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Research Note

Controllability of linear fractional stochastic systems

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Abstract. This paper discusses the controllability condition for linear stochastic fractional systems. The definition of fractional stochastic controllability is given. The α-controllability matrix has been presented to derive the required theorems for necessary and sufficient conditions of complete and approximate fractional stochastic controllability. The equivalency of fractional stochastic controllability to fractional controllability is also investigated. An example has been given to examine the effectiveness of the theory.

1. Introduction

Fractional calculus is dealing with integration and derivation of non-integer order [1,2]. It has been used increasingly in variety of fields of sciences and engineering [3-8]. The history of fractional calculus returns back to 18th century in the very basic works of Euler and Lagrange and also systematic studies of Liouville, Riemann and Holmgren in the 19th century. Nowadays this tool is used to model so many systems in variety of fields, such as viscoelastic structures [9], vibrations and suspensions [10], fractional conservation of mass [11] and diffusion wave [12].

On the other hand, the concept of controllability has the key role in control theory. This concept has been fully investigated in deterministic systems [13,14] as well as stochastic systems [15-18]. In the mentioned works for linear and non-linear systems, necessary and sufficient conditions have been presented for some types of controllability. For fractional systems this situation is not satisfactory at all. There are a few numbers of contributions in the deterministic case. Some results on controllability of fractional systems and the rank condition for these systems are discussed in [19], while the controllability and observability of linear discrete-time fractional-order systems are also studied elsewhere [20]. In [21], the robust controllability for interval fractional order linear time invariant systems has been investigated, whereas [22] shows the controllability condition for some classes of linear and nonlinear fractional systems.

There is no significant works in the literature concerning the controllability condition in stochastic fractional systems. In the present paper, for the first time, a systematic investigation on complete and approximate controllability of linear stochastic fractional systems is presented as a generalization for deterministic case. First, some notations and definitions have been generalized from stochastic non-fractional systems via Mittag-Leffler matrix functions and then the concepts of complete and approximate controllability have been introduced. We then present some theorems for necessary and sufficient conditions in linear stochastic fractional systems. The mentioned conditions then have been adapted to the results in deterministic systems derived by [19] and [22] to find the rank condition. An example is then presented to illustrate the effectiveness of these theorems.

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2. Preliminaries

2.1. Notations

In this paper, the following notations have been used:

- (Ω, F, P):= The probability space with probability measure P on Ω;
- $\{F_t : 0 \leq t \leq T\}$:= The filtration generated by $\{w(s) : 0 \leq s \leq t\}$ and $F = F_t$;
- $L_2(\Omega, F_T, \mathbb{R}^n)$:= The Hilbert space of all F_t-measurable square integrable variables with values in $\mathbb{R}^n$;
- $L_2^F([0,T], \mathbb{R}^n)$:= The Hilbert space of all square integrable and F_t-measurable process with values in $\mathbb{R}^n$;
- $C([0,T], L_2(\Omega, F, P, X))$:= The Banach space of continuous maps from $[0,T]$ into $L_2(\Omega, F, P, X)$ satisfying the condition $\sup_{t\in[0,T]} E||x(t)||^2 < \infty$;
- X_s:= The Banach space with norm topology given by $||x||_s^2 = \sup_{t\in[0,s]} E||x(t)||^2 < \infty$ which is a closed subspace of $C([0,T], L_2(\Omega, F, P, X))$ consisting of measurable and F_t-adapted processes $x(t)$;
- U_s:= The Banach space with norm topology given by $||u||_s^2 = \sup_{t\in[0,s]} E||u(t)||^2 < \infty$ which is a closed subspace of $C([0,T], L_2(\Omega, F, P, X))$ consisting of measurable and F_t-adapted processes $u(t)$;
- $L(X, Y)$:= The space of all linear bounded operators from a Banach space X to a Banach space Y.

