expanded. In other words, increasing in uncertainty interval of the cost parameters leads to increasing in the optimal total cost of robust model. Also, the in-control ARL decreases when the uncertainty in the interval data of the cost parameters increases.

6. Sensitivity analysis

In this section, first we present sensitivity analysis on the uncertainty interval of cost parameters. We increase the length of uncertainty intervals of cost parameters and evaluate the effect of increasing in uncertainty of input parameters on total cost of robust economic-statistical design of the MEWMA control chart. Furthermore, we present sensitivity analysis on some cost parameters of Lorenzen and Vance cost function that are not considered uncertain, and study the effect of these parameters on the best solution of robust economic-statistical model of the MEWMA control chart.

Sensitivity analysis on the solution of the robust economic-statistical model of the MEWMA control chart under different values of upper bound of uncertainty interval for cost parameters C_0, C_1, F and W are done, and the results are summarized in Tables 3 to 6, respectively. In other words, to study the effect of the changing interval data of the cost parameter

Table 3. Sensitivity analysis of robust economic-statistical model of the MEWMA control chart under different uncertainty intervals for C_0 and the expected value approach.

	C_0	Cost	ARL_0	ARL_1	n	l	h	r
Interval data	[957.65, 1025]	1073.1	695.68	1.63	8	9.77	0.37	0.67
	[957.65, 1035]	1080.6	621.45	1.49	6	11.73	0.43	0.52
	[957.65, 1045]	1089.3	537.65	1.36	4	13.37	0.46	0.43
Expected value	957.65	1007.75	635.45	1.71	3	8.97	0.36	0.68

Table 4. Sensitivity analysis of robust economic-statistical model of the MEWMA control chart under different uncertainty intervals for C_1 and the expected value approach.

	C_1	Cost	ARL_0	ARL_1	n	l	h	r
Interval data	[1925.78, 2000]	1073.1	695.68	1.63	8	9.77	0.37	0.67
	[1925.78, 2010]	1074.1	917.69	1.16	11	11.81	0.52	0.44
	[1925.78, 2020]	1074.9	1024.6	1.1	12	6.2	0.6	0.24
Expected value	1925.78	1070.58	822.74	1.032	10	7.445	0.88	0.71

Table 5. Sensitivity analysis of robust economic-statistical model of the MEWMA control chart under different uncertainty intervals for F and the expected value approach.

	F	Cost	$\mathbf{ARL_0}$	$\mathbf{ARL_1}$	n	l	h	r
Interval data	[50, 55]	1073.1	695.68	1.63	8	9.77	0.372	0.674
	[50, 60]	1074.5	838.77	1.52	6	11.19	0.42	0.73
	[50, 65]	1075.7	1056.4	1.4	12	9.8	0.5	0.804
Expected value	50	1071.4	709.44	1.44	6	13.2	0.54	0.43

Table 6. Sensitivity analysis of robust economic-statistical model of the MEWMA control chart under different uncertainty intervals for W and the expected value approach.

	W	Cost	$\mathbf{ARL_0}$	$\mathbf{ARL_1}$	n	l	h	r
Interval data	[25, 30]	1073.1	695.68	1.63	8	9.77	0.372	0.67
	[25, 35]	1073.9	904.66	1.47	8	9.43	0.5156	0.73
	[25, 40]	1074.5	1210.8	1.4	10	8.2	0.59	0.79
Expected value	25	1072.08	587.65	1.39	7	11.84	0.465	0.23

C_0, the parameter C_0 is taken from different interval data in Table 3 and the cost parameters C_1, F and W are taken from the same defined interval data of case (a) in the previous section. Table 3 shows that with increasing in uncertainty interval of the parameter C_0, the optimal total cost of interval robust optimization approach (*cost*) increases. Also, the optimal sampling interval (h) and optimal control limit (l) increase when the uncertainty interval of the parameter C_0 increases. The results show that the in-control ARL and out-of-control ARL decrease when the uncertainty interval of the parameter C_0 increases.

