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A revision on area ranking and deviation degree methods of ranking fuzzy numbers

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Abstract. Recently two important methods [Wang Zh., X., Liu, Y.J. and Feng, B. "Ranking L-R fuzzy number based on deviation degree", *Information Science*, pp. 2070-2077 (2009); and Wang, Y.M. and Luo, Y. "Area ranking of fuzzy numbers based on positive and negative ideal points", *Computers and Mathematics with Applications*, pp. 1769-1779 (2009)] have been proposed for ranking fuzzy numbers. But we have found that they both have a same basic disadvantage. In this paper, after a short review on different proposed fuzzy number ranking methods, we explain the drawback on deviation degree and the area ranking methods and provide an improvement method to overcome this shortage. Our approach is based on the maximization set and minimization set methods concepts. The results show the superiority of the proposed method in comparison with other ranking methods, especially when the ranking of the inverse and the symmetry of the fuzzy numbers are of interest.

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1. Introduction

Under fuzzy environment, ranking Fuzzy Numbers (FNs) is an important part of decision making process. Following the Zadeh's paper [1] on fuzzy set theory and then Jain's paper [2] and Dubois and Prade's paper [3] on FNs, the fuzzy theory and its application have grown explosively. There exist many different methods in ranking FNs. The earliest method of ranking the FNs was proposed by Jain [4]. For the triangular and trapezoidal FNs, Liou and Wang [5] used the concepts of the integral values for ranking normal and non-normal FNs. Cheng [6] indicated that Liou and Wang's [5] method has a defect in ranking normal and non-normal triangular/trapezoidal FNs because it consider the normal and non-normal triangular/trapezoidal FNs equal. Chu and Tsao [7] proposed a new approach for ranking FN that considered the area between the centroid and original points. But, it has been known that their approach has some drawbacks. Also, Wang et al. [8] explained that the centroid formulae provided by Cheng [6] is not always true and leads to some misapplications with some FNs. They gave the revised centroid formulae for ranking FNs. Abbasbandy and Asadi [9] performed a modification on distance based methods and proposed a new fuzzy number ranking method, called sign distance method. Asady and Zendehman [10] used the nearest point of support function for ranking FNs. Wang and Lee [11] gave a revision on Chu and Tsao's approach [7] to overcome its defects. Chu and Lin [12] applied an interval arithmetic method based on fuzzy TOPSIS model for ranking fuzzy numbers. Sadi-Nezhad and Damghani [13] introduced use of the preference ratio with a moderate modification as an efficient method for ranking negative fuzzy numbers. Ramli and Mohamad [14] used Jaccard

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similarity measure index with degree of optimism. Xu and Zhai [15] proposed an improved method for ranking FNs by distance minimization. Zhang and Yu [16] proposed a pairwise comparison based method for ranking L-R fuzzy numbers. A review list on different fuzzy number ranking methods was provided in Table 1. For more details, readers are referred to the references.

Wang et al. [17], introduced an approach to ranking L-R fuzzy numbers based on a deviation degree. Also Wang and Luo [18] proposed the positive and the negative ideal point concept for ranking FNs. They defined two new alternative indices for the purpose of ranking. The two new indices are defined for ranking based on idioms of a Decision Maker (DM)'s policy towards risks and the left and the right areas between FNs and the two ideal points. Though these two methods acted well, we found that they both have a same basic disadvantage with some FNs. In this paper, we explain this defect and provide an improvement approach to overcome this shortage. This approach is based on considering the maximization and minimization sets concepts and DM's risk attitudes.

The rest of this paper is organized as follows. Section 2 introduces some basic concepts and definitions of the FNs. Section 3 briefly introduces the two debatable methods and explains their shortage. Section 4 proposes an improved approach to overcome the shortage of these methods. In section 5, some numerical examples are explained to verify the efficiency of the proposed improvement approach. Finally, this study concludes in Section 6.

2. Preliminaries

Definition 1. Let X be as a universe set. A fuzzy subset A of X is defined with a membership function $\mu_A(x)$ mapping each element x in A to a real number in the interval [0,1]. The membership function of a FN A is defined as follows (see [8]):

$$\mu_A(x)=\begin{cases} f_A^L(x) & a\le x\le b\\ 1 & b\le x\le c\\ f_A^R(x) & c\le x\le d\\ 0 & \text{Otherwise,}\end{cases} \tag{1}$$

where $\mu_A^L(x):[a,b]\ [0,1]$ and $\mu_A^R(x):[c,d]\rightarrow[0,1]$ are two continuous functions mapping from the real line R to the closed interval [0,1]. The former is a strictly increasing function called the left membership function and the latter is a monotonically decreasing function called the right membership function. If $\mu_A^R(x)$ and $\mu_A^L(x)$ are both linear, then A is referred to as a trapezoidal FN and is usually denoted by $A=(a,b,c,d)$. In particular, when $b=c$, the trapezoidal FN is reduced to a triangular FN, denoted by $A=(a,b,d)$. So, triangular FNs are special cases of trapezoidal FNs. The set of all these fuzzy numbers is denoted by E.