2.2. Fractional stochastic definitions

We consider the following class of linear fractional stochastic system in the interval $[0,T]$:

$$ {}_0^C D_t^\alpha x(t) = Ax(t) + Bu(t) + \Theta z(t); \; x(0) = x_0, \quad (1) $$

where $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $B \in L(\mathbb{R}^m, \mathbb{R}^n)$, $\Theta \in L(\mathbb{R}^d, \mathbb{R}^n)$ and $z(t)$ is an n-dimensional Gaussian white noise. ${}_0^C D_t^\alpha$ is the Caputo derivative of fractional order defined as below:

$$ {}_{t_0}^C D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & n-1<\alpha<n \\ D^n f(t) & \alpha = n \end{cases} \quad (2) $$

One may rewrite the n-dimensional Gaussian white noise $z(t)$ as:

$$ z(t) = \frac{dw(t)}{dt}, \quad (3) $$

where $w(t)$ is an n-dimensional Wiener process i.e. its mean value is zero; $E[w(t)] = 0$. Thus Eq. (1) is rewritten in fractional-Ito integral form as:

$$ x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (Ax(s) + Bu(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Theta dw(s). \quad (4) $$

The solution of the above system can be written as [1]:

$$ x(t) = E_{\alpha,1}(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) Bu(s)ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \Theta dw(s), \quad (5) $$

where the Mittag-Leffler function with two parameters is defined as [1]:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}. \quad (6) $$

2.3. Operators and sets definitions

The definitions here are appropriate extension of classical stochastic systems presented in [15]. Thus the following operators and sets are introduced.

- The operator ${}^\alpha L_0^T \in L(L_2^F([0,T], \mathbb{R}^m), L_2(\Omega, F_T, \mathbb{R}^n))$ is defined by:
$$ L_0^T u = \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha) Bu(s)ds. \quad (7) $$
- The operator $({}^\alpha L_0^T)^* : L_2(\Omega, F_T, \mathbb{R}^n) \to L_2^F([0,T], \mathbb{R}^m)$ as adjoint to operator L_0^T is defined by:
$$ (L_0^T)^* y = B^* E_{\alpha,\alpha}^* (A(T-t)^\alpha) E[y|F_t], \quad (8) $$
where * is the transpose operator.
- The α-controllability operator ${}^\alpha\Pi_s^T \in L(L_2(\Omega, F_T, \mathbb{R}^n), L_2(\Omega, F_T, \mathbb{R}^n))$ is defined by:
$$ {}^\alpha\Pi_s^T y = \int_s^T (T-t)^{\alpha-1} E_{\alpha,\alpha}(A(T-t)^\alpha) BB^* E_{\alpha,\alpha}^* (A(T-t)^\alpha) E[y|F_t]dt. \quad (9) $$
- The α-controllability matrix ${}^\alpha\Gamma_s^T \in L(\mathbb{R}^n\mathbb{R}^n)$ is defined by:
$$ {}^\alpha\Gamma_s^T = \int_s^T (T-t)^{\alpha-1} E_{\alpha,\alpha}(A(T-t)^\alpha) BB^* E_{\alpha,\alpha}^* (A(T-t)^\alpha)\, dt. \quad (10) $$
- The set of all states attainable from x_0 in time $t > 0$ can be defined as:
$$ \mathfrak{R}_t(x_0) = \{x(t; x_0, u) : u(.) \in L_2(\Omega, F_T, \mathbb{R}^n)\}, \quad (11) $$
where $x(t; x_0, u)$ is the solution of Eq. (4) corresponding to $x_0 \in \mathbb{R}^n$ and $u(.) \in L_2(\Omega, F_T, \mathbb{R}^n)$.

3. Fractional stochastic controllability

Definition 1. The stochastic system represented in Eq. (4) is approximately controllable on $[0,T]$ if:

$$\overline{\Re_T(x_0)} = L_2(\Omega, F_T, \mathbb{R}^n) \quad \forall x_0, \tag{12}$$

that is given an arbitrary $\varepsilon > 0$. It is possible to move from any point x_0 to within a distance ε of every point in the state space $L_2(\Omega, F_T, \mathbb{R}^n)$ at time T.