Also, for studying the effect of changing interval data of the cost parameter C_1, the parameter C_1 is taken from different interval data in Table 4 and the cost parameters C_0, F and W are taken from the same defined interval data of case (a) in the previous section. Results given in Table 4 show that with increasing in uncertainty interval of the parameter C_1, the optimal total cost of robust model (*cost*) increases a bit. Also, the optimal sampling interval (h) increases when the uncertainty interval of the parameter C_1 increases. Moreover, the in-control ARL increases and out-of-control ARL decreases when the uncertainty interval of the parameter C_1 increases.

In addition, to evaluate the effect of changing interval data of the cost parameter F, the parameter F is taken from different interval data in Table 5 and the cost parameters C_0, C_1 and W are taken from the same defined interval data of case (a) in the previous section. Also, to investigate the effect of changing interval data of the cost parameter W, the parameter W is taken from different interval data in Table 6, and the cost parameters C_0, C_1 and F are taken from the same defined interval data of case (a) in the previous section.

The results of sensitivity analyses on the solutions of the robust economic-statistical model of the MEWMA control chart under different intervals for the cost parameters F and W in Tables 5 and 6 shows that with increasing interval data of the parameters F and W, the optimal total cost of interval robust optimization approach (*cost*) increases a bit. Also, the optimal sampling interval (h) increases when the uncertainty intervals of parameters F and W increase. Moreover, the results show that the in-control ARL increases and out-of-control ARL decreases when the uncertainty intervals of the parameters F and W increase.

Note that increasing in the uncertainty interval of the parameter C_0 has the most effect on the total cost of developed interval robust model with respect to the other uncertain parameters C_1, F and W. In other words, increasing the uncertainty interval of the parameter C_0 has the most effect on increasing in the total cost, i.e. with increasing 1% in the interval data of the cost parameter C_0, the total cost of the developed model increases 0.7% while with increasing 1% in the interval data of the cost parameter C_1, the total cost of the developed model increases 0.17%. Furthermore, with increasing 1% in the interval data of the cost parameter F, the total cost of the developed model increases 0.014% and also with increasing 1% in the interval data of the cost parameter W, the total cost of the developed model increases 0.004%.

Moreover, the results of the expected value approach on the solutions of the proposed robust model for the cost parameters (C_0, C_1, F, W) are reported in Tables 3 to 6. In the expected value approach, the midpoint of interval data is used as the input parameter for calculating the total cost. This value is the same value of the parameter in certain situation when interval data is symmetric. In this paper, the lower bounds of the intervals are considered equal to

Table 7. Sensitivity analysis of robust economic-statistical model of the MEWMA control chart under different values for the fixed cost (a) and variable cost of sampling (b).

a	b	Cost	$\mathbf{ARL_0}$	$\mathbf{ARL_1}$	n	l	h	r
0.5	0.1	1073.1	695.68	1.63	8	9.77	0.37	0.67
	0.3	1075.1	786.75	1.75	9	10.43	0.79	0.71
0.7	0.1	1073.9	1120.2	1.2	10	12.3	0.5	0.54
	0.3	1075.8	827.31	1.67	5	10.72	0.46	0.36

the value of the parameters in certain situation (refer to the interval data definition in paragraph 2 of Section 3.2). Therefore, the lower bounds of the intervals data are the same value of input parameter for calculating the total cost by using the expected value approach. In other words, in this approach for example, the parameter F is taken from the lower bound of the interval data in Table 5 and the cost parameters C_0, C_1 and W are taken from the same defined interval data of case (a) in the previous section. The results show the total cost decreases in the expected value approach versus the interval data approach.

Finally, we investigate sensitivity analysis on the fixed cost (a) and the variable cost of sampling (b) of Lorenzen and Vance cost function that are not considered uncertain in the robust economic-statistical model of the MEWMA control chart. The results are given in Table 7.

The results in Table 7 show that with increasing in the fixed cost and the variable cost of sampling, total cost of proposed robust economic-statistical model increases a bit. In addition, the variable cost of sampling (b) has more effect on the total cost rather than the fixed cost of sampling (a). Also, results show that with increasing in the fixed cost of sampling (a), value of in-control ARL increases and value of out-of-control ARL decreases while with increasing in the variable cost of sampling (b), value of out-of-control ARL increases.

