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Nonlinear free vibration analysis of clamped circular fiber metal laminated plates

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Abstract. Nonlinear free vibration of symmetric circular Fiber Metal Laminated (FML) hybrid plates is investigated. Considering the Von Karman geometric nonlinearity, the First order Shear Deformation Theory (FSDT) is used to obtain the equations of motion. For the first time, five equations of motion of circular FML plates are derived in terms of plate displacements. The obtained equations are simplified for analyzing the first mode of symmetric circular plates. Using the Galerkin method, five coupled nonlinear Partial Differential Equations (PDEs) of motion are transformed to a single nonlinear Ordinary Differential Equation (ODE), which is solved analytically by the multiple time scales method, and an analytical relation is found for the nonlinear frequency of these plates. The obtained results are compared with the published results and good agreement is found. Moreover, the effects of several parameters on linear and nonlinear frequencies and the free vibration response are investigated.

1. Introduction

FMLs are laminated composites in which isotropic and orthotropic layers are combined in alternate form, where isotropic layers are the outer layers protecting the inner layers from impact and environmental conditions, such as humidity and erosion. Because of their low weight, high strength and stiffness and long life, they are good alternatives for traditional metals in aerospace and military applications.

Many parts of military, aerospace, naval, and nuclear machinery can be modeled as plates, which is why the buckling and vibration of plates have been investigated extensively. The theories of plates can be divided into two main groups: the Classical Plate Theory (CPT) in which the shear stress is ignored, and shear deformation theories. The simplest shear deformation theory is the FSDT [1]. Although Higher order Shear Deformation Theories (HSDT), like Reddy's Third order Shear Deformation Theory (TSDT) [1], give more accurate results, they result in much more complicated equations of motion.

There are many studies dealing with the linear vibration of circular plates using different pate theories [2-20]. On the other hand, nonlinear vibration studies are relatively few. Sarma and co-workers [21] studied the nonlinear free vibration of two layered hybrid plates. Nageswara and Pillai [22] analyzed a simply-supported rectangular plate using CPT theory and Von Karman nonlinear strains. The obtained three coupled nonlinear PDEs were solved numerically. Experimental and CPT-based theoretical nonlinear vibration analysis of a Glare 3 hybrid plate was done by Harras and co-workers [23]. Using the Galerkin method [24] and performing two-mode analysis [25], inclusion of in-plane displacements and rotary inertia effects in equations of motion [26], analysis of orthotropic circular plate with an embedded isotopic core layer [27] and the study of transition

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from periodic to chaotic vibrations [28] were also considered. Peng and co-workers [29] performed a semi-analytical analysis of nonlinear vibration of circular isotropic plates using the Differential Quadrature Method (DQM). The nonlinear free vibration of simply-supported rectangular FML plates, based on FSDT, and using the method of multiple time scales, was investigated by Shooshtari and Razavi [30]. He and co-workers [31] presented a general perturbation solution for large-deflection analysis of circular plates having different moduli in tension and compression. Liu and Chen [32] reviewed the axi-symmetric vibration of polar orthotropic circular plates using the axisymmetric finite element method. Civalek [33-34] studied the nonlinear static and dynamic response of thin plates using HDQ and FD methods and, in another work, introduced a coupled methodology for numerical solution of the same problem. Ducceschi et al. [35] investigated the interaction of modes of thin rectangular plates in nonlinear free and forced vibrations. Zheng et al. [36] investigated nonlinear free vibrations of axisymmetric polar orthotropic circular membranes using large deflection theories. Shooshtari and Razavi [37] studied the nonlinear free and forced vibration of anti-symmetric hybrid laminated rectangular plates based on FSDT and the multiple scales method. Nonlinear vibration of laminated rectangular plates under several boundary conditions was studied by Amabili and Karazis [38] using three different plate theories. Large amplitude vibration of thin rubber rectangular plates was studied by Breslavsky et al. [39] in which both geometric and physical nonlinear effects were considered. Rahimi et al. [40], using DQM, investigated the linear free vibration of FML annular plates. To the best knowledge of the authors of this article, there is no investigation dealing with the nonlinear vibration of circular FML plates. So, this study is performed to fill the gap in the literature.

In this work, nonlinear free vibration of symmetric circular FML plates is investigated analytically, considering the Von Karman geometric nonlinearity, and based on the FSDT. For the first time, five equations of motion of circular FML plates are derived in terms of plate displacements. The obtained equations are simplified for analyzing the first mode of symmetric circular plates. Using the Galerkin method, five coupled nonlinear PDEs of motion are transformed to a single nonlinear ODE, which is solved analytically by the multiple time scales method, and an analytical relation is found for the nonlinear frequency of these plates. The obtained results are compared with the published results and good agreement is found. The effects of several parameters on linear and nonlinear frequencies and the free vibration response are investigated, too.

2. Formulation of the plate motion

The displacements field of the FSDT in polar coordinates is:

$$u_r(r,\theta,z,t) = u(r,\theta,t) + z\varphi_r(r,\theta,t),$$

$$u_\theta(r,\theta,z,t) = v(r,\theta,t) + z\varphi_\theta(r,\theta,t),$$

$$u_z(r,\theta,z,t) = w(r,\theta,t), \tag{1}$$

where u, v and w are the displacements of the mid-plane along r, θ, and z axes, respectively; u_r, u_θ, and u_z are the displacements of any point of the plate along r, θ, and z axes, respectively. $\varphi_r = \frac{\partial u}{\partial z}$ and $\varphi_\theta = \frac{\partial v}{\partial z}$ are rotations of mid-plane about θ and r axes, respectively (Figure 1). Using the above displacements field in Von Karman nonlinear strains [41], the following strains field is obtained:

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \gamma_{r\theta} \\ \gamma_{\theta z} \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} u_{,r} + \frac{1}{2}(w_{,r})^2 \\ \frac{1}{r}\left(u + v_{,\theta} + \frac{1}{2r}(w_{,\theta})^2\right) \\ \frac{1}{r}\left(u_{,\theta} - v + rv_{,r} + (w_{,r}w_{,\theta})\right) \\ \varphi_\theta + \frac{1}{r}w_{,\theta} \\ \varphi_r + w_{,r} \end{bmatrix}$$

$$+z \begin{bmatrix} \varphi_{r,r} \\ \frac{1}{r}(\varphi_r + \varphi_{\theta,\theta}) \\ \frac{1}{r}(\varphi_{r,\theta} - \varphi_\theta + r\varphi_{\theta,r}) \\ 0 \\ 0 \end{bmatrix}, \tag{2}$$

where subscript (,) denotes the differential with respect to the following parameter.