Definition 2. An L-R fuzzy number $A=(m,n,\alpha,\beta)_{LR}$, $m\le n$ and $\alpha,\beta\ge 0$ is defined as follows (see [18]):

$$\mu_A(x)=\begin{cases} L\left(\frac{m-x}{\alpha}\right) & -\infty<x<m\\ 1 & m\le x\le n\\ R\left(\frac{x-n}{\beta}\right) & n<x<\infty\end{cases} \tag{2}$$

Definition 3. For fuzzy set A, the support set of A is defined as (see [17]):

$$s(A)=\{x\in R|\mu_A(x)>0\}. \tag{3}$$

Definition 4. Let $x_{\min}$ and $x_{\max}$ be the infimum and supremum of the support set of an arbitrary group of L-R fuzzy numbers, $A_1,A_2,...,A_n$, respectively. Then $A_{\min}$ and $A_{\max}$ are defined as the minimization set, and the maximization set respectively, and their membership function is given by (see [17]):

$$\mu_{A_{\min}}=\begin{cases}\frac{x_{\max}-x}{x_{\max}-x_{\min}} & x\in s\\ 0 & \text{otherwise,}\end{cases} \tag{4}$$

and:

$$\mu_{A_{\max}}=\begin{cases}\frac{x-x_{\min}}{x_{\max}-x_{\min}} & x\in s\\ 0 & \text{otherwise,}\end{cases} \tag{5}$$

where S is the support set of these FNs, i.e. $s=\bigcup_{i=1}^n s(A_i)$.

3. The shortage of the area ranking and the deviation degree ranking methods

In this section, we briefly introduce Wang et al.'s [17] method and Wang and Luo's [18] method. Then we analytically discuss on their shortage.

3.1. Ranking L-R fuzzy numbers based on deviation degree

Wang et al. [17] utilized maximization and minimization sets and the left and right deviation degree concepts in their method.

Definition 5. For a group of L-R fuzzy numbers, $A_1,A_2,...,A_n$ the left deviation degree and the right deviation degree of $A_i (i\in 1,2,...,n)$ that is defined by Wang et al. [17] are calculated as follows (see [17] and

Table 1. An itemized list of different methods proposed for ranking the FNs.

Paper title	Author (s)	Year	Method for ranking	Notes
An improved method for ranking fuzzy numbers by distance minimization	Xu, R-N and Zhai, X.-Y. [15]	2012	An improved method for ranking FNs by distance minimization	
New pairwise comparison based method of ranking L-R fuzzy numbers	Zhang, M. and Yu, F.A. [16]	2011	This paper proposed a new approach for ranking L-R FNs based on pairwise comparison	
On the Jaccard index with degree of optimism in ranking fuzzy numbers	Ramli, N., and Mohamad, D. [14]	2010	The authors used Jaccard similarity measure index with degree of optimism for FN ranking	
Triangular approximations of fuzzy numbers using alpha-weighted valuations	Abbasbandy, S., Ahmady, E. and Ahmady, N. [19]	2010	A fuzzy triangular approximation using a-weighted valuations is introduced and the nearest approximation by the minimization technique is obtained	
Ranking L-R fuzzy number based on deviation degree	Wang Zh., X., Liu, Y.J., and Feng, B. [17]	2009	This paper introduced an approach to ranking L-R fuzzy numbers based on deviation degree. In their approach, the maximal and minimal reference sets are constructed to measure the L-R deviation degree of FNs	
A new method for analyzing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers	Sanguansat, K.A.T.A. and Chen, S.M. [20]	2009	The proposed method calculates the areas on the positive side, the areas on the negative side, the spreads and the heights of the generalized fuzzy numbers to evaluate ranking scores of the generalized fuzzy numbers	Generalized FNs
An interval arithmetic method based on fuzzy TOPSIS model	Chu, T.C. and Lin, Y.C. [12]	2009	Ranking fuzzy numbers by means of removals	
Application of a fuzzy TOPSIS method base on modified preference ratio and fuzzy distance measurement in assessment of traffic police centers performance	Sadi-Nezhad, S. and Khalili Damghani, K. [13]	2009	Preference ratio with a moderate modification for negative fuzzy numbers was used as an efficient ranking method for fuzzy numbers in a relative manner	Generalized fuzzy numbers
The satisfaction degree of the fuzzy numbers and ranking of the fuzzy numbers	Shi, Y.Y. and Xue-hai, Y. [21]	2009	In this paper a new concept of the satisfaction degree of the fuzzy number is presented	
Area ranking of fuzzy numbers based on positive and negative ideal points	Wang, Y.M. and Luo, Y. [18]	2009	This paper presents an alternative ranking approach for fuzzy numbers called area ranking based on positive and negative ideal points	
Fuzzy risk analysis based on ranking fuzzy numbers using a-cuts, belief features and signal/noise ratios	Chen, S.M. and Wang, C.H. [22]	2009	This paper proposed a new method for ranking fuzzy numbers using the a-cuts, the belief feature and the signal/noise ratios	
Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads	Chen, S.M. and Chen, J.H. [23]	2009	The proposed method considers the defuzzified values, the heights and the spreads for ranking generalized fuzzy numbers	Generalized fuzzy numbers
Ranking of fuzzy numbers, some recent and new formulas	Abbasbandy, S. [24]	2009	Method of D-distance Method of min distance Method of magnitude	
Fuzzy risk analysis based on fuzzy numbers with different shapes and different deviations	Lee, L.W. and Chen, S.M. [25]	2008	This paper represented a new method for ranking trapezoidal fuzzy numbers based on their shapes and deviations	Trapezoidal fuzzy numbers
Preference ratio-based maximum operator approximation and its application in fuzzy flow shop scheduling	Sadi Nezhad, S. and Ghaleh Assadi, R. [26]	2008	This paper introduced an appropriate approximation for the maximum operator in which the weak-dominance fuzzy numbers are ranked based on the concept of preference ratio	Triangular and trapezoidal fuzzy numbers

Table 1. An itemized list of different methods proposed for ranking the FNs (continued).

