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LP modelling for the two dimensional nonlinear Fredholm integral equations

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Abstract. A different numerical approach for the two dimensional nonlinear Fredholm integral equations of the second kind with the continuous kernel is considered. The main idea is to convert the integral equation into an optimization problem. Then by using an embedding method, the class of admissible trajectories is replaced by a class of positive Borel measures. The optimization problem in measure space is then approximated by a finite dimensional Linear Programming (LP) problem. Some examples demonstrate the effectiveness of the method.

1. Introduction

In this paper, we are concerned with an optimization method for two-dimensional nonlinear Fredholm integral equations of the second kind, i.e.:

$$u(x,y)=f(x,y)+\int_a^b\int_c^d k(x,y,s,t,u(s,t))dtds,\qquad (x,y)\in D, \tag{1}$$

where $u(x,y)$ is an unknown function, $f(x,y)$ and $k(x,y,s,t,u)$ are given continuous functions definitions, respectively, on:

$$D=[a,b]\times[c,d],$$

and:

$$E=D\times D\times(-\infty,\infty),$$

with $k(x,y,s,t,u)$ nonlinear in u. We assume throughout this paper that the integral equation (Eq. (1)) has a unique solution. Integral equations are often involved in the mathematical formulation of physical phenomena, and can be encountered in various fields of science such as physics [1], biology [2] and engineering [3,4]. It can also be used in numerous applications, such as biomechanics, control, economics, elasticity, electrical engineering, electrodynamics, electrostatics, filtration theory, fluid dynamics, game theory, heat and mass transfer, medicine, oscillation theory, plasticity, queuing theory, etc. [5]. Fredholm and Volterra integral equations of the second kind were shown up in studies that includes airfoil theory [6], elastic contact problems [7,8], fracture mechanics [9], combined infrared radiation and molecular conduction [10] and so on.

There has been much work on developing and analyzing the numerical methods for solving integral equations (see, for example [11]-[25] and the references cited therein). But among them, the analysis of computational methods for multi-dimensional Fredholm integral equations seems to have been discussed in only a few papers, especially in the nonlinear case. Using Gaussian radial basis function, Alipanah and Esmaeili [11] also, successfully, solved the two-dimensional Fredholm integral equations using Gaussian radial basis function.

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Recently, Babolian et al. have considered the use of a basis of rationalized Haar functions for the numerical solution of nonlinear two-dimensional Volterra and Fredholm integral equations [15]. In [18], Han and Wang applied the iterated Galerkin method to the solution of nonlinear two-dimensional Fredholm integral equations of the second kind. Han and Jiong [19] considered this problem by the Nystrm method. Xie and Lin [25] also proposed a fast numerical solution method for linear two-dimensional Fredholm integral equations of the second kind.

Motivated by the above discussions, in this paper, we intend to present a numerical optimization scheme for extracting approximate solution for the nonlinear Fredholm integral equation (Eq. (1)) by an extended measure theory-based approach established in [26]. The advantages of the proposed method are in the fact that the method is not iterative, it is self-starting and is not restricted to differentiable cost functions. Because of these features, this method has been extended to solve a variety of control and optimization problems. In this connection we may refer to the numerical estimation of the distributed control of a diffusion equation [27], an optimal shape design formulation for inhomogeneous dam problems [28], determining optimal shape of the pole of an electromagnet [29], time optimal control problem of the heat [30,31] and wave equations [32], the shape variation design problem of the planar contraction nozzle [33], optimal shape design for a thin airfoil [34], optimal designing for a two dimensional nozzle [35], shape optimization of cylindrical bar cross-sections [36] and the time optimal control problem in the case of multiple targets [37].

2. Moment problem

Let $\Delta_1 = \{x_0, x_1, \ldots, x_M\}$ and $\Delta_2 = \{y_0, y_1, \ldots, y_M\}$ be two equidistance partitions of $I = [a,b]$ and $J = [c,d]$, where:

$$h_1 = x_{i+1} - x_i, \ i = 0, 1, \cdots, M-1,$$

and:

$$h_2 = y_{j+1} - y_j, \ j = 0, 1, \cdots, N-1,$$

are the discretization parameters of the partitions. For the partitions:

$$\Delta_1 = \{x_0, x_1, \ldots, x_M\},$$

and:

$$\Delta_2 = \{y_0, y_1, \ldots, y_M\},$$

on $I \times J$, the integral equation (Eq. (1)) can be discretized in the following form:

$$\begin{cases} \int_a^b \int_c^d k(x_0, y_0, s, t, u(s,t)) dtds - u(x_0, y_0) \\ \qquad = -f(x_0, y_0), \\ \int_a^b \int_c^d k(x_0, y_1, s, t, u(s,t)) dtds - u(x_0, y_1) \\ \qquad = -f(x_0, y_1), \\ \vdots \\ \int_a^b \int_c^d k(x_0, y_N, s, t, u(s,t)) dtds - u(x_0, y_N) \\ \qquad = -f(x_0, y_N), \\ \vdots \\ \int_a^b \int_c^d k(x_M, y_0, s, t, u(s,t)) dtds - u(x_M, y_0) \\ \qquad = -f(x_M, y_0), \\ \int_a^b \int_c^d k(x_M, y_1, s, t, u(s,t)) dtds - u(x_M, y_1) \\ \qquad = -f(x_M, y_1), \\ \vdots \\ \int_a^b \int_c^d k(x_M, y_N, s, t, u(s,t)) dtds - u(x_M, y_N) \\ \qquad = -f(x_M, y_N). \end{cases} \tag{2}$$