7. Conclusion

In this paper, by using the interval robust optimization approach, the robust economic-statistical model of the MEWMA control chart for monitoring the mean vector of a process was developed. The proposed interval robust model was solved by the genetic algorithm. The results showed that the total cost of robust economic-statistical model of the MEWMA control chart increases when the cost parameters of Lorenzen and Vance cost model are taken from uncertain interval rather than the certain data. The performance of the proposed robust model was illustrated by a numerical example, and a comparison with certain situation was done. In addition, a sensitivity analysis was performed to study the effects of changing uncertainty interval of cost parameters on total cost and parameters of the MEWMA control chart. Furthermore, the effects of increasing deterministic cost parameters on the best solution were investigated. The obtained results showed that increasing in the uncertainty interval of the parameter C_0 has the most effect on the total cost of developed interval robust model with respect to the other uncertain parameters C_1, F and W. Hence, if we determine interval data of the parameter C_0 with more precise in real application, the optimal solution achieved under uncertainty is close to the optimal solution obtained under certain data. In addition, the results of sensitivity analysis expressed that the variable cost of sampling has more effect on the total cost of the developed model rather than the fixed cost of sampling. Note that the uncertainty in the other parameters of the Lorenzen and Vance cost function including the time parameters (T's) could be occurred. Hence, the proposed model can be developed to model uncertainty in these parameters as a future research.

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Appendix

The in-control and out-of-control ARL of the MEWMA control chart

In this appendix, computing the in-control and out-of-control ARL of the MEWMA control chart is explained based on the paper by Runger and Prabhu [28]. In the Markov chain approach, assumes that $S(r)$ is a p-dimensional sphere with radius r and $\mathbf{Z}$ is a p-dimensional random vector with standard normal distribution. Then, the p-dimensional random vector $r\mathbf{Z}/\|\mathbf{Z}\|$ has uniform distribution on $S(r)$ ($\|Z\|$ is the size of vector $\mathbf{Z}$) and the uniform random variable on $S(1)$ is denoted by U. The p-dimensional random vector $\mathbf{Z}$ is partitioned to a $(p-1)$-dimensional random vector and a random variable in which the partitions are independent and the shift only affects the random variable. So, $S(\text{UCL})$ is divided to $m+1$ spherical shells as the size of each shell is:

$$g = 2(\text{UCL})/(m+1). \tag{A.1}$$

So, transition probability from state i ($i = 1, 2, \cdots, m$) to state j (j is non-zero), $p(i,j)$, is calculated by the Eq. (A.2) (q_t is each point in the sphere that is shown by the center point of each shell):

$$\begin{aligned} p(i,j) =& P(q_t \text{ in state } j | q_{t-1} \text{ in state } i) \\ =& P\{(j-.5)g < r\mathbf{X}_t + (1-r)\mathbf{Z}_{t-1} \\ & < (j+.5)g | q_{t-1} = gi\}, \end{aligned} \tag{A.2}$$

where, $\mathbf{X}_t$ is a p-dimensional vector and follows a spherical distribution. Now, Z_{t-1} and igU have similar distribution by condition $q_{t-1} = gi$ and therefore:

$$p(i,j) = P\{(j-0.5)g < rX_t + (1-r)igU < (j+0.5)g\}. \tag{A.3}$$

Since $\mathbf{U}$ and $\mathbf{X}_t$ are independent spherical random vectors, by using a non-central chi-squared distribution with p degrees of freedom and the non-centrality parameter $c = [(1-r)ig/r]^2$, transition probability, $p(i,j)$, is calculated:

$$p(i,j) = P\{(j-0.5)^2 g^2/r^2 < \chi^2(p,c) < (j+0.5)^2 g^2/r^2\}. \tag{A.4}$$

For state $j = 0$ we have:

$$p(i,0) = P\{\chi^2(p,c) < (0.5)^2 g^2/r^2\}. \tag{A.5}$$

Finally, the average run length of the MEWMA control chart in the in-control state is calculated as follows:

$$\text{ARL} = \mathbf{s}^{\mathbf{T}}(\mathbf{I} - \mathbf{O})^{-1}\mathbf{1}, \tag{A.6}$$