Paper title	Author (s)	Year	Method for ranking	Notes
Ranking of intuitionistic fuzzy numbers	Nayagam, V.L.G., Venkateshwari, G. and Sivaraman, G. [27]	2008	In this paper, a new method of intuitionistic fuzzy scoring to intuitionistic fuzzy numbers that generalizes Chen and Hwang's scoring method for ranking of intuitionistic fuzzy numbers has been introduced and studied	Intuitionistic Fuzzy Numbers
Similarity measure between generalized fuzzy numbers using quadratic-mean operator	Chen, S.J. [28]	2008	This paper presents a novel similarity measure that is based on the quadratic-mean operator to solve similarity measurement problems that involve generalized fuzzy numbers	Generalized fuzzy Numbers
The revised method of ranking fuzzy numbers with an area between the centroid and original points	Wang, Y.J. and Lee, H.S. [29]	2008	The paper proposed the revised method of ranking fuzzy numbers with an area between the centroid and original points	
Trapezoidal approximations of fuzzy numbers preserving the expected interval - Algorithms and properties	Grzegorzewski, P. [30]	2008	The Algorithms for calculating the proper approximations are proposed and some properties of the approximation operators are discussed	
Ranking fuzzy numbers by distance minimization	Asady, A. and Zendehnam, A. [10]	2007	The authors proposed use of the nearest point of support function for ranking FNs	
Distance and similarity measures for fuzzy operators	Balopoulos, V., Hatzimichailidis, A.G. and Papadopoulos, B.K. [31]	2007	This paper suggests a new family of normalized distance measures between fuzzy sets, based on binary operators and matrix norms	
Fuzzy risk analysis based on the ranking of generalized trapezoidal fuzzy numbers	Chen, S.J. and Chen, S.M. [32]	2007	The proposed method considers the centroid points and the standard deviations of generalized trapezoidal fuzzy numbers for ranking generalized trapezoidal fuzzy numbers	Generalized trapezoidal fuzzy numbers
A new approach for ranking fuzzy numbers based on fuzzy simulation analysis method	Sun, H. and Wu, J. [33]	2006	A combination of methods including computer and math application is developed	
A new similarity measure of generalized fuzzy numbers based on geometric-mean averaging operator	Chen, S.J. [34]	2006	This paper presented a new similarity measure based on the geometric-mean averaging operator to handle the similarity measure problems of generalized fuzzy numbers	Generalized fuzzy numbers
A theoretical development on a fuzzy distance measure for fuzzy numbers	Chakraborty. C. and Chakraborty. D. [35]	2006	This paper introduced a fuzzy distance measure for generalized fuzzy numbers	It computes the fuzzy distance between two generalized fuzzy numbers and also LR-type fuzzy numbers
On the centroids of fuzzy numbers	Wang, Y.M., Yang, J.B., Xu, D.L. and Chin, K.S. [36]	2006	This paper presented the correct centroid formulae for fuzzy numbers and justified them from the viewpoint of analytical geometry	
The nearest trapezoidal form of a generalized left right fuzzy number	Abbasbandy, S. and Amirfakhrian, M. [37]	2006	This paper proposed a new approach to assigning distance between fuzzy numbers	Generalized LR fuzzy number
Selecting IS personnel use fuzzy GDSS based on metric distance method	Chen, L.S. and Ching., H.C. [38]	2005	This paper proposed a new approach to rank fuzzy numbers by metric distance	

Table 1. An itemized list of different methods proposed for ranking the FNs (countinued).

Paper title	Author (s)	Year	Method for ranking	Notes
Ranking fuzzy numbers with preference weighting function expectations	Liu, X.W. and Han, S.L. [39]	2005	This paper extends the centroid expectation approach and proposes a preference weighting function expectation method to rank FNs	
The nearest trapezoidal fuzzy number to a fuzzy quantity	Abbasbandy, S. and Asady, B. [40]	2004	This paper introduced a fuzzy trapezoidal approximation using the metric (distance) between two fuzzy numbers	Trapezoidal fuzzy number
A new method for handling multi criteria fuzzy decision-making problems using FN-IOWA operators	Chen, S.J. and Chen, S.M. [41]	2003	The method used fuzzy numbers to extend the traditional induced ordered weight averaging (IOWA) operator to present the fuzzy-number IOWA (FN-IOWA) operator, wherein fuzzy numbers are used to describe the argument values and the weights of the FN-IOWA operator, and the aggregation results are obtained by using fuzzy-number arithmetic operations	This method presented a new method for ranking fuzzy numbers
Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers	Chen, S.Y. and Chen, S.M. [42]	2003	This paper represented a method called the simple center of gravity method (SCGM) to calculate the center-of-gravity (COG) points of generalized fuzzy numbers. Then, it used the SCGM to propose a new method to measure the degree of similarity between generalized fuzzy numbers	Generalized fuzzy numbers
Ranking fuzzy numbers with an area between the centroid point and original point	Chu, T.C., Tsao, C.T. [7]	2002	The authors proposed their new approach for ranking FN that considered the area between the centroid and original points	
An approximate approach for ranking fuzzy numbers based on left and right dominance	Chen, L.H. and Lu, H.W. [43]	2001	The proposed approach only requires a few left and right spreads at some a-levels of fuzzy numbers to determine the respective dominance of one fuzzy number over the other	
Ranking fuzzy numbers by preference ratio	Modarres, M. and Sadi-Nezhad, S. [44]	2001	In this method a preference function is denied by which fuzzy numbers are measured point by point and at each point the most preferred number is identified. Then, these numbers are ranked on the basis of their preference ratio	Triangular fuzzy numbers
Reasonable properties for the ordering of fuzzy quantities (I)	Wang, X. and Kerre, E.E. [45]	2001		
A method for ranking fuzzy numbers and its application to decision-making	Lee-Kwang, H. and Lee, J.H. [46]	1999	A new method for ranking fuzzy numbers is proposed. The method considers the overall possibility distributions of fuzzy numbers in their evaluations for ranking FNs	This method evaluates fuzzy numbers with a satisfaction function and the viewpoint given by a user.
A model and algorithm of fuzzy product positioning	Hsieh, C.H. and Chen, S.H. [47]	1999	A modified geometrical distance method is presented to measure the distance between two fuzzy numbers	Triangular fuzzy numbers
A new fuzzy arithmetic	Ma, M., Friedman, M. and Kandel, A. [48]	1999	This paper presented fuzzy numbers with a new parametric form. Based on this representation, a new fuzzy arithmetic is defined	