We define an approximating optimization problem corresponding to the integral equation (Eq. (1)) as follows:

$$\text{minimize} \quad \int_a^b \int_c^d g(s, t, u(s,t)) \ dtds, \tag{3}$$

subject to :

$$\int_a^b \int_c^d k(x_i, y_j, s, t, u(s,t)) dtds - u(x_i, y_j)$$
$$= -f(x_i, y_j),$$
$$(i = 0, 1, \cdots, M), \quad (j = 0, 1, \cdots, N), \tag{4}$$

where $g(s,t,u(s,t))$ is a continuously differentiable function. Without loos of generality, throughout this paper we assume $g(s,t,u(s,t)) = 0$.

Proposition 1. *Finding a solution for the approximated system (Eq. (2)) of the integral equation (Eq. (1)) is equivalent to find a solution of the optimization problem (Eqs. (3)-(4)).*

Proof. The proof is clear, since the problem (Eq. (1)) has a unique solution.

Definition 1. The trajectory function $u(\cdot,\cdot) : [a,b] \times [c,d] \to I\!R$ is called admissible if it is absolutely continuous and Constrains (4) are satisfied. We denote the set of all admissible trajectories by U_{ad} which is also nonempty.

Now integral equation problem (Eq. (1)) is reduced to find a solution $u \in U_{ad}$ satisfying:

$$\text{minimize} \quad \int_a^b \int_c^d g(s,t,u(s,t))dtds, \tag{5}$$

subject to :

$$\int_a^b \int_c^d k_{ij}dtds = a_{ij},$$

$$(i = 0,1,\cdots,M),\ (j = 0,1,\cdots,N), \tag{6}$$

where for simplicity, we denote:

$$a_{ij} = u(x_i,y_j) - f(x_i,y_j)$$

$$k_{ij} = k(x_i,y_j,s,t,u(s,t)),$$

$$(i = 0,1,\cdots,M),\ (j = 0,1,\cdots,N).$$

In the next section, we proceed to enlarge the set U_{ad}.

3. Metamorphosis

In general, it may be difficult to characterize the optimal trajectory in U_{ad}; necessary conditions are not always helpful because the information that they give may be impossible to interpret [30]. It appears that these situations may become more favorable if the set U_{ad} could somehow be made larger. In the following, we use a transformation to enlarge the set U_{ad}.

Let $\Omega = I \times J \times U$, where U is a known compact sets in $\mathbb{R}$ such that the trajectory u gets its values for each $(x,y) \in I \times J$ in this set, and $C(\Omega)$ is the space of all real-valued continuously differentiable functions on Ω. For each admissible trajectory $u \in U_{ad}$, we correspond the following linear continuous functional:

$$\Lambda : h \longrightarrow \int_a^b \int_c^d h(s,t,u(s,t))dtds,\ \forall\ h \in C(\Omega). \tag{7}$$

Some aspects of this mapping are useful; it is well defined, and positive [31].

Proposition 2. *Transformation $u \to \Lambda$ of an admissible trajectory in U_{ad} into the linear mapping Λ defined in Eq. (7) is an injection.*

Proof. We must show that if $u_r \neq u_q$, then $\Lambda_r \neq \Lambda_q$. Indeed, if u_r and u_q are different admissible trajectories, then there is a subinterval of I, say N_L, where $u_r(s,t) \neq u_q(s,t)$ for $(s,t) \in N_L$. A continuous positive function h can be constructed on I so that the right-hand side of Eq. (7), corresponding to u_r and u_q, are not equal. For instance, assume for all $(s,t) \in N_L$, the function h is positive on the appropriate portion of the graph of $u_r(\cdot,\cdot)$, and zero on $u_q(\cdot,\cdot)$. Then the corresponding linear functionals are not equal.