where, $\mathbf{s}$ is a $(m+1)$-dimensional vector in which the component related to the starting state of the chain is 1 and the other components are 0; $\mathbf{O}$ is a $(m+1)\times(m+1)$ matrix of transitional probability from state i to state j; and $\mathbf{1}$ is a $(m+1)$-dimensional vector of all 1s. For estimating the out-of-control ARL, the weighted vector ($\mathbf{Z}_t$) is first partitioned to a $(p-1)$-dimensional random vector with mean zero and a random variable shown as:

$$q_t = \|Z_t\| = (Z_{t1}^2 + Z_{t2}^T Z_{t2})^{1/2}. \tag{A.7}$$

In state of out-of-control, $\|Z_{t2}\|$ has mean zero and can be estimated using the method explained above. To model Z_{t1}, the part that is between -UCL and UCL is divided to the $2m_1+1$ states with the length $g_1 = 2\text{UCL}/(2m_1+1)$. The states are labeled as h_α for $\alpha = 1, 2, \cdots, 2m_1+1$, and state h_{m1+1} has center point zero. Since in the out-of-control state $\mu = \delta e$, $\mathbf{Z_{t2}}$ is a $(p-1)$-dimensional spherical with mean zero, the $(p-1)$ hyper planes are orthogonal to e and pass through the center point of each h_α, then each hyper plane is divided to the m_2+1 states that the width of each shell is $g_2 = 2\text{UCL}/(m_2+1)$ and the width of the first shell is $g_2/2$. The shells are labeled as v_β for $\beta = 0, 1, \cdots, m_2$ and v_0 has zero center point. Each point in $S(\text{UCL})$ is based on the distance in the direction of e and its radius distance that is perpendicular to e belongs to a h_α and a v_β. The pair (α, β) is accepted for a state in the Markov chain if its related point is inside $S(\text{UCL})$. In other words, $(\alpha - (m_1+1))^2 g_1^2 + \beta^2 g_2^2 < \text{UCL}^2$.

The transition probability of Z_{t1} from state i to j, $h(i,j)$, is as follows:

$$\begin{aligned} h(i,j) =& P(Z_{t1} \text{ in state } j | Z_{t-1,1} \text{ in state } i) \\ =& P(-\text{UCL} + (j-1)g < rX_{t1} \\ & + (1-r)Z_{t-1,1} < -\text{UCL} + jg | Z_{t-1,1} = c_i) \\ =& P[(-\text{UCL} + (j-1)g - (1-r)c_i)/r - \delta < X_{t1} \\ & - \delta < (-\text{UCL} + jg - (1-r)c_i)/r - \delta], \end{aligned} \tag{A.8}$$

where for $i = 1, 2, \cdots, 2m_1+1$, $c_i = -\text{UCL} + (i - 0.5g)$. The transition probability of $\|Z_{t2}\|$ from state i to state j, $v(i,j)$ is obtained similar to the in-control state by replacing p by $p-1$ as follows:

$$\begin{aligned} v(i,j) =& P\{(j-0.5)^2 g^2/r^2 < \chi^2(p-1,c) \\ & < (j+0.5)^2 g^2/r^2\}. \end{aligned} \tag{A.9}$$

Since Z_{t1} and Z_{t2} are independent, the transition probability of the bivariate Markov chain (Z_{t1}, Z_{t2}) from state (i_x, i_y) to (j_x, j_y) is as follows:

$$p[(i_x, j_x), (j_x, j_y)] = h(i_x, j_x) v(i_y, j_y). \tag{A.10}$$

So, the average run length of the MEWMA control chart in the out-of-control state is calculated as follows:

$$\text{ARL} = \mathbf{s}^{\mathbf{T}}(\mathbf{I} - \mathbf{O})^{-1}\mathbf{1}, \tag{A.11}$$

where, $\mathbf{s}$ is the first vector of the chain and $\mathbf{1}$ is a vector of all 1s, $\mathbf{O}$ is a $(2m_1+1)(m_2+1)$ matrix of transitional probability from state i to state j. For an in-control state we let $\delta = 0$ in the above bivariate Markov chain to estimate the in-control ARL.

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