Table 1. An itemized list of different methods proposed for ranking the FNs (countinued).

Paper title	Author (s)	Year	Method for ranking	Notes
A simple fuzzy group decision making method	Cheng, C.H. [49]	1999	First the intuition ranking method, then the alpha cut method and finally the fuzzy mean and spread are used for ranking	The defuzzification value of the trapezoidal fuzzy numbers is used for ranking FNs
On a canonical representation of fuzzy numbers	Delgado, M., Vila, M.A. and Voxman, W. [50]	1998	This paper used two parameters, value and ambiguity parameters to obtain canonical representations and to deal with fuzzy numbers in decision-making problems	
Some remarks on distances between fuzzy numbers	Voxman, W. [51]	1998	A fuzzy distance between fuzzy numbers is introduced and its basic properties are studied	
A new approach for ranking fuzzy numbers by distance method	Cheng, C.H. [6]	1998	This paper indicated that Liou and Wang's (1992) [5] method has a defect in ranking normal and provided a method improvement	
Ranking fuzzy numbers with integral value	Liou, T.S, and Wang, M.J.J. [5]	1992	This paper used the concepts of the integral values for ranking normal and non-normal FNs	
A new index for ranking fuzzy numbers	Choobineh, F. and Huishen, L. [52]	1990	A procedure for ranking discrete fuzzy numbers is presented. proposed in F. Choobineh and Huishen. Li (1990)	

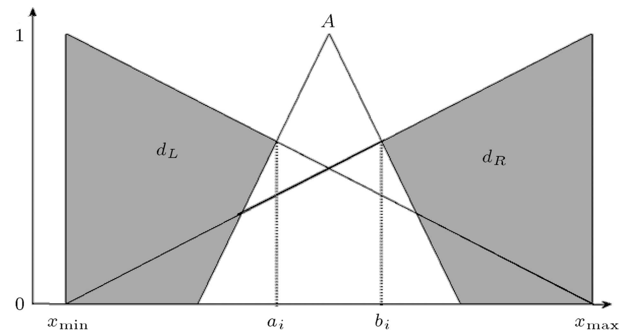

Figure 1. The left and right deviation degrees of A_i.

Figure 1):

$$d_L^i = \int_{x_{\min}}^{a_i} \left(\mu_{A_{\min}}(x) - \mu_{A_i^L}(x)\right) dx, \tag{6}$$

and:

$$d_L^i = \int_{i}^{x_{\max}} \left(\mu_{A_{\max}}(x) - \mu_{A_i^R}(x)\right) dx, \tag{7}$$

where a_i is the abscissa of the crossover point of $\mu_{A_i^L}$ and $\mu_{A_{\max}}(x)$ and b_i is that of $\mu_{A_{\max}}(x)$ and $\mu_{A_i^R}(x)$, $i = 1, 2, ..., n$.

Wang et al. [17] considered an indice for ranking by using these deviation degrees. They calculated the expectation value of centroid of each FN and constructed the transfer coefficient of each FN $A_i (i \in 1, 2, ..., n)$ as follows:

$$\lambda_i = \frac{M_i - M_{\min}}{M_{\max} - M_{\min}}, \tag{8}$$

where M_i is the expectation value of centroid of L-R fuzzy number $A_i = (m_i, n_i, \alpha_i, \beta_i)$ defined as:

$$M_i = \frac{\int_{m_i-\alpha_i}^{n_i+\beta_i} x\mu_{A_i}(x)dx}{\int_{m_i-\alpha_i}^{n_i+\beta_i} \mu_{A_i}(x)dx}, \tag{9}$$

and $M_{\min} = \min(m_1, m_2, ..., m_n)$ and $M_{\max} = \max(m_1, m_2, ..., m_n)$ and $M_{\max} \neq M_{\min}$.

Ultimately the ranking measure is given by:

$$d_i = \begin{cases} \frac{\lambda_i d_i^L}{1+(1-\lambda_i)d_i^R} & M_{\max} \neq M_{\min} \quad i = 1, 2, ..., n \\ \frac{d_i^L}{1+d_i^R} & M_{\max} = M_{\min} \quad i = 1, 2, ..., n \end{cases} \tag{10}$$

where the greater the d_i is, the larger is the FN A_i.