Definition 2. The stochastic system represented in Eq. (4) is completely controllable on $[0,T]$ if:

$$\Re_T(x_0) = L_2(\Omega, F_T, \mathbb{R}^n) \quad \forall x_0. \tag{13}$$

It means that all points in the state space $L_2(\Omega, F_T, \mathbb{R}^n)$ can be reached from any point x_0 at time T.

Lemma 1. *For every $y \in L_2(\Omega, F_T, \mathbb{R}^n)$ there exists a process $\varphi \in L_2^F\left([0,T], L(\mathbb{R}^d, \mathbb{R}^n)\right)$ such that:*

$$y = E[y] + \int_0^T \varphi(s) dw(s). \tag{14}$$

Proof. The proof can be found in [23].■

Lemma 2. *For every $y \in L_2(\Omega, F_T, \mathbb{R}^n)$ there exists a process $\varphi \in L_2^f\left([0,T], L(\mathbb{R}^d, \mathbb{R}^n)\right)$ such that:*

$$^{\alpha}\Pi_s^T y = {}^{\alpha}\Gamma_s^T E[y] + \int_0^T {}^{\alpha}\Gamma_{T-s}^T \varphi(s) dw(s). \tag{15}$$

Proof. The proof can be found in [23].■

Theorem 1. *For arbitrary $h \in L_2(\Omega, F_T, \mathbb{R}^n)$ the control:*

$$\begin{aligned} u^{\beta} =& B^* E^*_{\alpha,\alpha}\left(A(T-t)^{\alpha}\right)(\beta I + {}^{\alpha}\Gamma_0^T)^{-1} \\ & (E[h] - E_{\alpha,1}(AT^{\alpha})x_0) - B^* E^*_{\alpha,\alpha}\left(A(T-t)^{\alpha}\right) \\ & \int_0^t (\beta I + {}^{\alpha}\Gamma_s^T)^{-1}[(T-s)^{\alpha-1} E^*_{\alpha\alpha}\left(A(T-s)^{\alpha}\right) \\ & \Theta - \varphi(s)] dw(s), \end{aligned} \tag{16}$$

transfers the system of Eq. (4) from x_0 to:

$$\begin{aligned} x^{\beta}(T) =& h - \beta(\beta I + {}^{\alpha}\Gamma_0^T)^{-1}\left(E[h] - E_{\alpha,1}(AT^{\alpha})x_0\right) \\ & + \beta \int_0^T (\beta I + {}^{\alpha}\Gamma_s^T)^{-1}[(T-s)^{\alpha-1} E^*_{\alpha,\alpha} \\ & (A(T-s)^{\alpha})\Theta - \varphi(s)] dw(s), \end{aligned} \tag{17}$$

at time T where $\varphi \in L_2^F\left([0,T], L(\mathbb{R}^d, \mathbb{R}^n)\right)$, and selects in a way that Eq. (14) be satisfied for h.

Proof. By substituting Eq. (16) in Eq. (5), one can write:

$$\begin{aligned} x^{\beta}(t) =& E_{\alpha,1}(At^{\alpha})x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t-s)^{\alpha}\right) \\ & BB^* E^*_{\alpha,\alpha}\left(A(T-s)^{\alpha}\right)(\beta I + {}^{\alpha}\Gamma_0^T)^{-1}(E[h] - \\ & E_{\alpha,1}(AT^{\alpha})x_0) ds - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \\ & \left(A(t-s)^{\alpha}\right) BB^* E^*_{\alpha,\alpha}\left(A(T-s)^{\alpha}\right) \\ & \int_0^s (\beta I + {}^{\alpha}\Gamma_s^T)^{-1}[(T-r)^{\alpha-1} E^*_{\alpha,\alpha} \\ & \left(A(T-r)^{\alpha}\right)\Theta - \varphi(r)] dw(r) ds \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t-s)^{\alpha}\right)\Theta dw(s) \\ & = E_{\alpha,1}(AT^{\alpha})x_0 + \int_0^t (t-s)^{\alpha-1} \\ & E_{\alpha,\alpha}\left(A(t-s)^{\alpha}\right) BB^* E^*_{\alpha,\alpha}\left(A(T-s)^{\alpha}\right) \\ & (\beta I + {}^{\alpha}\Gamma_0^T)^{-1}(E[h] - E_{\alpha,1}(AT^{\alpha})x_0) ds \\ & - \int_0^t \int_r^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t-s)^{\alpha}\right) \\ & BB^* E^*_{\alpha,\alpha}\left(A(T-s)^{\alpha}\right)(\beta I + {}^{\alpha}\Gamma_r^T)^{-1} \\ & [(T-r)^{\alpha-1} E^*_{\alpha,\alpha}\left(A(T-r)^{\alpha}\right)\Theta - \varphi(r)] \\ & ds dw(r) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t-s)^{\alpha}\right) \\ & \Theta dw(s). \end{aligned} \tag{18}$$