The equations of motion of plate are derived by the Hamilton principal, i.e.:

$$\delta W = \int_0^t (\delta\Pi - \delta T)\, dt = 0, \tag{3}$$

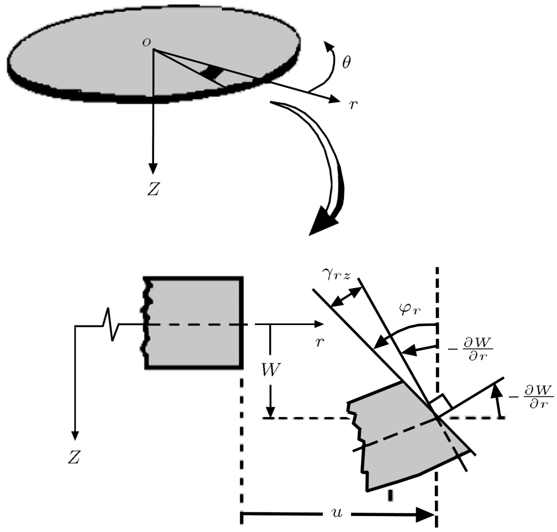

Figure 1. Undeformed and deformed geometries of an edge of a circular plate in the FSDT.

where δT and $\delta\Pi$ are the virtual kinetic energy and the virtual strain energy, respectively. So, the following five equations of motion are obtained [41]:

$$N_{rr,r}+\frac{1}{r}N_{r\theta,\theta}+\frac{1}{r}(N_{rr}-N_{\theta\theta})=I_0u_{,tt}+I_1\varphi_{r,tt}, \quad (4)$$

$$N_{r\theta,r}+\frac{1}{r}N_{\theta\theta,\theta}+\frac{2}{r}N_{r\theta}=I_0v_{,tt}+I_1\varphi_{\theta,tt}, \quad (5)$$

$$\begin{aligned}&Q_{r,r}+\frac{1}{r}Q_{\theta,\theta}+\frac{1}{r}Q_r+N_{r\theta}\left(\frac{2}{r}w_{,r\theta}\right)\\&\quad+N_{rr}\left(\frac{1}{r}w_{,r}+w_{,rr}\right)+N_{\theta\theta}\left(\frac{1}{r^2}w_{,\theta\theta}\right)\\&\quad+w_{,r}\left(N_{rr,r}+\frac{1}{r}N_{r\theta,\theta}\right)\\&\quad+\frac{1}{r}w_{,\theta}\left(N_{r\theta,r}+\frac{1}{r}N_{\theta\theta,\theta}\right)=I_0w_{,tt},\end{aligned} \quad (6)$$

$$\begin{aligned}&M_{rr,r}+\frac{1}{r}M_{r\theta,\theta}+\frac{1}{r}(M_{rr}-M_{\theta\theta})-Q_r\\&\quad=I_2\varphi_{r,tt}+I_1u_{,tt},\end{aligned} \quad (7)$$

$$M_{r\theta,r}+\frac{1}{r}M_{\theta\theta,\theta}+\frac{2}{r}M_{r\theta}-Q_\theta=I_2\varphi_{\theta,tt}+I_1v_{,tt}, \quad (8)$$

where N_{rr}, $N_{\theta\theta}$, and $N_{r\theta}$ are stress resultants; M_{rr}, $M_{\theta\theta}$, and $M_{r\theta}$ are torque resultants; I_0, I_1, and I_2 are moments of inertia; and Q_r and Q_θ are shear stress resultants. These parameters are obtained by the following relations:

$$\begin{Bmatrix}N_{rr}\\N_{\theta\theta}\\N_{r\theta}\\M_{rr}\\M_{\theta\theta}\\M_{r\theta}\end{Bmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\begin{Bmatrix}\sigma_{rr}\\\sigma_{\theta\theta}\\\sigma_{r\theta}\\z\sigma_{rr}\\z\sigma_{\theta\theta}\\z\sigma_{r\theta}\end{Bmatrix}dz,$$

$$\begin{Bmatrix}I_0\\I_1\\I_2\end{Bmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\begin{Bmatrix}1\\z\\z^2\end{Bmatrix}\rho dz,$$

$$\begin{Bmatrix}Q_r\\Q_\theta\end{Bmatrix}=\overline{K}\int_{-\frac{h}{2}}^{\frac{h}{2}}\begin{Bmatrix}\sigma_{rz}\\\sigma_{\theta z}\end{Bmatrix}dz, \quad (9)$$

where $\overline{K}$ is the shear correction factor.

To obtain the equations of motion in terms of displacement, Hooke's law is used, which gives:

$$\begin{aligned}&A_{11}\left\{(u_{,rr}+w_{,r}w_{,rr})+\frac{1}{r}\left(u_{,r}+\frac{1}{2}w_{,r}{}^2\right)\right\}\\&+A_{12}\left\{\frac{1}{r}\left(v_{,\theta r}-\frac{1}{2}w_{,r}{}^2\right)+\frac{1}{r^3}\left(rw_{,\theta}w_{,\theta r}-\frac{1}{2}w_{,\theta}{}^2\right)\right\}\\&+A_{16}\left\{\frac{1}{r}(2u_{,r\theta}+rv_{,rr}+w_{,\theta}w_{,rr}+2w_{,r}w_{,\theta r})\right\}\\&+A_{22}\left\{\frac{-1}{r^2}\left(v_{,\theta}+u+\frac{1}{2r}w_{,\theta}{}^2\right)\right\}\\&+A_{26}\left\{\frac{1}{r^2}\left(v_{,\theta\theta}-rv_{,r}+v+\frac{1}{r}w_{,\theta}w_{,\theta\theta}-w_{,\theta}w_{,r}\right)\right\}\\&+A_{66}\left\{\frac{1}{r^2}(rv_{,r\theta}+u_{,\theta\theta}-v_{,\theta}+w_{,r}w_{,\theta\theta}+w_{,\theta}w_{,\theta r})\right\}\\&+B_{11}\left\{\varphi_{r,rr}+\frac{1}{r}\varphi_{r,r}\right\}+B_{12}\left\{\frac{1}{r}\varphi_{\theta,\theta r}\right\}\\&+B_{16}\left\{\varphi_{\theta,rr}+\frac{2}{r}\varphi_{r,r\theta}\right\}+B_{22}\left\{-\frac{1}{r^2}(\varphi_{\theta,\theta}+\varphi_r)\right\}\\&+B_{26}\left\{\frac{1}{r^2}(\varphi_{\theta,\theta\theta}+\varphi_\theta)-\frac{1}{r}\varphi_{\theta,r}\right\}\\&+B_{66}\left\{\frac{1}{r^2}(\varphi_{r,\theta\theta}-\varphi_{\theta,\theta})+\frac{1}{r}\varphi_{\theta,r\theta}\right\}\\&=I_0u_{,tt}+I_1\varphi_{r,tt},\end{aligned} \quad (10)$$