Thus, solving Eqs. (5)-(6) is equivalent to find Λ in functional space $C^*(\Omega)$ (C^* is the dual space), such that:

$$\text{minimize} \quad \Lambda(g), \tag{8}$$

subject to :

$$\Lambda(k_{ij}) = a_{ij},\ (i = 0,1,\cdots,M),\ (j = 0,1,\cdots,N). \tag{9}$$

By Riesz representation theorem [38], there exists a unique positive Radon representing the measure μ on Ω, such that:

$$\Lambda(h) = \int_\Omega h d\mu = \mu(h),\ h \in C(\Omega). \tag{10}$$

These measures μ are required to have certain properties which are abstracted from the definition of admissible trajectories. First, from Eq. (10), we have:

$$|\mu(h)| \leq ST \sup_\Omega |h(s,t,u(s,t))|,$$

where $S = b - a$ and $T = d - c$. Hence $\mu(1) \leq ST$. From Eqs. (9) and (10), we see that the measures μ satisfy:

$$\mu(k_{ij}) = a_{ij},\quad (i = 0,1,\cdots,M),\ (j = 0,1,\ldots,N).$$

Next, suppose that $\eta \in C(\Omega)$ does not depend on u, that is:

$$\eta(s,t,u_1) = \eta(s,t,u_2),$$

for all $s \in [a,b]$, $t \in [c,d]$, and $u_1, u_2 \in U$, where $u_1(\cdot,\cdot) \neq u_2(\cdot,\cdot)$. Then the measures μ must satisfy:

$$\int_\Omega \eta d\mu = \int_a^b \int_c^d \eta(s,t,u(s,t))dtds = \alpha_\eta,$$

where u is an arbitrary number in the set U, and α_η is the Lebesgue integral of $\eta(\cdot,\cdot,u)$ over I.

Let $M^+(\Omega)$ be the set of all positive Radon measures on Ω. We topologize the space $M^+(\Omega)$ by the weak*-topology and define the set Q as a subset of $M^+(\Omega)$ as follows:

$$Q = S_1 \cap S_2 \cap S_3,$$

where:

$$S_1 = \{\mu \in M^+(\Omega) : \mu(1) \leq ST\},$$

$$S_2 = \{\mu \in M^+(\Omega) : \mu(k_{ij}) = a_{ij}, (i = 0,1,\cdots,M),$$
$$(j = 0,1,\cdots,N)\},$$

$$S_3 = \{\mu \in M^+(\Omega) : \mu(\eta) = \alpha_\eta,$$
$$\eta \in C(\Omega) \text{ independent of } u\}.$$

So one may change the problem (8)-(9) in functional space, to the following optimization problem in measure space:

$$\text{minimize } I(\mu) = \int_{\Omega} d\mu \equiv \mu(g) \tag{11}$$

subject to :

$$\mu \in Q. \tag{12}$$

Theorem 1. *The set Q is compact in $M^+(\Omega)$.*

Proof. The set S_1 is compact and the set S_2 can be written as:

$$S_2 = \bigcap_{i=1}^{M} \bigcap_{j=1}^{N} \{\mu \in \mathcal{M}^+(\Omega) : \mu(k_{ij}) = a_{ij}\} = \bigcap_{i=1}^{M} \bigcap_{j=1}^{N} W_{ij},$$

where each $W_{ij} = \{\mu \in \mathcal{M}^+(\Omega) : \mu(k_{ij}) = a_{ij}\}$ is closed, because it is the inverse image of a closed set on the real line, the set $\{a_{ij}\}$, under a continuous map. By a similar argument, it is easy to show that S_3 is closed. Thus Q is a closed subset of the compact set, S_1, and then Q is compact.

Theorem 2. *The measure-theoretical problem, which consists of finding the minimum of the functional (11) over the set Q of $M^+(\Omega)$, possesses a minimizing solution μ^*, say, in Q.*

Proof. The proof is clear, since μ is a linear functional on a compact set Q, therefor it attains its minimum.

In the next sections, we shall establish a method for estimating numerically trajectories which approximate the action of the optimal measures.

4. Approximation of the optimal measure

In this section, we obtain an approximation to the optimal measure μ^* satisfying Eqs. (11)-(12).

It is clear that the measure-theoretical problem (11)-(12), can be written in the following form

$$\text{minimize } I(\mu) = \mu(g), \tag{13}$$

subject to :

$$\begin{cases} \mu(1) \le ST, \\ \mu(k_{ij}) = a_{ij}, \ (i = 0, 1, \cdots, M), \ (j = 0, 1, \cdots, N), \\ \mu(\eta) = \alpha_\eta, \ \eta \in C(\Omega) \ \text{ independent of } u. \end{cases} \tag{14}$$

The minimizing problem of Eqs. (13)-(14) is an infinite-dimensional LP problem and we are mainly interested in approximating it. It is possible to approximate the nearly trajectory function of the problem (Eqs. (13)-(14)) by the solution of a finite dimensional LP of sufficiently large dimension.

First we consider the minimization of Eq. (13) not only over the set Q, but also over a subset of it defined by requiring that only a finite number of Constraints (14) be satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces, and then selecting a finite number of them.