3.2. Area ranking of fuzzy numbers based on positive and negative ideal points

Wang and Luo [18] considered a positive ideal point and a negative ideal point. For a set of FNs $A_1, A_2, .., A_n$ they defined the positive and negative ideal points as $x_{\max} = \sup S$ and $x_{\min} = \inf S$, where $s = \bigcup_{i=1}^{n} s_i$ and s_i is the support set of the A_i. Let $A_i = (a_i, b_i, c_i, d_i)$ be one of the FNs to be compared whose membership function is defined by Eq. (1). The areas $S_L(i)$ and

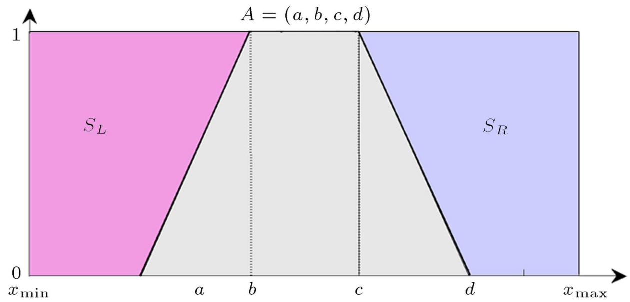

Figure 2. Area ranking based on Wang and Lou's [18] method.

$S_R(i)$, as shown in Figure 2, are computed as:

$$S_L(i)=\int_{x_{\min}}^{a_i} dx+\int_{a_i}^{b_i}\left(1-f_{A_i}^L(x)\right)dx$$

$$=(b_i-x_{\min})-\int_{a_i}^{b_i} f_{A_i}^L(x)dx, \tag{11}$$

$$S_R(i)=\int_{c_i}^{d_i}\left(1-f_{A_i}^R(x)\right)dx+\int_{d_i}^{x_{\max}} dx$$

$$=(x_{\max}-c_i)-\int_{c_i}^{d_i} f_{A_i}^R(x)dx. \tag{12}$$

In particular, for trapezoidal FNs, the two areas are computed by:

$$S_L(i)=\frac{a_i+b_i}{2}-x_{\min}, \tag{13}$$

$$S_R=(i)=x_{\max}-\frac{c_i+d_i}{2}. \tag{14}$$

By considering DM's risk treatment, they introduced two Ranking Indices based on Areas (RIA) to rank FNs by the following equations:

$$\mathrm{RIA}_1(i)=\frac{1}{2}\left[\left(\frac{S_L(i)}{x_{\max}-x_{\min}}\right)r_L(i)+\left(1-\frac{S_R(i)}{x_{\max}-x_{\min}}\right)r_R(i)\right], \tag{15}$$

$$\mathrm{RIA}_2(i)=\frac{S_L(i)r_L(i)}{S_L(i)r_L(i)+S_R(i)r'_R(i)}, \tag{16}$$

$r_L(i)$ is a left risk factor; both $r_R(i)$ and $r'_R(i)$ are right risk factors defined for different purposes and are computed as:

$$r_L(i)=1+(\alpha-0.5)\frac{b_i-a_i}{x_{\max}-x_{\min}}, \tag{17}$$

$$r_R(i)=1+(\alpha-0.5)\frac{d_i-c_i}{x_{\max}-x_{\min}}, \tag{18}$$

$$r'_R(i)=1-(\alpha-0.5)\frac{d_i-c_i}{x_{\max}-x_{\min}}, \tag{19}$$

where $0\leq\alpha\leq 1$.

3.3. Analysis of the shortage

The main disadvantage in both of these methods is that all of FNs, having infimum equal to $x_{\min}$ and left spread equal to zero, are considered equal, but this is not true. We will explain this in the following.

Consider Wang et al.'s [17] method. In this method for every L-R fuzzy number $A_i=(m_i,n_i,\alpha_i,\beta_i)$, a_i, the abscissa of the crossover point of $\mu_{A_i^L}(x)$ and $\mu_{A_{\min}}(x)$, is equal to:

$$a_i=\frac{x_{\max}(m_i-\alpha_i)+m_i(1-x_{\min})-\alpha_i}{x_{\max}-x_{\min}+1-\alpha_i}, \tag{20}$$

if $m_i=x_{\min}$ and $\alpha_i=0$ then:

$$a_i=\frac{x_{\max}x_{\min}+x_{min}(1-x_{min})-\alpha_i}{x_{\max}-x_{\min}+1}$$

$$=x_{\min},$$

so:

$$d_i^L=\int_{x_{\min}}^{x_{\min}}\Big(\mu_{A_{\min}}(x)-\mu_{A_i^L}(x)\Big)dx$$

$$=0.$$

Therefore, by Eq. (10), for every couple of this described FN, we reach to:

$$d_i=0.$$

Also in Wang and Luo's [18] method for a set of FNs $A_1,A_2,\cdots,A_n$, $A_i=(a_i,b_i,c_i,d_i)$, if $a_i=x_{\min}$ ($x_{\min}=\inf S$, where $S=\bigcup_{i=1}^n s_i$ and s_i is the support set of the A_i) and the left spread is equal to zero ($b_i=a_i$), then:

$$S_l(i)=\int_{x_{\min}}^{x_{\min}} dx+\int_{x_{\min}}^{x_{\min}}\left(1-f_{A_i}^L(x)\right)dx$$

$$=0,$$

and so:

$$\mathrm{RIA}_2(i)=\frac{S_L(i)r_L(i)}{S_L(i)r_L(i)+S_R(i)r'_R(i)},$$

and therefore, all of these FNs are considered equal.

Therefore, in both of these methods, all of the fuzzy numbers, having the infimum equal to $x_{\min}$ and the left spread equal to zero, are considered equal. For example consider L-R fuzzy numbers $A=(2,0,0)$, $B=(2,0,3)$, $C=(2,0,6)$ and $D=(2,5,0,3)$ as shown in Figure 3. By Wang et al.'s [17] method as $d_A^L=d_B^L=d_C^L=d_D^L=0$ so $d_A=d_B=d_C=d_D=0$. Therefore, using this method results in $A=B=C=D$. Also in Wang and Luo's [18] method, as $S_L(A)=S_L(B)=S_L(C)=S_L(D)=0$, using RIA_2 results in $A=B=C=D$. However, Figure 3, intuitively, shows that $A\prec B\prec C\prec D$.