Evaluating at time $t = T$, one may readily find that:

$$\begin{aligned} x^{\beta}(T) =& E_{\alpha,1}(AT^{\alpha})x_0 + \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha} \\ & (A(T-s)^{\alpha}) BB^* E^*_{\alpha,\alpha}(A(T-s)^{\alpha}) \\ & (\beta I + {}^{\alpha}\Gamma_0^T)^{-1}(E[h] - E_{\alpha,1}(AT^{\alpha})x_0) ds \\ & - \int_0^T \int_r^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^{\alpha}) \\ & BB^* E^*_{\alpha,\alpha}(A(T-s)^{\alpha})(\beta I + {}^{\alpha}\Gamma_r^T)^{-1} \\ & [(T-r)^{\alpha-1} E^*_{\alpha,\alpha}(A(T-r)^{\alpha})\Theta - \varphi(r)] \end{aligned}$$

$$dsdw(r)+\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^\alpha)$$

$$\Theta dw(s)=E_{\alpha,1}(AT^\alpha)x_0+{}^\alpha\Gamma_0^T(\beta I+{}^\alpha\Gamma_0^T)^{-1}$$

$$(E[h]-E_{\alpha,1}(AT^\alpha)x_0)-\int_0^T {}^\alpha\Gamma_r^T(\beta I+{}^\alpha\Gamma_r^T)^{-1}$$

$$[(T-r)^{\alpha-1}E^*_{\alpha,\alpha}(A(T-r)^\alpha)\Theta-\varphi(r)]dw(r)$$

$$+\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^\alpha)\Theta dw(s)$$

$$=E_{\alpha,1}(AT^\alpha)x_0+(\beta I+{}^\alpha\Gamma_0^T)(\beta I+{}^\alpha\Gamma_0^T)^{-1}$$

$$(E[h]-E_{\alpha,1}(AT^\alpha)x_0)-\beta(\beta I+{}^\alpha\Gamma_0^T)^{-1}$$

$$(E[h]-E_{\alpha,1}(AT^\alpha)x_0)-\int_0^T(\beta I+{}^\alpha\Gamma_r^T)$$

$$(\beta I+{}^\alpha\Gamma_r^T)^{-1}[(T-r)^{\alpha-1}E^*_{\alpha,\alpha}(A(T-r)^\alpha)$$

$$\Theta-\varphi(r)]dw(r)+\beta\int_0^T(\beta I+{}^\alpha\Gamma_r^T)^{-1}$$

$$[(T-r)^{\alpha-1}E^*_{\alpha,\alpha}(A(T-r)^\alpha)\Theta-\varphi(r)]dw(r)$$

$$+\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^\alpha)\Theta dw(s)$$

$$=E[h]+\int_0^T\varphi(r)dw(r)-\beta(\beta I+{}^\alpha\Gamma_0^T)^{-1}$$

$$(E[h]-E_{\alpha,1}(AT^\alpha)x_0)+\beta\int_0^T(\beta I+{}^\alpha\Gamma_r^T)^{-1}$$

$$[(T-r)^{\alpha-1}E^*_{\alpha,\alpha}(A(T-r)^\alpha)\Theta-\varphi(r)]dw(r), \tag{19}$$

which via Lemma 1, it results in Eq. (17).■

Theorem 2. *The system in Eq. (4) is completely controllable if and only if there exists* $\gamma>0$ *which* ${}^\alpha\Gamma_s^T\geq\gamma I$ *for every* $s\in[0,T]$*; i.e.* ${}^\alpha\Gamma_s^T$ *is positive-definite for every* $s\in[0,T]$.