Proposition 3. *Let $Q(MN, GH)$ be a subset of $M^+(\Omega)$ consisting of all measures which satisfy:*

$$\begin{cases} \mu(1) \le ST, \\ \mu(k_{ij}) = a_{ij}, \quad (i = 0, 1, ..., M, \ j = 0, 1, ..., N), \\ \mu(\eta_{vw}) = \alpha_{\eta_{vw}}, \quad (v = 1, 2, ..., G, \ w = 1, 2, ..., H). \end{cases}$$

As M, N, G and H tend to infinity, $\varrho(MN, GH) = \inf_{Q(MN,GH)} \mu(g)$ tends to $\varrho = \inf_Q \mu(g)$.

Proof. The proof is similar to Proposition 2 in [35].

This is the first stage of the approximation. As the second stage, from Theorem (A.5) of [26], we can characterize a measure, say μ^*, in the set $Q(MN, GH)$ at which the function $\mu \to \mu(g)$ attains its minimum. Proposition 4 follows a result of Rosenbloom [39].

Proposition 4. *The measure μ^* in the set $Q(MN, GH)$, at which the function $\mu \to \mu(g)$ attains its minimum, has the following form:*

$$\mu^* = \sum_{k=1}^{MN+GH} \rho_k^* \delta(z_k^*), \tag{15}$$

with $z_k^* \in \Omega$ and $\rho_k^* \ge 0$, $k = 1, 2, \cdots, MN + GH$. Here $\delta_\Omega(z^*)$ is unitary atomic measure concentrated at $z^* \in \Omega$, characterized by $\delta(z^*)(F) = F(z^*)$, where $F \in C(\Omega)$.

Based on Eq. (15), the measure theoretical optimization problem (13)-(14) is equivalent to the following nonlinear optimization problem:

$$\text{minimize } \sum_{k=1}^{MN+GH} \rho_k^* g(z_k^*), \tag{16}$$

subject to :

$$\sum_{k=1}^{MN+GH} \rho_k^* k_{ij}(z_k^*) - u(x_i, y_j) = -f(x_i, y_j),$$

$$(i = 0, 1, \cdots, M), (j = 0, 1, \ldots, N), \tag{17}$$

$$\sum_{k=1}^{MN+GH} \rho_k^* \eta_{vw}(z_k^*) = \alpha_{\eta_{vw}},$$

$$(v = 1, \cdots, G), \ (w = 1, \cdots, H), \tag{18}$$

$$\sum_{k=1}^{MN+GH} \rho_k^* \leq ST, \tag{19}$$

$u(x_i, y_j)$ is free,

$$(i = 0, 1, \ldots, M), \ (j = 0, 1, \ldots, N), \tag{20}$$

$$\rho_k^* \geq 0, \ (k = 1, 2, \ldots, MN + GH), \tag{21}$$

where the unknowns are the coefficients ρ_k^*, supports z_k^* $(k = 1, 2, \ldots, MN + GH)$, and $u(x_i, y_j)$ $(i = 0, 1, \ldots, M)$ $(j = 0, 1, \ldots, N)$.

It would be computationally convenient if we could minimize the function $\mu \to \mu(g)$ only with respect to the coefficients ρ_k^* $(k = 1, 2, \ldots, MN + GH)$, and $u(x_i, y_j)$ $(i = 0, 1, \ldots, M)$ $(j = 0, 1, \ldots, N)$, which leads to a finite-dimensional LP problem. However, we do not know the supports of the optimal measure. The answer lies in a meaningful approximation of this support, by introducing a dense subset in Ω.

Proposition 5. *Let σ be a countable dense subset of Ω. Given $\epsilon > 0$, a measure $\bar{\mu} \in M^+(\Omega)$ can be found such that:*

$$|(\mu^* - \bar{\mu})(g)| \leq \epsilon,$$

$$|(\mu^* - \bar{\mu})(k_{ij})| \leq \epsilon, \ (i = 0, 1, \ldots, M) \ (j = 0, 1, \ldots, N),$$

$$|(\mu^* - \bar{\mu})(\eta_{vw})| \leq \epsilon, \ (v = 1, \cdots, G), \ (w = 1, \cdots, H),$$

the measure $\bar{\mu}$ has the form:

$$\bar{\mu} = \sum_{k=1}^{MN+GH} \rho_k^* \delta(z_k), \tag{22}$$

where the coefficients ρ_k^* are the same as in the optimal measure (15) and $z_k \in \sigma$.

Proof. See the proof of Proposition III.3 in [26].