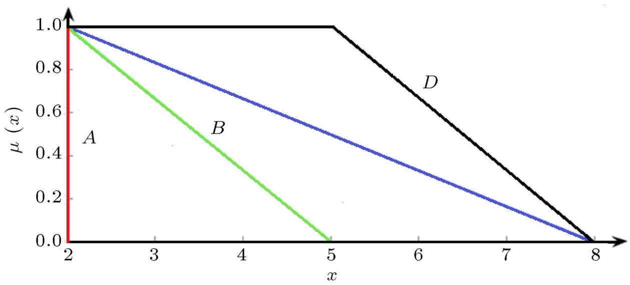

Figure 3. L-R fuzzy numbers in Example 1.

4. Improvement in the area ranking and deviation degree methods by considering decision maker's risk attitudes to ranking

In this section an improved approach is proposed to overcome the shortage of Wang et al. [17] and Wang and Luo [18] methods. For a set of FNs $A_1, A_2, \cdots, A_n$, consider $x_{\max} = \sup S$ and $x_{\min} = \inf S$, where $S = \bigcup_{i=1}^{n} s_i$ and s_i is the support set of the A_i. The proposed method is based on considering two areas. One area is under minimization set from $x_{\min}$ to the crossover point of minimization set and $f_A^R(x)$. Another is the area under maximization set between two points a_i and $x_{\max}$ (see Figure 4). However, to take into account the decision maker's(DM's) risk attitude, the number $0 \leq \varepsilon \leq 1$ is considered to show the DM's attitude toward risks, so that the greater is the ε, the more is the DM's attitude toward the risk, and vice versa. To consider the DM's risk attitude into ranking the ε is engaged in the maximization and minimization sets by shifting the minimization set as ε to the left and the maximization set as $1-\varepsilon$ to the right. This makes the first and second areas, as mentioned above, to be larger and smaller, respectively. The areas are shown in Figure 4. By engaging the number $0 \leq \varepsilon \leq 1$ in the minimization and the maximization sets, the two new sets are obtained which are named by $t_{\max}$ and $t_{\min}$, respectively, and their membership functions are:

$$\mu_{t_{\min}}(x) = \begin{cases} \frac{x_{\max}-x+1-\varepsilon}{x_{\max}-x_{\min}+1} & x_{\min}-\varepsilon \leq x \leq x_{\max}+1-\varepsilon \\ 0 & \text{else} \end{cases} \tag{21}$$

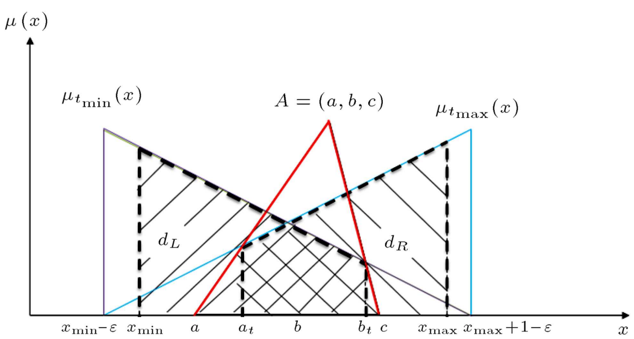

Figure 4. Areas under $t_{\min}$ and $t_{\max}$ between two determined points.

$$\mu_{t_{\max}}(x) = \begin{cases} \frac{x-x_{\min}+\varepsilon}{x_{\max}-x_{\min}+1} & x_{\min}-\varepsilon \leq x \leq x_{\max}+1-\varepsilon \\ 0 & \text{else} \end{cases} \tag{22}$$

and $0 \leq \varepsilon \leq 1$.

It is obvious that for every fuzzy number A, if A is larger, the area d_L is larger and the area d_R is smaller.

For a trapezoidal fuzzy number $A = (a, b, c, d)$, a_t and b_t, the crossover points, are:

$$a_t = \frac{a(x_{\max}+1-\varepsilon) - b(x_{\min}-\varepsilon)}{x_{\max}-x_{\min}+a-b+1}, \tag{23}$$

$$b_t = \frac{c(x_{\max}+1+\varepsilon) - d(x_{\min}+\varepsilon)}{x_{\max}-x_{\min}+c-d+1}. \tag{24}$$

For a triangular fuzzy number $A = (a, b, c)$, a_t is the same as Eq. (23) and b_t is easily obtained by replacing c and d by b and c, in Eq. (24).

The areas d_L and d_R are computed as follows:

$$d_L = \int_{x_{\min}}^{b_t} \mu_{t_{\min}}(x)dx, \tag{25}$$

$$d_R = \int_{a_t}^{x_{\max}} \mu_{t_{\max}}(x)dx. \tag{26}$$

It is obtained that for $0 \leq \varepsilon \leq 1$, d_L and d_R are monotonically increasing and decreasing functions of ε, respectively. For a trapezoidal fuzzy number $A = (a, b, c, d)$, the ranking measure value(A) is defined as:

$$\text{Value}(A) = \frac{a+d+d_L}{1+d_R}, \tag{27}$$

and for a triangular FN $A = (a, b, c)$, an area ranking measure is defined as:

$$\text{Value}(A) = \frac{a+c+d_L}{1+d_R}, \tag{28}$$

where the larger is Value(A), the greater the fuzzy number A is.

5. Numerical examples

In this section, we represent numerical examples to verify the validity of the proposed approach. The results are compared with some other methods.