$$\begin{aligned}&A_{12}\left\{\frac{1}{r}(u_{,r\theta}+w_{,r}w_{,r\theta})\right\}\\&+A_{16}\left\{\frac{2}{r}\left(u_{,r}+\frac{1}{2}w_{,r}{}^2\right)+(u_{,rr}+w_{,r}w_{,rr})\right\}\\&+A_{22}\left\{\frac{1}{r^2}(v_{,\theta\theta}+u_{,\theta})+\frac{1}{r^3}w_{,\theta}w_{,\theta\theta}\right\}\\&+A_{26}\left\{\frac{1}{r^2}(u+u_{,\theta\theta}+w_{,r}w_{,\theta\theta}+2w_{,\theta}w_{,r\theta})\right.\\&\left.+\frac{1}{r}(u_{,r}+2v_{,r\theta})\right\}+A_{66}\left\{\frac{1}{r^2}(u_{,\theta}-v+w_{,r}w_{,\theta})\right.\\&\left.+\frac{1}{r}(v_{,r}+u_{,\theta r}+rv_{,rr}+w_{,rr}w_{,\theta}+w_{,r}w_{,r\theta})\right\}\\&+B_{12}\left\{\frac{1}{r}\varphi_{r,r\theta}\right\}+B_{16}\left\{\varphi_{r,rr}+\frac{2}{r}\varphi_{r,r}\right\}\\&+B_{22}\left\{\frac{1}{r^2}(\varphi_{\theta,\theta\theta}+\varphi_{r,\theta})\right\}\\&+B_{26}\left\{\frac{1}{r^2}(\varphi_r+\varphi_{r,\theta\theta})+\frac{1}{r}(\varphi_{r,r}+2\varphi_{\theta,\theta r})\right\}\end{aligned}$$

$$+ B_{66}\left\{\frac{1}{r^2}(\varphi_{r,\theta} - \varphi_\theta) + \frac{1}{r}(\varphi_{r,\theta r} + \varphi_{\theta,r}) + \varphi_{\theta,rr}\right\} = I_0 v_{,tt} + I_1\varphi_{\theta,tt}, \quad (11)$$

$$A_{11}\left\{\frac{1}{r}\left(rw_{,r}(u_{,r} + \frac{1}{2}w_{,r}{}^2)\right)_{,r}\right\} + A_{12}\left\{\frac{1}{r}\left(w_{,r}(u + v_{,\theta} + \frac{1}{2r}w_{,\theta}{}^2)\right)_{,r} + \frac{1}{r^2}\left(w_{,\theta}(u_{,r} + \frac{1}{2}w_{,r}{}^2)\right)_{,\theta}\right\}$$
$$+ A_{45}\left\{\varphi_{\theta,r} + \frac{1}{r}(\varphi_\theta + \varphi_{r,\theta} + 2w_{,r\theta})\right\} + A_{22}\left\{\frac{1}{r^3}\left(w_{,\theta}(u + v_{,\theta} + \frac{1}{2r}w_{,\theta}{}^2)\right)_{,\theta}\right\}$$
$$+ A_{16}\left\{\frac{1}{r}\left(w_{,r}(u_{,\theta} + rv_{,r} - v + w_{,\theta}w_{,r})\right)_{,r} + \frac{1}{r}\left(w_{,\theta}(u_{,r} + \frac{1}{2}w_{,r}{}^2)\right)_{,r} + \frac{1}{r}\left(w_{,r}(u_{,r} + \frac{1}{2}w_{,r}{}^2)\right)_{,\theta}\right\}$$
$$+ A_{26}\left\{\frac{1}{r^3}\left(w_{,\theta}(u_{,\theta} + rv_{,r} - v + w_{,\theta}w_{,r})\right)_{,\theta} + \frac{1}{r^2}\left(w_{,r}(u + v_{,\theta} + \frac{1}{2r}w_{,\theta}{}^2)\right)_{,\theta} + \frac{1}{r}\left(\frac{w_{,\theta}}{r}(u + v_{,\theta} + \frac{1}{2r}w_{,\theta}{}^2)\right)_{,r}\right\}$$
$$+ A_{66}\left\{\frac{1}{r}\left(\frac{w_{,\theta}}{r}(u_{,\theta} + rv_{,r} - v + w_{,\theta}w_{,r})\right)_{,r} + \frac{1}{r^2}\left(w_{,r}(u_{,\theta} + rv_{,r} - v + w_{,\theta}w_{,r})\right)_{,\theta}\right\}$$
$$+ A_{55}\left\{\varphi_{r,r} + w_{,rr} + \frac{1}{r}(\varphi_r + w_{,r})\right\} + B_{11}\left\{\frac{1}{r}\left(r(w_{,r}\varphi_{r,r})\right)_{,r}\right\}$$
$$+ B_{12}\left\{\frac{1}{r}\left(w_{,r}(\varphi_r + \varphi_{\theta\theta})\right)_{,r} + \frac{1}{r^2}(w_{,\theta}\varphi_{r,r})_{,\theta}\right\}$$
$$+ B_{16}\left\{\frac{1}{r}\left(w_{,r}(\varphi_{r,\theta} - \varphi_\theta)\right)_{,r} + \frac{1}{r}\left(r(w_{,r}\varphi_{\theta,r})\right)_{,r} + \frac{1}{r}(w_{,\theta}\varphi_{r,r})_{,r} + \frac{1}{r}(w_{,r}\varphi_{r,r})_{,\theta}\right\}$$
$$+ B_{22}\left\{\frac{1}{r^3}\left(w_{,\theta}(\varphi_r + \varphi_{\theta,\theta})\right)_{,\theta}\right\}$$
$$+ B_{26}\left\{\frac{1}{r^3}\left(w_{,\theta}(\varphi_{r,\theta} - \varphi_\theta)\right)_{,\theta} + \frac{1}{r^2}(w_{,\theta}\varphi_{\theta,r})_{,\theta} + \frac{1}{r^2}\left(w_{,r}(\varphi_r + \varphi_{\theta,\theta})\right)_{,\theta} + \frac{1}{r}\left(\frac{w_{,\theta}}{r}(\varphi_r + \varphi_{\theta\theta})\right)_{,r}\right\}$$
$$+ B_{66}\left\{\frac{1}{r}(w_{,\theta}\varphi_{\theta,r}) + \frac{1}{r}(w_{,r}\varphi_{\theta,r})_{,\theta} + \frac{1}{r^2}\left(w_{,\theta}(\varphi_{r,\theta} - \varphi_\theta)\right)_{,\theta} + \frac{1}{r}\left(\frac{w_{,\theta}}{r}(\varphi_{r,\theta} - \varphi_\theta)\right)_{,r}\right\}$$
$$+ A_{44}\left\{\frac{1}{r}\left(\varphi_{\theta,\theta} + \frac{1}{r}w_{,\theta\theta}\right)\right\} + q = I_0 w_{,tt}, \quad (12)$$