Finally, the above results enable us to approximate the problem via finite dimensional LP problem:

$$\text{minimize} \ \sum_{k=1}^{L} \rho_k g(z_k), \tag{23}$$

subject to :

$$\sum_{k=1}^{L} \rho_k k_{ij}(z_k) - u(x_i, y_j) = -f(x_i, y_j),$$

$$(i = 0, 1, \cdots, M), \quad (j = 0, 1, \ldots, N), \tag{24}$$

$$\sum_{k=1}^{L} \rho_k \eta_{vw}(z_k) = \alpha_{\eta_{vw}},$$

$$(v = 1, \cdots, G), \quad (w = 1, \cdots, H), \tag{25}$$

$$\sum_{k=1}^{L+1} \rho_k = ST, \tag{26}$$

$u(x_i, y_j)$ is free,

$$(i = 0, 1, \ldots, M), \ (j = 0, 1, \ldots, N), \tag{27}$$

$$\rho_k^* \geq 0 \ (k = 1, 2, \ldots, MN + GH), \tag{28}$$

where $L >> MN + GH$ and $z_k, \ k = 1, ...L$ are fixed in σ. It is to be noted that we added a slack variable ρ_{L+1} for obtaining equality in Eq. (19).

In the problem (23)-(28), Ω is partitioned into L subregions $\Omega_1, \Omega_2, ..., \Omega_L$ where $\Omega = \bigcup_{k=1}^{L} \Omega_k$ and z_k is chosen in Ω_k. To this means, assume that $I = [a, b]$ is divided to m portions, $J = [c, d]$ to n portions and U to p portions, that is $L = mnp$. In application, the functions η_{vw} in Eq. (25) are chosen as piecewise constant. Let us define:

$$\eta_{vw}(s, t, u) = \begin{cases} 1 & \text{if } \ t \in J_{vw} \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

where:

$$J_{vw} = \left[\frac{(v-1)S}{m}, \frac{vS}{m} \right] \times \left[\frac{(w-1)T}{n}, \frac{wT}{n} \right],$$

$$(v = 1, 2, \ldots, m), \ (w = 1, 2, \ldots, n),$$

and we set $G = m$ and $H = n$. In the right-hand side of Eq. (25), $\alpha_{\eta_{vw}}$ is the integral of $\eta_{vw}(t, u)$ on $[a, b] \times [c, d]$, so by Eq. (29) we have:

$$\alpha_{\eta_{vw}} = \int_{J_{vw}} \eta_{vw}(s, t, u(s, t)) dt = \frac{ST}{mn},$$

$$(v = 1, 2, \ldots, m) \ (w = 1, 2, \ldots, n). \tag{30}$$

From the above relations and expanding Eq. (25), we have:

$$\sum_{k=1}^{p} \rho_k = \frac{ST}{mn},$$

$$\sum_{k=p+1}^{2p} \rho_k = \frac{ST}{mn},$$

$$\vdots$$

$$\sum_{k=(mn-2)p+1}^{(mn-1)p} \rho_k = \frac{ST}{mn},$$

$$\sum_{k=(mn-1)p+1}^{mnp} \rho_k = \frac{ST}{mn}.$$

Adding the above equalities leads to:

$$\sum_{k=1}^{L} \rho_k = ST. \tag{31}$$

Comparing Eqs. (26) and (31) guarantees that $\rho_{L+1} = 0$.

From the above analysis, problem (23)-(28) can be converted to the following LP problem:

$$\text{minimize} \sum_{k=1}^{L} \rho_k g(z_k), \tag{32}$$

$$\text{subject to :}$$

$$\begin{cases} \sum_{k=1}^{L} \rho_k k_{ij}(z_k) - u(x_i, y_j) = -f(x_i, y_j), \\ \qquad (i = 0, 1, \cdots, M), \ (j = 0, 1, \ldots, N), \\ \sum_{k=1}^{L} \rho_k = ST, \\ u(x_i, y_j) \text{ is free}, (i = 0, 1, \ldots, M), \\ \qquad (j = 0, 1, \ldots, N), \\ \rho_k \geq 0, (k = 1, 2, \ldots, L). \end{cases} \tag{33}$$

5. Computer simulations

In this section, we propose our method to obtain approximate solution of two dimensional Fredholm integral equations. To compare the solutions, we define an extension of error function proposed in [14]:

$$e(x_i, y_j) = u(x_i, y_j) - u^*(x_i, y_j),$$

$$i = 0, 1, ..., M, \tag{34}$$

where we suppose $u(x, y)$ be exact solution of nonlinear Fredholm integral equation (1) and $u^*(x_i, y_j)$, $i = 0, 1, ..., M$, $j = 1, ..., N$ be a solution obtained by solving the final LP problem.