Proof. First we will show that if there exists $\gamma>0$ which ${}^\alpha\Gamma_s^T\geq\gamma I$ for every $s\in[0,T]$, the fractional stochastic system is completely controllable. One may write for arbitrary positive β and for every $s\in[0,T]$ that:

$$0\leq\beta(\beta I+{}^\alpha\Gamma_s^T)^{-1}\leq\frac{\beta}{\beta+\gamma}I. \tag{20}$$

One may find that:

$$\lim_{\beta\to0}\beta(\beta I+{}^\alpha\Gamma_s^T)^{-1}\leq\lim_{\beta\to0}\frac{\beta}{\beta+\gamma}I=0. \tag{21}$$

which results in:

$$\lim_{\beta\to0}\beta(\beta I+{}^\alpha\Gamma_s^T)^{-1}=0. \tag{22}$$

Thus considering Theorem 1, the following is obtained:

$$\lim_{\beta\to0}x^\beta(T)=h. \tag{23}$$

On the other hand, according to definition of Eq. (10), ${}^\alpha\Gamma_s^T$ is positive semi definite. If matrix ${}^\alpha\Gamma_s^T$ is not positive definite, there should be a non-zero y such that:

$$y^{*\alpha}\Gamma_s^Ty=0, \tag{24}$$

that is:

$$\int_s^T y^*(T-t)^{\alpha-1}E_{\alpha,\alpha}(A(T-t)^\alpha)$$

$$BB^*E^*_{\alpha,\alpha}(A(T-t)^\alpha)ydt=0, \tag{25}$$

thus:

$$y^*E_{\alpha,\alpha}(A(T-t)^\alpha)B=0,\quad 0\leq t\leq T. \tag{26}$$

Now assume that the initial condition is:

$$x_0=E_{\alpha,1}(At^\alpha)^{-1}y. \tag{27}$$

By assumption, there exists an input u such that it moves x_0 to any point. Here we choose:

$$x_1=\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\Theta dw(s), \tag{28}$$

thus:

$$x_1=\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\Theta dw(s)$$

$$=E_{\alpha,1}(At^\alpha)x_0+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)$$

$$Bu(s)ds+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\Theta dw(s)$$

$$=y+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)Bu(s)ds$$

$$+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\Theta dw(s), \tag{29}$$

which can be rewritten as:

$$0=y+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)Bu(s)ds. \tag{30}$$

Multiplying both sides of Eq. (30) by y^*, one may find:

$$0 = y^* y + \int_0^t y^*(t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) Bu(s) ds. \tag{31}$$

The second term is zero based on Eq. (26) and thus $y^* y = 0$. This is a contradiction to $y \neq 0$. Thus ${}^\alpha\Gamma_s^T$ is positive definite.■

Theorem 3. *The system in Eq. (4) is approximate controllable by the controller of Eq. (16), if and only if* ${}^\alpha\Gamma_s^T > 0$ *for every* $s \in [0, T]$.

Proof. The proof is the same as [15,16] and follows the same pattern in Theorem 2 of this paper.■

Let us now define the fractional deterministic version of Eq. (1) as:

$${}_0^C D_t^\alpha x(t) = Ax(t) + Bu(t); \quad x(0) = x_0. \tag{32}$$

The solution of Eq. (32) can be found as:

$$\begin{aligned} x(t) &= E_{\alpha,1}(At^\alpha)x_0 + \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(A\tau^\alpha) Bu(t-\tau) d\tau \\ &= E_{\alpha,1}(At^\alpha)x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^\alpha) \\ &\quad Bu(\tau) d\tau. \end{aligned} \tag{33}$$