$$B_{11}\left\{(u_{,rr} + w_{,r}w_{,rr}) + \frac{1}{r}\left(u_{,r} + \frac{1}{2}w_{,r}{}^2\right)\right\} + B_{12}\left\{\frac{1}{r}\left(v_{,\theta r} - \frac{1}{2}w_{,r}{}^2\right) + \frac{1}{r^3}\left(rw_{,\theta}w_{,\theta r} - \frac{1}{2}w_{,\theta}{}^2\right)\right\}$$
$$+ B_{16}\left\{\frac{1}{r}(2u_{,r\theta} + rv_{,rr} + w_{,\theta}w_{,rr} + 2w_{,r}w_{,\theta r})\right\} + B_{22}\left\{\frac{-1}{r^2}\left(v_{,\theta} + u + \frac{1}{2r}w_{,\theta}{}^2\right)\right\}$$
$$+ B_{26}\left\{\frac{1}{r^2}\left(v_{,\theta\theta} - rv_{,r} + v + \frac{1}{r}w_{,\theta}w_{,\theta\theta} - w_{,\theta}w_{,r}\right)\right\} + D_{66}\left\{\frac{1}{r^2}(\varphi_{r,\theta\theta} - \varphi_{\theta,\theta}) + \frac{1}{r}\varphi_{\theta,r\theta}\right\}$$
$$+ B_{66}\left\{\frac{1}{r^2}(rv_{,r\theta} + u_{,\theta\theta} - v_{,\theta} + w_{,r}w_{,\theta\theta} + w_{,\theta}w_{,\theta r})\right\}$$
$$+ D_{11}\left\{\varphi_{r,rr} + \frac{1}{r}\varphi_{r,r}\right\} + D_{12}\left\{\frac{1}{r}\varphi_{\theta,\theta r}\right\} + D_{16}\left\{\varphi_{\theta,rr} + \frac{2}{r}\varphi_{r,r\theta}\right\}$$
$$+ D_{22}\left\{-\frac{1}{r^2}(\varphi_{\theta,\theta} + \varphi_r)\right\} + D_{26}\left\{\frac{1}{r^2}(\varphi_{\theta,\theta\theta} + \varphi_\theta) - \frac{1}{r}\varphi_{\theta,r}\right\}$$
$$- A_{55}\left\{\varphi_r + w_{,r}\right\} + A_{45}\left\{\varphi_\theta + \frac{1}{r}w_{,\theta}\right\}$$

$$= I_1 u_{,tt} + I_2 \varphi_{r,tt}, \tag{13}$$

$$B_{12}\left\{\frac{1}{r}(u_{,r\theta}+w_{,r}w_{,r\theta})\right\} + B_{16}\left\{\frac{2}{r}\left(u_{,r}+\frac{1}{2}{w_{,r}}^2\right)+(u_{,rr}+w_{,r}w_{,rr})\right\} + B_{22}\left\{\frac{1}{r^2}(v_{,\theta\theta}+u_{,\theta})+\frac{1}{r^3}w_{,\theta}w_{,\theta\theta}\right\} - A_{44}\left\{\varphi_\theta+\frac{1}{r}w_{,\theta}\right\} - A_{45}\left\{\varphi_r+w_{,r}\right\} + D_{22}\left\{\frac{1}{r^2}(\varphi_{\theta,\theta\theta}+\varphi_{r,\theta})\right\} + B_{26}\left\{\frac{1}{r^2}(u+u_{,\theta\theta}+w_{,r}w_{,\theta\theta}+2w_{,\theta}w_{,r\theta}) + \frac{1}{r}(u_{,r}+2v_{,r\theta})\right\} + B_{66}\left\{\frac{1}{r^2}(u_{,\theta}-v+w_{,r}w_{,\theta}) + \frac{1}{r}(v_{,r}+u_{,\theta r}+rv_{,rr}+w_{,rr}w_{,\theta}+w_{,r}w_{,r\theta})\right\} + D_{12}\left\{\frac{1}{r}\varphi_{r,r\theta}\right\} + D_{16}\left\{\varphi_{r,rr}+\frac{2}{r}\varphi_{r,r}\right\} + D_{26}\left\{\frac{1}{r^2}(\varphi_r+\varphi_{r,\theta\theta})+\frac{1}{r}(\varphi_{r,r}+2\varphi_{\theta,\theta r})\right\} + D_{66}\left\{\frac{1}{r^2}(\varphi_{r,\theta}-\varphi_\theta)+\frac{1}{r}(\varphi_{r,\theta r}+\varphi_{\theta,r})+\varphi_{\theta,rr}\right\}$$

$$= I_1 v_{,tt} + I_2 \varphi_{\theta,tt}, \tag{14}$$

in which A_{ij} are extensional stiffnesses; B_{ij} are bending-extension coupling stiffnesses; and D_{ij} are bending stiffnesses [1].

Eqs. (10) - (14) are the equations of a motion of an arbitrary circular laminated plate. We can ignore derivatives with respect to θ, due to the existence of axi-symmetry in the first mode of motion. For a symmetric ($B_{ij}=0$) cross-ply ($D_{26}=D_{16}=A_{26}=A_{16}=A_{45}=0$) lay-up, by neglecting the in-plane inertia terms ($u_{,tt}=v_{,tt}=0$) [41], and assuming ρ as an even function of thickness, z (i.e., $I_1=0$), the equations of motions are reduced to:

$$A_{11}\left(\frac{1}{r}\left(\frac{r}{2}(w_{,r})^3\right)_{,r}\right) + A_{55}\left(\varphi_{r,r}+w_{,rr}+\frac{1}{r}(\varphi_r+w_{,r})\right) = I_0 w_{,tt}, \tag{15}$$

$$D_{11}\left(\varphi_{r,rr}+\frac{1}{r}\varphi_{r,r}\right) - \left(\frac{D_{22}}{r^2}+A_{55}\right)\varphi_r - A_{55}w_{,r} = I_2\varphi_{r,tt}, \tag{16}$$

$$D_{66}\left(\frac{-1}{r^2}\varphi_\theta+\frac{1}{r}\varphi_{\theta,r}+\varphi_{\theta,rr}\right) - A_{44}\varphi_\theta = I_2\varphi_{\theta,tt}. \tag{17}$$

Eq. (17) can be solved individually by separation of variables. Substitution of $\varphi_\theta = p(t)W(r)$ into this equation gives:

$$\varphi_\theta(r,t) = a_n \cos\left(\frac{\lambda t}{\sqrt{I_2}}+\psi\right) J_1\left(\sqrt{\frac{(\lambda^2-A_{44})}{D_{66}}}r\right), \tag{18}$$

where λ is the frequency parameter and obtained by using the boundary condition; $J_1()$ is the Bessel function of order one; and a_n and ψ are found by using the initial conditions. Since the differential equation of Eq. (17) is decoupled from the two coupled equations of motion (i.e., Eqs. (15) and (16)), the natural frequency of $\lambda/\sqrt{I_2}$ is not related to the natural frequency of the first mode of the plate. That is, rotation of the plate about the r-axis (φ_θ) oscillates on its surface as an independent degree of freedom.

The Galerkin method ($\iint_A L_{(r,t)}.g_{(r)}.rdr = 0$) is used to convert the PDEs of motion to a single ODE, where $L_{(r,t)}$ is the differential equation that is to be solved, and $g_{(r)}$ is an unknown spatial function, which should be assumed in a way that satisfies the boundary conditions ($w_{,r}(R,t)=0, w(R,t)=0$) and the mode shape of the plate.