Example 1. Consider the following two-dimensional nonlinear Fredholm integral equation with the exact solution $u(x, y) = \frac{1}{(1+x+y)^2}$ [11]:

$$u(x, y) = f(x, y) + \int_0^1 \int_0^1 \frac{x}{1+y}(1+s+t)u^2(s, t)dtds,$$

where:

$$f(x, y) = \frac{1}{(1+x+y)^2} - \frac{x}{6(1+y)}.$$

We choose $M = N = 4$, $m = p = 10$ and $n = 5$. Thus $\Omega = [0, 1] \times [0, 1] \times [-1.5, 1.5]$ is divided to $L = 250$ equal subintervals. We select $z_k = (s_k, t_k, u_k)$ and $k = 1, 2, ..., 250$, as:

$$k = e + 5(f-1) + 25(g-1),$$

$$(e = 1, 2, ..., 10), \quad (f, g = 1, 2, \cdots, 5),$$

$$z_k = \begin{cases} s_k = \frac{1}{5}f, \\ t_k = \frac{1}{10}e, \\ u_k = -1.5 + \frac{3}{5}g. \end{cases}$$

Thus the corresponding LP model is:

$$\begin{cases} \text{minimize } \mathbf{0}^t \rho \\ \text{subject to} \\ -\sum_{k=1}^{250} \rho_k \frac{x_i}{1+y_j}(1 + s_k + t_k)u_k^2 + u(x_i, y_j) \\ \qquad = \frac{1}{(1+x_i+y_j)^2} - \frac{x_i}{6(1+y_j)}, \ (i, j = 0, 1, 2, 3, 4), \\ \sum_{k=1}^{250} \rho_k = 1 \\ \rho_k \geq 0, \ k = 1, 2, \cdots, 250, \ \mathbf{0}^t = (\underbrace{0, 0, ...0}_{250}), \\ \qquad \rho^t = (\rho_1, \rho_2, ...\rho_{250}). \end{cases}$$

One can compare the approximate and exact solutions of the integral equation in Figures 1 and 2. The error function (Eq. (34)) can be seen in Figure 3. The numerical results are also compared in Table 1.

Example 2. As the second example, consider the following integral equation with the exact solution $u(x, y) = -\log\left(1 + \frac{xy}{(1+y^2)}\right)$ [19]:

$$u(x, y) = f(x, y) + \int_0^1 \int_0^1 k(x, y, s, t, u(s, t))dtds,$$

where:

$$f(x, y) = \frac{x}{16(1+y)} - \log\left(1 + \frac{xy}{(1+y^2)}\right),$$

$$k(x, y, s, t, u(s, t)) = \left(\frac{x(1-t^2)}{(1+y)(1+s^2)}\right)(1 - e^{-u(s,t)}).$$

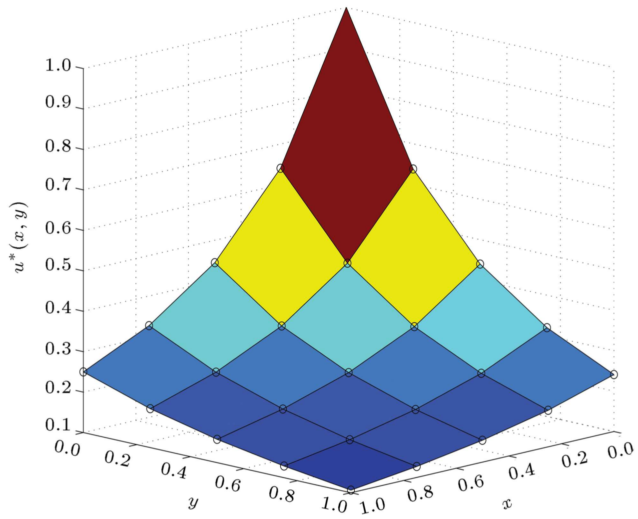

Figure 1. The surface of the approximate solution $u^*(x,y)$ in Example 1.

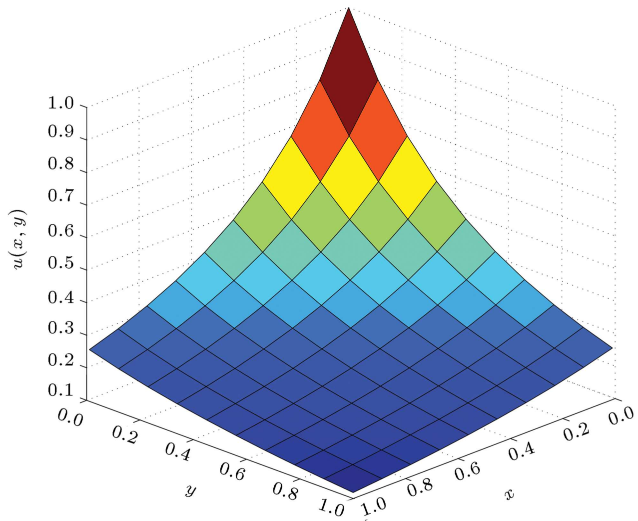

Figure 2. The surface of the exact solution $u(x,y)$ in Example 1.