The deterministic version can also be derived by applying the expectation to Eq. (5). To this end first, let us find the Caputo fractional derivatives of an expectation of a state as:

$$\begin{aligned} {}_{t_0}^C D_t^\alpha (E[x(t)]) &= \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{\frac{d^n \left(\int_{-\infty}^{+\infty} x(\tau,\omega) p(\omega) d\omega\right)}{d\tau^n}}{(t-\tau)^{\alpha+1-n}} d\tau \\ &= \int_{t_0}^t \frac{\int_{-\infty}^{+\infty} x^{(n)}(\tau,\omega) p(\omega) d\omega}{\Gamma(n-\alpha)(t-\tau)^{\alpha+1-n}} d\tau \\ &= \int_{t_0}^t \int_{-\infty}^{+\infty} \frac{x^{(n)}(\tau,\omega) p(\omega)}{\Gamma(n-\alpha)(t-\tau)^{\alpha+1-n}} d\omega d\tau \\ &= \int_{-\infty}^{+\infty} \int_{t_0}^t \frac{x^{(n)}(\tau,\omega)}{\Gamma(n-\alpha)(t-\tau)^{\alpha+1-n}} d\tau p(\omega) d\omega \\ &= \int_{-\infty}^{+\infty} {}_{t_0}^C D_t^\alpha \left(x(t,\omega)\right) p(\omega) d\omega \\ &= E\left[{}_{t_0}^C D_t^\alpha (x(t))\right]. \end{aligned} \tag{34}$$

The formula for expectation of state can be found in [23]. Thus, by applying the expectation on Eq. (1), the deterministic version of stochastic fractional differential equation is obtained as:

$$E[{}_0^C D_t^\alpha x(t)] = {}_0^C D_t^\alpha E[x(t)] = AE[x(t)] + BE[u(t)];$$
$$E[x(0)] = E[x_0]. \tag{35}$$

Theorem 4. *The fractional deterministic system in Eq. (32) is controllable if and only if the* α*-controllability matrix* ${}^\alpha\Gamma_0^T \in L(\mathbb{R}^n, \mathbb{R}^n)$ *defined in Eq. (10) is positive definite; i.e. there exist* $\gamma > 0$ *which* ${}^\alpha\Gamma_0^T \geq \gamma I$.

Proof. The proof has been presented in [22].■

Theorem 5. The following conditions are equivalent:

(a) The α-controllability matrix ${}^\alpha\Gamma_s^T \in L(\mathbb{R}^n, \mathbb{R}^n)$ defined in Eq. (10) is positive definite for every $s \in [0, T]$;

(b) For every $y \in L_2(\Omega, F_T, \mathbb{R}^n)$, there is a $\gamma > 0$ where $E\langle {}^\alpha\Pi_s^T y, y\rangle \geq \gamma ||y||^2$;

(c) The fractional deterministic system (32) is controllable on every $[s, T]$, $0 \leq s \leq T$;

(d) The fractional stochastic system (4) is completely controllable;

(e) The fractional deterministic system (32) is controllable on every $[0, r]$, $0 \leq r \leq T$;

(f) The matrix $[B|AB|A^2B|...|A^{n-1}B]$ has rank equal to n.

Proof. (a) $\Leftrightarrow$ (b) is presented in the following equation using Lemma 2:

$$\begin{aligned} E\left\langle {}^\alpha\Pi_s^T y, y\right\rangle &= E\left\langle {}^\alpha\Gamma_s^T E[y] \right. \\ &\quad \left. + \int_0^T {}^\alpha\Gamma_{T-s}^T \varphi(s) dw(s), y\right\rangle \\ &= E\left\langle {}^\alpha\Gamma_s^T E[y], y\right\rangle \\ &\quad + E\left\langle \int_0^T {}^\alpha\Gamma_{T-s}^T \varphi(s) dw(s), y\right\rangle \\ &= {}^\alpha\Gamma_s^T E\left\langle y, y\right\rangle = {}^\alpha\Gamma_s^T ||y||^2. \end{aligned} \tag{36}$$