Finding φ_r from Eq. (15) and substituting it into Eq. (16) gives $L_{(r,t)}$:

$$L_{(r,t)} = D_{11}\left[\left\{-w_{,r}+\frac{1}{A_{55}}\left(\frac{-A_{11}}{2}{w_{,r}}^3+\frac{I_0}{r}\int_0^r w_{,tt}\mathfrak{R}d\mathfrak{R}\right)\right\}_{,rr} + \frac{1}{r}\left\{-w_{,r}+\frac{1}{A_{55}}\left(\frac{-A_{11}}{2}{w_{,r}}^3 + \frac{I_0}{r}\int_0^r w_{,tt}\mathfrak{R}d\mathfrak{R}\right)\right\}_{,r}\right] - \left(\frac{D_{22}}{r^2}+A_{55}\right) \left\{-w_{,r}+\frac{1}{A_{55}}\left(\frac{-A_{11}}{2}{w_{,r}}^3+\frac{I_0}{r}\int_0^r w_{,tt}\mathfrak{R}d\mathfrak{R}\right)\right\} - A_{55}w_{,r} - I_2\left\{-w_{,r}+\frac{1}{A_{55}}\left(\frac{-A_{11}}{2}{w_{,r}}^3 + \frac{I_0}{r}\int_0^r w_{,tt}\mathfrak{R}d\mathfrak{R}\right)\right\}_{,tt}, \tag{19}$$

in which $\mathfrak{R}$ is used for definite integration. The only unknown parameter in Eq. (19) is w. Mbakogu and

Pavlovic [14] used the following value for w to obtain the natural frequency of the first mode, which is also used in this study. However, the rotation is proposed for the first time:

$$w(r,t)=h\left\{\frac{3-K-4\left(\frac{r}{R}\right)^{1+K}+(1+K)\left(\frac{r}{R}\right)^{4}}{(9-K^2)(1+K)}\right\}f(t)$$

$$\varphi_r(r,t)=g(r)f(t)=\left(1-\left(\frac{r}{R}\right)^{m}\right)f(t), \tag{20}$$

where $K=\sqrt{E_\theta/E_r}$ is the degree of orthotropy, and m is an unknown parameter. Performing the Galerkin method, the following nonlinear ODE is obtained:

$$Z_1\ddot{f}(t)+Z_2f(t)+Z_3f(t)^3+Z_4f(t)^2\ddot{f}(t)+Z_5f(t)\dot{f}(t)^2=0, \tag{21}$$

where the constant coefficients of $Z_i(i=1,...,5)$ are functions of material and geometric properties of the plate, and the natural frequency of the plate is simply obtained by $\overline{\omega}=\sqrt{Z_2/Z_1}$.

Introducing dimensionless time in the form of $\tau=t/\vartheta$, where $\vartheta=R^2\sqrt{I_0/D_{11}}$, Eq. (21) is transformed into the following dimensionless form:

$$f_{,\tau\tau}+\omega^2f+\alpha^2f^3+\beta^2f_{,\tau\tau}f^2+\gamma^2f_{,\tau}^2f=0, \tag{22}$$

where:

$$\omega^2=\frac{Z_2}{Z_1}\vartheta^2=\overline{\omega}^2\vartheta^2, \qquad \alpha^2=\frac{Z_3}{Z_1}\vartheta^2,$$

$$\beta^2=\frac{Z_4}{Z_1}, \qquad \gamma^2=\frac{Z_5}{Z_1}=2\beta^2. \tag{23}$$

In order to find the unknown parameter of m in Eq. (20), m is found for different values of K and h/R in such a way that the best approximation for the natural frequency is obtained. Then, the regression between these optimum points gives the value of m, in terms of K and h/R, which is $m=\left(5.5-22.0452\frac{h}{2}\right)(0.232.0.7543K)$.

3. Implementation of the multiple time scales method

To solve Eq. (22) by the method of multiple time scales, the nonlinear terms of this equation are multiplied by a positive dimensionless parameter denoted by ε which will be equated to unity after obtaining the nonlinear frequency [42]. So:

$$f_{,\tau\tau}+\omega^2f+\varepsilon\left(\alpha^2f^3+\beta^2f_{,\tau\tau}f^2+\gamma^2f_{,\tau}^2f\right)=0. \tag{24}$$

In this method, the independent time variables are $T_n=\tau\varepsilon^n(n=0,1,2,...)$, and the solution of Eq. (24) can be written as $f(\tau,\epsilon)=\sum_{n=0}^{i}\varepsilon^nf_n(T_0,T_1,...,T_i)$. In this study, it is considered that $i=2$. So, using the chain rule, the differentials of Eq. (24) can be rewritten, with respect to T_n, as:

$$\frac{\partial}{\partial\tau}=D_0+\epsilon D_1+\varepsilon^2D_2,$$

$$\frac{\partial^2}{\partial\tau^2}=D_0^2+2\varepsilon D_0D_1+\varepsilon^2(D_1^2+2D_0D_2). \tag{25}$$

Substitution of Eq. (25), along with the assumed $f(\tau,\epsilon)$, into Eq. (24), and equating the like powers of ε with each other, gives:

$$D_0^2f_0+\omega^2f_0=0, \tag{26}$$

$$D_0^2f_1+\omega^2f_1=-2D_0D_1f_0-\alpha^2f_0^3-\beta^2f_0^2(D_0^2f_0)-\gamma^2f_0(D_0f_0)^2, \tag{27}$$

$$D_0^2f_2+\omega^2f_2=-\left(D_1^2+2D_0D_2\right)f_0-2D_0D_1f_1-3\alpha^2f_0^2f_1-\beta^2\left\{2f_0^2(D_0D_1f_0)+f_0^2(D_0^2f_1)+2f_0f_1(D_0^2f_0)\right\}-\gamma^2\left\{f_1(D_0f_0)^2+2f_0(D_0f_0)(D_0f_1)+2f_0(D_0f_0)(D_1f_0)\right\}. \tag{28}$$

The solution of Eq. (26) is $f_0=A_1(T_1,T_2)\exp(i\omega T_0)+cc$, where A_1 is an unknown complex function of T_1 and T_2, which is to be found, and cc denotes the complex conjugate of the preceding terms. By substituting f_0 into Eq. (27) and noting that $\gamma^2=2\beta^2$, the following is obtained:

$$D_0^2f_1+\omega^2f_1=\left(-2i\omega D_1A_1-3A_1^2\overline{A}_1\alpha^2+\omega^2A_1^2\overline{A}_1\beta^2\right)\exp(i\omega T_0)+\left(3\omega^2\beta^2-\alpha^2\right)A_1^3\exp(3i\omega T_0)+cc, \tag{29}$$

in which $\overline{A}_1$ is the complex conjugate of A_1. To have a periodic solution for f_1, the coefficient of $\exp(i\omega T_0)$ must be equated to zero, which gives:

$$-2i\omega D_1A_1-3A_1^2\overline{A}_1\alpha^2+\omega^2A_1^2\overline{A}_1\beta^2=0. \tag{30}$$