Example 1. Consider triangular fuzzy numbers $A = (2, 2, 2)$, $B = (2, 2, 8)$, and trapezoidal fuzzy number $C = (2, 2, 3, 4)$ in Figure 5. From Figure 5 it is obvious that $A \prec C \prec B$, but in Wang and Luo's [18] method, by regarding to measure RIA$_2$ and also by using Wang et al.'s [17] method, the results will be $A = B = C$.

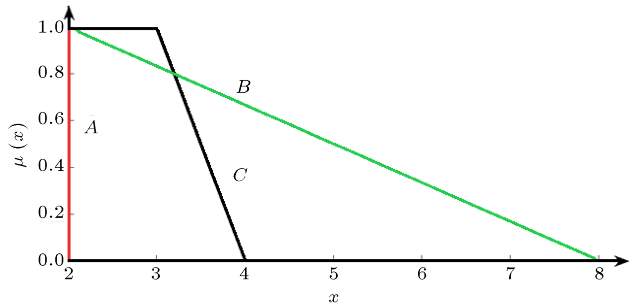

Figure 5. FNs in Example 1.

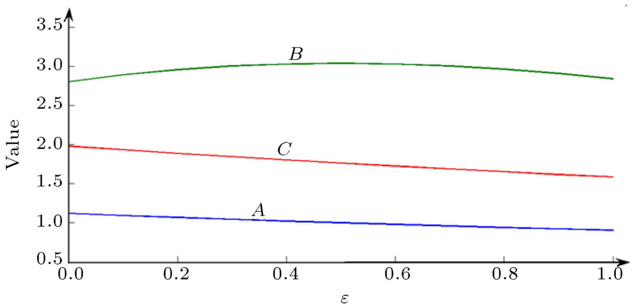

Figure 6. Ranking results for FNs A, B and C in Example 1.

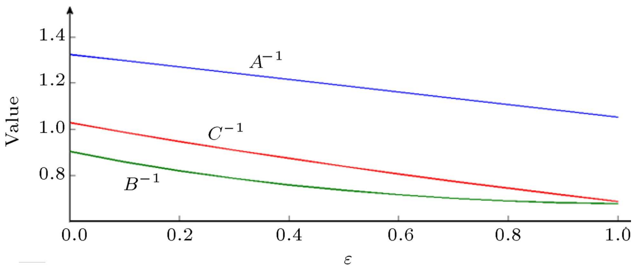

Figure 7. Ranking results for FNs A^{-1}, B^{-1}, C^{-1} in Example 1.

Our result for ranking these FNs is shown in Figure 6 and for every $0 \leq \varepsilon \leq 1$ we see that $A \prec B \prec C$.

Also the inverse and symmetry of these FNs are $A^{-1} = (0.5, 0.5, 0.5)$, $B^{-1} = (0.125, 0.5, 0.5, 0.5)$, $C^{-1} = (0.25, 0.33, 0.5, 0.5)$, and $(-A) = (-2, -2, -2)$, $(-B) = (-8, -2, -2, -2)$, $(-C) = (-4, -3, -2, -2)$. Our approach's results for inverse and symmetry of FNs are shown in Figures 7 and 8 and in both of them we reach to ranking $A^{-1} \succ C^{-1} \succ B^{-1}$ and $(-A) \succ (-C) \succ (-B)$ which are true.

Example 2. Consider two triangular fuzzy numbers $A = (2, 3, 3)$ and $B = (2, 2, 5)$, as show in Figure 9. In Wang and Luo's [18] method, by regarding the indice RIA_2 the ranking is $A \succ B$, and by noting measure RIA_1 for $\alpha = 0.5$ the result is $A \sim B$. Also Wang et al.'s [17] method results $A \succ B$. All of these rankings are false, but our approach's result that is shown in Figure 10, for every $0 \leq \varepsilon \leq 1$ reaches to $B \succ A$ that is true.

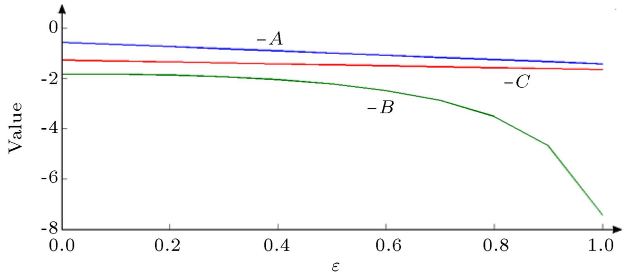

Figure 8. Ranking results for FNs $-A$, $-B$ and $-C$ in Example 1.

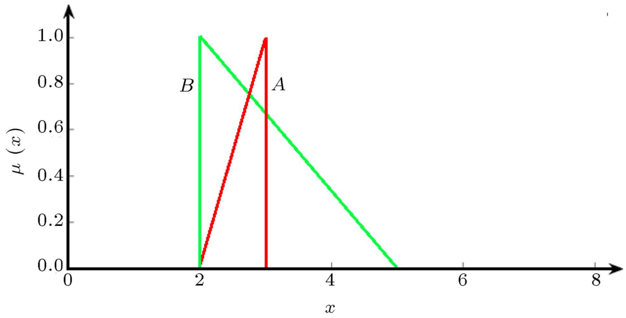

Figure 9. FNs in Example 2.

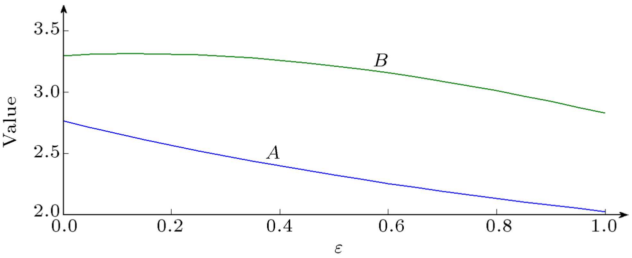

Figure 10. Ranking results for FNs A and B in Example 2.