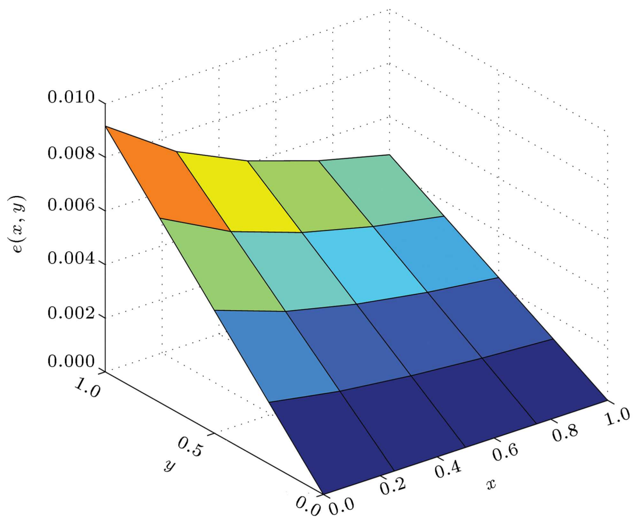

Figure 3. The error function $e(x,y)$ in Example 1.

Table 1. The results for Example 1 with $(x_i,y_j)=(\frac{i}{4},\frac{j}{4})(i,j=0,1,...,4)$.

(x_i,y_j)	$u^*(x_i,y_j)$	$u(x_i,y_j)$	$e(x_i,y_j)$
(0, 0)	1.0000	1.0000	0.0000
(0, 0.25)	0.6400	0.6400	0.0000
(0, 0.5)	0.4444	0.4444	0.0000
(0, 0.75)	0.3265	0.3265	0.0000
(0, 1)	0.2500	0.2500	0.0000
(0.25, 0)	0.6400	0.6400	0.0023
(0.25, 0.25)	0.4426	0.4444	0.0018
(0.25, 0.5)	0.3250	0.3265	0.0015
(0.25, 0.75)	0.24870	0.2500	0.0013
(0.25, 1)	0.1964	0.1975	0.0011
(0.5, 0)	0.4399	0.4444	0.0046
(0.5, 0.25)	0.3229	0.3265	0.0037
(0.5, 0.5)	0.2464	0.2500	0.0031
(0.5, 0.75)	0.1949	0.1975	0.0026
(0.5, 1)	0.1577	0.1600	0.0023
(0.75, 0)	0.3179	0.3265	0.0069
(0.75, 0.25)	0.2445	0.2500	0.0055
(0.75, 0.5)	0.1929	0.1975	0.0046
(0.75, 0.75)	0.1561	0.1600	0.0039
(0.75, 1)	0.1288	0.1322	0.0034
(1, 0)	0.2408	0.2500	0.0092
(1, 0.25)	0.1902	0.1975	0.0073
(1, 0.5)	0.1539	0.1600	0.0061
(1, 0.75)	0.1270	0.1322	0.0052
(1, 1)	0.1065	0.1111	0.0046

In this example, we choose $M = N = 4$ and $m = n = p = 10$. We employe again the LP model (33) and obtain Figures 4 and 5. The error function in Figure 6 also shows the precision of the approximate solution. The numerical results are summarized in Table 2.

To end this section, we answer a natural question: Are there advantages of our proposed method compared to the existing ones? To answer this, we summarize what we have observed from numerical experiments and theoretical results as below:

- Comparison of the results of the above examples with some other methods such as the proposed schemes in [11,15,18,19,25], shows the efficiency of this algorithm more clearly. This result is intuitive, since the results of this algorithm depend explicitly on the slack variables of the final LP problem (32) and (33).
- The proposed transformation method in this article can also allow us to transform easily and efficiently the different kinds of the integral equation problems into an optimization problem.

Table 2. The results for Example 2 with $(x_i, y_j) = (\frac{i}{4}, \frac{j}{4})(i, j = 0, 1, ..., 4)$.

(x_i, y_j)	$u^*(x_i, y_j)$	$u(x_i, y_j)$	$e(x_i, y_j)$
(0, 0)	0.0000	0.0000	0.0000
(0, 0.25)	0.0000	0.0000	0.0000
(0, 0.5)	0.0000	0.0000	0.0000
(0, 0.75)	0.0000	0.0000	0.0000
(0, 1)	0.0000	0.0000	0.0000
(0.25, 0)	0.0156	0.0000	-0.0156
(0.25, 0.25)	-0.0447	-0.0572	-0.0125
(0.25, 0.5)	-0.0849	-0.0953	-0.0104
(0.25, 0.75)	-0.1044	-0.1133	-0.0089
(0.25, 1)	-0.1100	-0.1178	-0.0078
(0.5, 0)	0.0312	0.0000	-0.0312
(0.5, 0.25)	-0.0862	-0.1112	-0.0250
(0.5, 0.5)	-0.1615	-0.1823	-0.0280
(0.5, 0.75)	-0.1973	-0.2151	-0.0178
(0.5, 1)	-0.2075	-0.2231	-0.0156
(0.75, 0)	0.0469	0.0000	-0.0469
(0.75, 0.25)	-0.1250	-0.1625	-0.0375
(0.75, 0.5)	-0.2311	-0.2624	-0.0312
(0.75, 0.75)	-0.2807	-0.3075	-0.0268
(0.75, 1)	-0.2950	-0.3185	-0.0234
(1, 0)	0.0625	0.0000	-0.0625
(1, 0.25)	-0.1613	-0.2113	-0.0500
(1, 0.5)	-0.2948	-0.3365	-0.0416
(1, 0.75)	-0.3563	-0.3920	-0.0357
(1, 1)	-0.3742	-0.4055	-0.0312