So, Eq. (29) gives $f_1=X\exp(3i\omega T_0)+cc$, where $X=\frac{1}{8\omega^2}(\alpha^2-3\omega^2\beta^2)A_1^3$. Substitution of f_0 and f_1 into Eq. (28) gives:

$$D_0{}^2f_2+\omega^2f_2=\left\{(\omega^2\beta^2-3\alpha^2)X\overline{A}_1{}^2-D_1^2A_1\right.$$

$$+i\omega(-2D_2A_1-4\beta^2A_1\overline{A}_1D_1A_1-2\beta^2{A_1}^2D_1\overline{A}_1)\Big\}$$

$$\exp(i\omega T_0)-\Big\{6\alpha^2A_1\overline{A}_1X-6i\omega D_1X$$

$$+\beta^2(-18\omega^2XA_1\overline{A}_1+6i\omega{A_1}^2D_1A_1$$

$$-2i\omega{A_1}^2D_1\overline{A}_1)\Big\}\exp(3i\omega T_0)+\Big\{-3\alpha^2{A_1}^2X$$

$$+25\beta^2\omega^2X{A_1}^2\Big\}\exp(5i\omega T_0)+cc. \tag{31}$$

To have a periodic solution for Eq. (31), again, the coefficient of $(i\omega T_0)$ is equated to zero, i.e.:

$$\Big\{(\omega^2\beta^2-3\alpha^2)(\alpha^2-3\omega^2\beta^2)A_1^3\overline{A}_1^2/(8\omega^2)\Big\}$$

$$-D_1^2A_1+i\omega\Big(-2D_2A_1-4\beta^2A_1\overline{A}_1D_1A_1$$

$$-2\beta^2A_1^2D_1\overline{A}_1\Big)=0. \tag{32}$$

Assuming $A_1(T_1,T_2)=\frac{1}{2}p(T_1,T_2)\exp(iq(T_1,T_2))$, where p and q are real functions of T_1 and T_2, and substituting it into Eqs. (30) and (32), gives the following differential equations:

$$\omega\frac{\partial p}{\partial T_1}=0,\qquad \omega\frac{\partial p}{\partial T_2}=0,$$

$$p\omega\frac{\partial q}{\partial T_1}+\frac{1}{8\omega^2}(\omega^2\beta^2-3\alpha^2)p^3=0,$$

$$\omega p\frac{\partial q}{\partial T_2}=-\left[\frac{(\omega^2\beta^2-3\alpha^2)(\alpha^2-3\omega^2\beta^2)}{256\omega^2}+\frac{(\omega^2\beta^2-3\alpha^2)^2}{128\omega^2}+\frac{(\omega^2\beta^2-3\alpha^2)\beta^2}{16}\right]p^5. \tag{33}$$

Solving these equations gives:

$$p=p_0,$$

$$q=\frac{{p_0}^2}{16\omega}\left\{2q_1\epsilon+\left(\frac{q_2}{2}+\frac{q_3}{8}+q_4\right)p_0^2\epsilon^2\right\}\tau+q_0, \tag{34}$$

in which:

$$q_1=3\alpha^2-\omega^2\beta^2,\qquad q_2=q_1(\alpha^2-3\omega^2\beta^2)\Big/(8\omega^2),$$

$$q_3=-{q_1}^2/\omega^2,\qquad q_4=-\beta^2q_1.$$

By solving Eq. (31), it is found that:

$$f_2=\frac{Y_1}{8\omega^2}{p_0}^2{A_1}^3\exp(3i\omega T_0)+\frac{Y_2}{24\omega^2}{A_1}^5\exp(5i\omega T_0)+cc,$$

where:

$$Y_1=\frac{3}{32\omega^2}(9\omega^4\beta^4-\alpha^4)-\frac{3}{4}\beta^2q_1,$$

$$Y_2=\frac{1}{8\omega^2}(\alpha^2-3\omega^2\beta^2)(3\alpha^2-256\omega^2\beta^2).$$

So, substitution of f_0, f_1, and f_2 into the assumed $f(\tau,\epsilon)=\sum_{n=0}^{2}\varepsilon^nf_n(T_0,T_1,T_2)$ gives:

$$f=p_0\cos\cos(\omega_{nl}\tau+q_0)+{p_0}^3Y_3\cos\cos(3\omega_{nl}\tau+3q_0)+{p_0}^5\Big\{Y_4\cos\cos(3\omega_{nl}\tau+3q_0)+Y_5\cos\cos(5\omega_{nl}\tau+5q_0)\Big\}, \tag{35}$$

where $Y_3=\frac{\alpha^2-3\omega^2\beta^2}{32\omega^2}$, $Y_4=\frac{Y_1}{32\omega^2}$ and $Y_5=\frac{Y_2}{384\omega^2}$. In this equation, ω_{nl} is the nonlinear frequency and is a function of the initial dimensionless amplitude (i.e., $p_0\equiv W_{\max}/h$), which is obtained to be:

$$\omega_{nl}=\omega+\frac{{p_0}^2}{16\omega}\left\{2q_1+\left(\frac{q_2}{2}+\frac{q_3}{8}+q_4\right){p_0}^2\right\}, \tag{36}$$

and the nonlinear frequency ratio can be written in the following form:

$$\frac{\omega_{nl}}{\omega}=\sqrt{\left(1+\frac{q_1{p_0}^2}{4\omega^2}+\frac{1}{8\omega^2}\Big(\frac{{q_1}^2}{8\omega^2}+\frac{q_2}{2}+\frac{q_3}{8}+q_4\Big){p_0}^4\right)}. \tag{37}$$

4. Results and discussion

4.1. Linear vibration

Determination of the shear correction factor ($\overline{K}$) for laminated structures is still an unresolved issue. The effect of the shear correction factor is to decrease the frequencies. That is, the smaller $\overline{K}$ is, the smaller are the frequencies [1]. In most situations, in particular moderately thick ($a/h\geq 10$) laminated plates, the classical shear correction factor, $\overline{K}=5/6$, provides results that are rather accurate [43]. So, in the present work, the shear correction factor is taken to be 5/6.

Table 1 gives the material properties of the studied plates. The dimensionless natural frequency of a clamped isotropic plate is shown in Table 2. It is seen that there is good agreement between the results of the present study and the results of highly accurate HSDT.

Table 1. Material properties used in this study [30].

Property	Aluminum alloy	Steel	Glass Fiber Reinforced Composite (GFRC)
E_1 (GPa)	72.4	200	55.8979
E_2 (GPa)	72.4	200	13.7293
G_{12} (GPa)	28	79	5.5898
G_{13} (GPa)	28	79	5.5898
G_{23} (GPa)	28	79	4.9033
ρ (kg/m^3)	2700	7860	2550
v	0.3333	0.2666	0.2777

The effect of degree of orthotropy (K) on the dimensionless natural frequency of circular plates is investigated, and the results are shown for a one-layered orthotropic plate in Table 3, which shows that higher degrees of orthotropy occur at higher dimensionless frequencies.