(a) $\Leftrightarrow$ (c) is according to Theorem 4.
(a) $\Leftrightarrow$ (d) is according to Theorem 2.
(c) $\Leftrightarrow$ (e); one may rewrite α-controllability matrix ${}^\alpha\Gamma_s^T \in L(\mathbb{R}^n, \mathbb{R}^n)$ defined in Eq. (10) via changing variable of $\tau = t - s$ and defining $r = T - s$ as:

$$\int_0^r (r-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(r-\tau)^\alpha) BB^* E_{\alpha,\alpha}^*(A(r-\tau)^\alpha) d\tau, \tag{37}$$

then using Theorem 4, the objective is achieved.
(e) $\Leftrightarrow$ (f) is according to [19].■

4. Example

Consider the fractional harmonic oscillator equation [22,24]:

$$ {}_0^C D_t^{2a}\psi(t) = -\psi(t) + u(t). \tag{38}$$

The stochastic version of above equation can be defined as:

$$ {}_0^C D_t^{2a}\psi(t) = -\psi(t) + u(t) + z(t). \tag{39}$$

To drive the controllability condition of the above system one may use the following auxiliary variables:

$$ x_1(t) = \psi(t); \quad x_2(t) = {}_0^C D_t^{a}\psi(t), \tag{40}$$

to obtain:

$$ {}_0^C D_t^{a} x_1(t) = x_2(t),$$

$$ {}_0^C D_t^{a} x_2(t) = {}_0^C D_t^{2a} x_1(t) = -x_1(t) + u(t) + z(t). \tag{41}$$

Thus, the above equation, has now the matrix form as presented in Eq. (4) with:

$$ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{42}$$

The corresponding two parameter Mittag-Leffler of the above system is given by [22]:

$$ E_{\alpha,\alpha}(A(T-t)^\alpha) = \begin{bmatrix} N_1(T-t) & N_2(T-t) \\ N_3(T-t) & N_4(T-t) \end{bmatrix}, \tag{43}$$

where:

$$ N_1(T-t) = \sum_{k=0}^{\infty} \frac{(-1)^k (T-t)^{2k\alpha}}{\Gamma(2k\alpha+\alpha)},$$

$$ N_2(T-t) = \sum_{k=0}^{\infty} \frac{(-1)^k (T-t)^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+\alpha)},$$

$$ N_3(T-t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (T-t)^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+\alpha)},$$

$$ N_4(T-t) = \sum_{k=0}^{\infty} \frac{(-1)^k (T-t)^{2k\alpha}}{\Gamma(2k\alpha+\alpha)}, \tag{44}$$

so the α-controllability matrix ${}^\alpha\Gamma_s^T \in L(\mathbb{R}^n, \mathbb{R}^n)$ can be found as:

$$ {}^\alpha\Gamma_s^T = \int_s^T (T-t)^{\alpha-1} E_{\alpha,\alpha}(A(T-t)^\alpha) BB^* E_{\alpha,\alpha}^*(A(T-t)^\alpha) dt = \int_s^T (T-t)^{\alpha-1} \begin{bmatrix} N_2^2(T-t) & N_1(T-t)N_2(T-t) \\ N_1(T-t)N_2(T-t) & N_1^2(T-t) \end{bmatrix} dt \tag{45}$$

which is positive definite for every $s \in [0,T]$, for every T and for every $0 < \alpha \leq 1$. To this end, one may rewrite Eq. (45) as:

$$ \int_0^{T-s} t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) BB^* E_{\alpha,\alpha}^*(At^\alpha) dt = \int_0^r t^{\alpha-1} \begin{bmatrix} N_2^2(t) & N_1(t)N_2(t) \\ N_1(t)N_2(t) & N_1^2(t) \end{bmatrix} dt. \tag{46}$$

The above matrix is positive-definite for every r. Thus via Theorem 2, the stochastic fractional system is completely controllable. One may also use Theorem 5 to find the controllability condition via rank condition. Thus:

$$ \mathrm{rank}[B|AB] = \mathrm{rank} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2, \tag{47}$$

which again shows the stochastic fractional system is completely controllable.

5. Conclusion

In this paper, the problem of complete and approximate controllability in linear fractional stochastic systems has been investigated via some theorems. The necessary and sufficient conditions have been investigated for these systems and the equivalency to linear fractional system controllability conditions has been provided. An example shows the effectiveness of this theory.

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