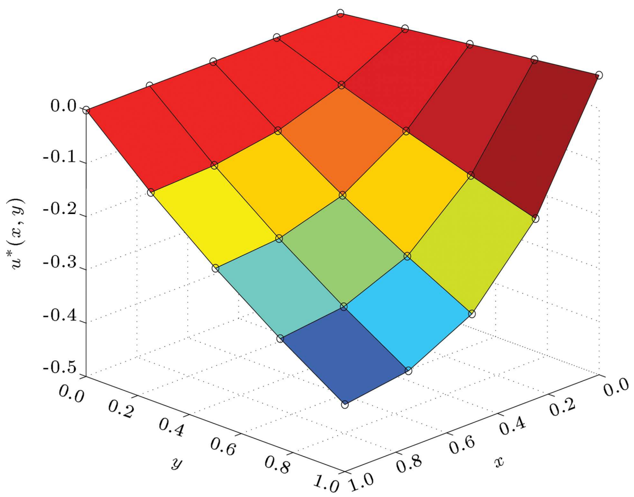

Figure 4. The surface of the approximate solution $u^*(x, y)$ in Example 2.

- Since the procedure of this algorithm is not iterative and does not need any initial guess of the solution, subsequently, appears that the applied method in this paper is very easy to use and straightforward in comparison with other numerical methods.

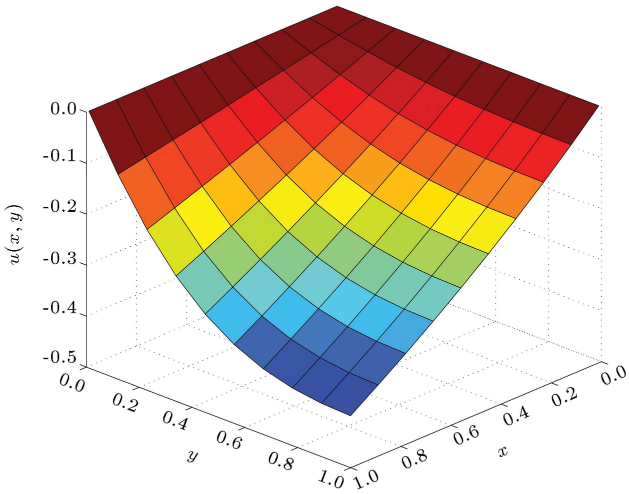

Figure 5. The surface of the exact solution $u(x, y)$ in Example 2.

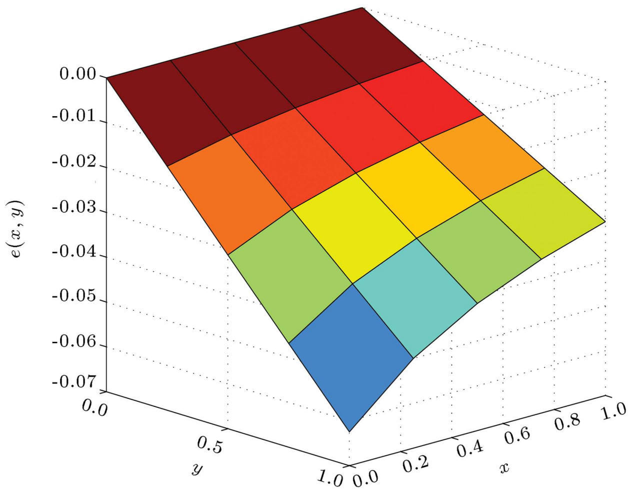

Figure 6. The error function $e(x, y)$ in Example 2.

6. Conclusion

In this paper, we investigated an optimization technique for solving two dimensional nonlinear Fredholm integral equations of the second kind. The integral equation problem was transformed into an approximating optimization problem, and the embedding method based on some principles of measure theory, functional analysis and linear programming was applied for solving this integral equation. The method is not iterative and it does not need any initial guess of the solution. Furthermore, in this approach the nonlinearity of the continuous kernels has not serious effects on the solution.

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Biography

Alireza Nazemi received his BSc degree from Sharif University of Technology, Tehran, Iran, in 2001, his MSc degree from Hakim Sabzevari University, Sabzevar, Iran, in 2003, and his PhD degree from Ferdowsi University of Mashhad, Mashhad, Iran, in 2008, all in applied mathematics. He is currently an Associate Professor at University of Shahrood, Iran. His current research interests include control and optimization, distributed parameter systems, and neural network theory.