The dimensionless natural frequency of a five-layered Glare 3 hybrid plate is compared with isotropic and laminated plates and shown in Table 4. The lay-up of a five-layered Glare 3 is Al(2024 - T3)/[0°/90°] GFRC/Al(2024-T3) /[90°/0°] GFRC/ Al(2024-T3). The total thicknesses of all layers are the same. In Glare 3, each aluminum sheet has 3 mm of thickness, and the thickness of each fiber reinforced layer is 2.25 mm (Figure 2). It is observed that the dimensionless frequency of Glare 3 and four-layered laminated composite plate is almost identical to that of the isotropic plates. This occurs, because in two laminated plates, the degree of orthotropy is unity, i.e., $K = 1$. In three-layered laminated plate, $K < 1$, the dimensionless frequency is smaller than the corresponding value of the isotropic plate, while, for the five-layered laminated plate, $K > 1$. Consequently, the dimensionless frequency is bigger than the isotropic plate frequency. It is also noticed that the thicker the plate is, the lower is the dimensionless frequency. Moreover, increasing the number of layers increases the dimensionless frequency.

The effect of composite layers thickness to total thickness ratio (h_c/h) of a five-layered FML plate on the dimensionless frequency is also investigated, and the results are shown for different thickness-to-radius ratios (h/R) in Table 5. It is seen that for thin plates,

Table 3. Effect of degree of orthotropy on the dimensionless frequency of a clamped orthotropic plate.

K	Method		
	Series solution*	Rayleigh [14]	Present study
0.2	7.7721	7.8622	7.700
0.4	8.3847	8.4785	8.3448
0.6	8.9975	9.0948	8.888
0.8	9.6107	9.7113	9.604
1	10.2244	10.328	10.225
1.2	10.8385	10.9449	10.842
1.4	11.4532	11.5622	11.4566
1.6	12.0684	12.1799	12.069
1.8	12.6842	12.798	12.68
2	13.3005	13.4146	13.289
2.2	13.9174	14.0352	13.899
2.4	14.5348	14.6545	14.508
2.6	15.1527	15.2741	15.11
2.8	15.7712	15.8941	15.72

* From reference [14].

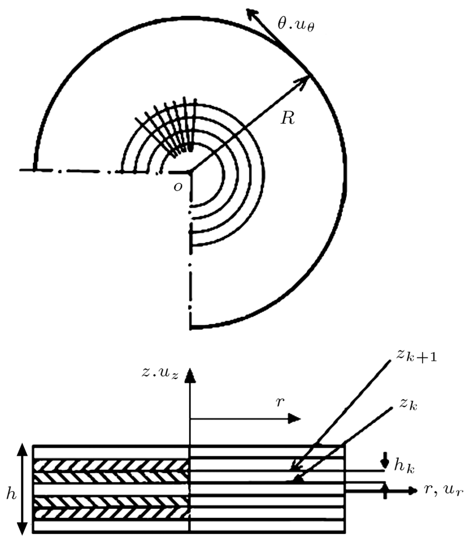

Figure 2. A circular symmetric fiber metal laminated composite plates and it's related number of lamina.

h_c/h ratio has, in fact, no effect on the dimensionless frequency, and its effect on thick plates is negligible, too. In fact, for a specified plate, the natural frequency is a function of its geometrical and material properties. As shown in Figures 3 and 4, change in orthotropic

Table 2. Dimensionless natural frequency of a clamped aluminum plate.

Method	h/R						
	0.001	0.01	0.05	0.1	0.15	0.2	0.25
HSDT [9]	10.2157	10.213	10.1459	9.9461	9.6419	9.2650	8.8463
Present	10.225	10.2142	10.1200	9.9315	9.6572	9.2926	8.8266
Error %	0.136	0.029	0.213	0.112	0.181	0.297	0.265

Table 4. Effect of thickness-to-radius ratio and number of layers on the dimensionless frequency of several plates.

Plate	h/R						
	0.001	0.01	0.02	0.04	0.06	0.08	0.1
One-layered steel	10.225	10.2115	10.1936	10.1486	10.0911	10.183	9.9374
One-layered aluminum	10.225	10.2142	10.1934	10.1476	10.089	10.016	9.9315
Five-layered Glare 3	10.225	10.2127	10.1984	10.1676	10.1338	10.0964	10.055
Three-layered [0°/90°/0°]GFRC	9.6506	9.6384	9.6376	9.5911	9.5538	9.5116	9.46417
Four-layered [0°/90°/90°/0°]GFRC	10.2250	10.2126	10.1978	10.1664	10.1289	10.0876	10.0413
Five-layered [90°/0°/90°/0°/90°] GFRC	10.6248	10.6123	10.6013	10.566	10.5291	10.4886	10.4428

Table 5. Effects of h/R and h_c/h ratios on the dimensionless frequency of a five-layered FML plate.

h_c/h	h/R					
	0.001	0.01	0.05	0.1	0.15	0.2
0.25	10.229	10.217	10.154	10.055	9.925	9.749
0.5	10.229	10.217	10.155	10.058	9.932	9.762
0.75	10.229	10.217	10.155	10.056	9.927	9.752

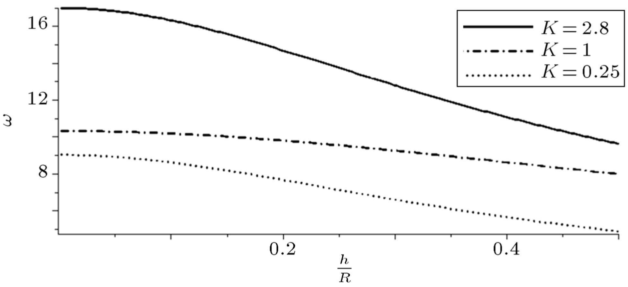

Figure 3. Effects of degree of orthotropy and thickness-to-radius ratio on the dimensionless natural frequency.

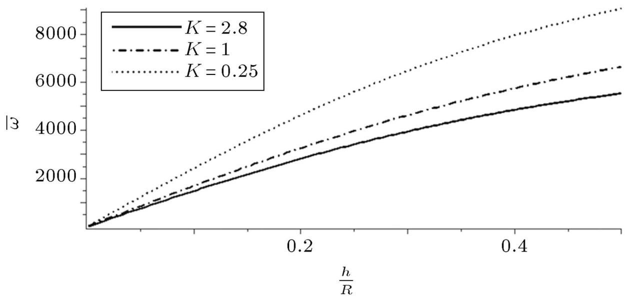

Figure 4. Effects of degree of orthotropy and thickness-to-radius ratio on the circular natural frequency.

ratio, K, results in the changing of natural frequency. However, in a symmetric FML plate, the variation of total orthotropic ratio ($\sqrt{D_{22}/D_{11}}$) is approximately negligible, while the h_c/h ratio changes. Hence, natural frequency is a weak function of this ratio.

Figures 3 and 4 show the effects of degree of orthotropy and ratio on the dimensionless and circular frequencies, respectively. It should be noticed that the inclusion of a non-dimensionalizing parameter, i.e., $\vartheta = R^2\sqrt{I_0/D_{11}}$, in the dimensionless frequency makes the slope of the curves negative, whereas the slopes of circular frequency curves are positive.