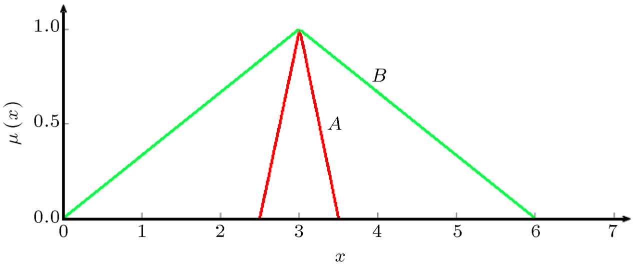

Figure 11. Fuzzy numbers in Example 3.

Example 3. Consider FNs $A = (2.5, 3, 3.5)$ and $B = (0, 3, 6)$ shown in Figure 11. It is obvious that the decision maker prefers the result $A \succ B$ because it obtains more precise information, but by Wang and Luo's [18] approach the different rankings are obtained for the different values of α. However our approach for every $0 \leq \varepsilon \leq 1$ reaches to $A \succ B$. The results for our method and Wang and Luo's [18] methods are summarized in Table 2. Also Table 3 summarizes the results obtained by some other methods.

Table 2. Ranking results of FNs A, B in Example 3 for our method and Wang and Luo's [18] method.

Proposed approach				Wang and Luo's [18] approach					
					RIA$_1$		RIA$_2$		
ε	A	B	Result	α	A	B	A	B	Result
0	2.778685	2.59875	$A \succ B$	0	0.479167	0.375	0.5	0.5	$A \succ B$
0.5	2.52798	2.261616	$A \succ B$	0.5	0.5	0.5	0.5	0.5	$A \sim B$
1	2.310817	2.002086	$A \succ B$	1	0.520833	0.625	0.5	0.5	$B \succ A$

Table 3. Ranking results of FNs A, B in Example 3 by different approaches.

Ranking approach	A	B	Result
Deviation degree [17]	0.5	0.89256288	$A \succ B$
Sign distance ($p = 1$) [9]	6	6	$A \sim B$
Sign distance ($p = 2$) [9]	4.32	4.8989	$B \succ A$
Chu and Tsao [7]	1.5	1.5	$A \sim B$
Cheng's distance index [6]	3.041	3.041	$A \sim B$
Cheng's CV index [6]	0.00833	0.3	$A \succ B$
Xu and Zhai [15]	$C(A) = 3,\ D(A) = 18.33$	$C(B) = 3,\ D(B) = 24$	$A \succ B$

Table 4. Proposed method's result for FNs in Example 4.

4	A	B	C	Result
0	7.875	9.2137	9.4167	$A \prec B \prec C$
0.5	6.647	8.1839	8.3687	$A \prec B \prec C$
1	5.961	7.3091	7.4545	$A \prec B \prec C$

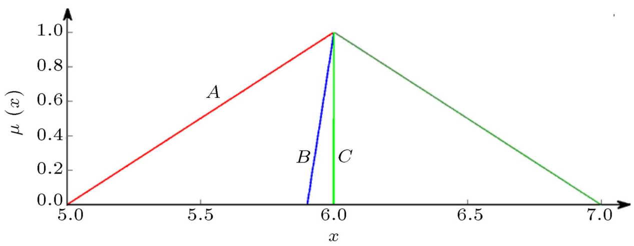

Figure 12. FNs in Example 4.

Example 4. Consider L-R FNs in Ref. [18], i.e. $A = (6, 6, 1, 1)_{LR}$, $B = (6, 6, 0.1, 1)_{LR}$ and $C = (6, 6, 0, 1)_{LR}$ as shown in Figure 12. Table 4 shows the results obtained by our approach.

Table 5 gives the ranking results obtained by some other methods. Also our approach's results for symmetry of these FNs are given in Table 6.

Table 6. Proposed approach's result for symmetry of FNs in Example 4.

ε	$-A$	$-B$	$-C$	Result
0	-4.35	-4.81755	-4.8667	$-A \succ -B \succ -C$
0.5	-4.4718	-4.8899	-4.9351	$-A \succ -B \succ -C$
1	-5.0196	-5.4431	-5.4901	$-A \succ -B \succ -C$

6. Conclusion

This paper introduced an improved approach to overcome the shortage of area ranking and deviation degree methods in ranking different fuzzy numbers. Considering decision maker's risk attitudes is an important point in ranking FNs that was considered in this paper. Whereas a number of approaches fail to rank the symmetry and inverse of FNs, this approach efficiently acts with these cases. The examples given in this paper shows that the proposed approach can

Table 5. Ranking results of L-R FNs A, B and C in Example 4 by different approaches.

Ranking approach	A	B	C	Ranking
Deviation degree [17]	0.25	0.5339	0.5625	$A \prec B \prec C$
Sign distance ($p = 1$) [9]	6.12	12.45	12.5	$A \prec B \prec C$
Sign distance ($p = 2$) [9]	8.52	8.82	8.85	$A \prec B \prec C$
Chu and Tsao [7]	3	3.126	3.085	$A \prec C \prec B$
Cheng's distance index [6]	6.021	6.349	6.7519	$A \prec B \prec C$
Cheng's CV index [6]	0.028	0.0098	0.0089	$C \prec B \prec A$
Xu and Zhai [15]	6	6.225	6.25	$A \prec B \prec C$

efficiently rank different FNs compared to other approaches.

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