4.2. Nonlinear vibration

In the present study, nonlinearity is because of large amplitudes of vibration and the system is of a hardening type. On the other hand, natural frequency can be introduced as the ratio of inner driving forces to restoring forces. For example, for a one Degree Of Freedom (DOF) linear mass-spring system, the natural frequency is $\sqrt{k/m}$, where the spring stiffness (k) represents the driving force, and the mass of the system (m) represents the inertia forces, which resist motion. The natural frequency of a $\sqrt{k/m}$ one DOF linear mass-spring system is always $\sqrt{k/m}$, because the driving and restoring forces do not change during the vibration of the system. However, this not true for nonlinear systems. Considering Figure 5, it is seen that increasing the transverse displacement (w) results in vertical components of normal stress, σ_x. So, in the nonlinear case, the driving force is the sum of vertical stresses and the vertical components of in-plane stresses, whereas the restoring force, which results from the mass of the system, does not change. That is why the nonlinear frequency is a function of the amplitude of motion, and increasing the amplitude of vibration increases the nonlinear frequency of the plate. This can be seen in Table 6, which gives the nonlinear frequency ratios of an isotropic plate. On the other hand, the effect of both h/R and w/h ratios on the nonlinear frequency of the Glare 3 plate is investigated in Table 7. It can be seen that nonlinear frequency is approximately constant for a specified w/h.

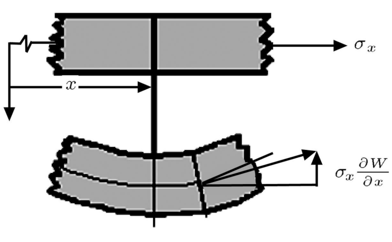

Figure 5. The normal stress (σ_x) has a vertical component because of the bending of the plate.

Table 6. Nonlinear frequency ratios of a circular aluminum plate ($h/R = 0.02$).

w/h	Method		
	Haterbouch and Benamar [25]	**Yamaki***	**Present study**
0.2	1.0072	1.0070	1.00813
0.4	1.0284	1.0278	1.03218
0.5	1.0439	1.0431	1.04990
0.6	1.0623	1.0614	1.07175
0.8	1.1073	1.1065	1.12342
1	1.1615	1.1617	1.18661
1.5	1.3255	1.3343	1.37114
2	1.5147	1.5423	1.53849

* From reference [25].

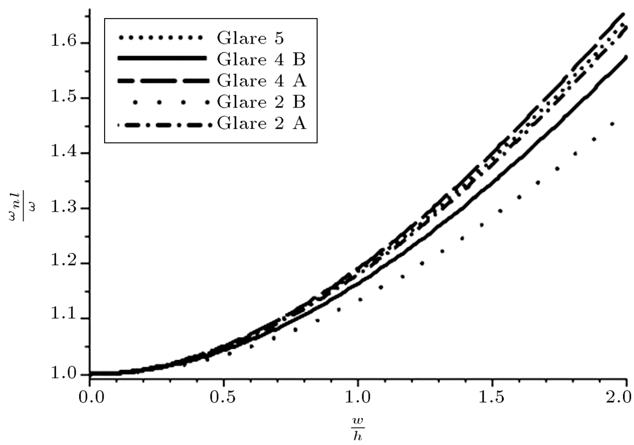

Figure 6. Back-bone curves of several grades of Glare hybrid laminated plate ($h/R = 0.05$).

Back-bone curves of several grades of Glare hybrid plate [44] are also obtained and shown in Figure 6. It is seen that Glare 4A and Glare 2B, respectively, exhibit the most and the least nonlinear behaviour compared with the other grades. The reason is that, regarding the type of layering in these plates [44], for cases where fibers lie in a radial direction (with the angle of [0]), the stiffness of the plate is increased. Figure 7 shows the back-bone curves of five-layered circular Glare 3 plates for different h/R ratios. This figure implicitly represents the effect of w/R ratio on the nonlinear frequency. The further the thickness

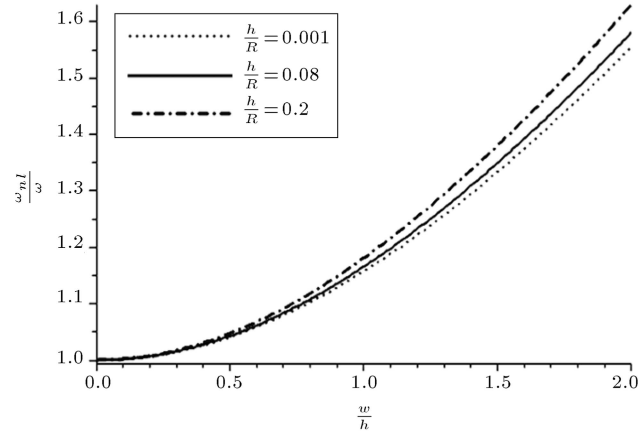

Figure 7. Back-bone curves of five-layered Glare 3 for different thickness-to-radius ratio.

of the plate is increased, for a specified radius, the ratio of w/R increases, too. Thus, greater ratios of h/R correspond to bigger values of amplitude, w. On the other hand, as depicted in Figure 5, the effects of nonlinear terms increase with an increase in plate slope curve (w/R). Consequently, as the plate becomes thicker, the nonlinear effects increase and the back-bone curve bends away more from w/h axis. In fact, the dimensional form of this figure shows that for bigger values of w/R, nonlinear terms are more effective.

5. Conclusion

In this paper, the nonlinear free vibration of Fiber Metal Laminated (FML) plates is investigated. The nonlinear equations of motion for FML circular plates are based on the First order Shear Deformation Theory (FSDT). Using the Galerkin method, along with the method of multiple time scales, analytical solution for nonlinear free vibration of these plates is obtained. A comparison of results with those published reveal that the present procedure can approximate linear and nonlinear frequencies of isotropic and composite circular plates with high precision. So, the presented procedure is extended to study FML plates.

It is observed that in the circular plates, material properties and, especially, the orthotropic ratio, have a significant effect on the mode shape of the plate. It is shown that the nonlinear terms are proportional to

Table 7. Effects of h/R and w/h ratios on the nonlinear frequency of a Glare 3 plate.

w/h	h/R									
	0.001	0.01	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
0.5	1.0413	1.0415	1.0418	1.0425	1.0433	1.0441	1.0449	1.0458	1.0470	1.0477
1	1.1579	1.1587	1.1600	1.1625	1.1652	1.1682	1.1713	1.1746	1.1787	1.1814
1.5	1.3341	1.3357	1.3385	1.3435	1.3489	1.3548	1.3611	1.3678	1.3760	1.3813
2	1.5554	1.5580	1.5625	1.5704	1.5791	1.5814	1.5985	1.6090	1.6221	1.6305

the slope curve of the plate and become more effective while the w/R ratio grows. Since in a FML plate and for symmetric cases, with respect to thickness, the orthotropic ratio approaches one, it is observed that plates with several types of FML and several ratios of h_c/h have almost similar behavior to each other.

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