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Intuitionistic random multi-criteria decision-making approach based on prospect theory with multiple reference intervals

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Abstract. With respect to Multi-Criteria Decision-Making (MCDM) problems under both stochastic and intuitionistic fuzzy uncertainties, this paper proposes an intuitionistic random MCDM approach based on prospect theory. The reference point in prospect theory is affected by many factors that result in difficulties for the determination of Decision Makers (DMs). This paper develops an approach to acquire multiple reference points in the form of interval numbers to support a certain alternative to be the most preferred one, thus, helping DM to find a satisfying solution by comparison with her/his own preference. Meanwhile, a novel score function of Intuitionistic Fuzzy (IF) number is proposed, based on DM's psychology of loss aversion, and a distance measure of the IF set is proposed as well, considering fully its actual meaning. Finally, we illustrate the effectiveness and practicality of the proposed method through a numerical example.

1. Introduction

Multi-Criteria Decision-Making (MCDM) problems are usually under various uncertainties, which can generally be considered using fuzzy set theory. Since Zadeh introduced the fuzzy set theory in 1965 [1], several extensions of fuzzy sets have been proposed, an important extension of which is the Intuitionistic Fuzzy (IF) set, introduced by Atanassov [2]. Compared with the fuzzy set, the IF set seems to be more suitable for expressing a very important factor, namely, the hesitation of the Decision Maker (DM), which should be taken into account in actual decision making problems.

Over the last decades, the IF set theory has been widely applied to MCDM problems [3-15]. In order to rank IF numbers, Chen and Tan [16] put forward a score function, which various researchers have continuously improved upon [17-22]. In 2012, Wang et al. [23] fully analyzed the limitations of these score functions and proposed a prospect score function, which showed greater priority over the others. However, the prospect score function is quite complex, and further, it is not flexible enough to make modifications according to DM's risk attitudes.

Another important measure of the IF set is the distance measure. The well acknowledged distance measure of IF sets was proposed by Szmidt and Kacprzyk [24], and has good geometric properties. However, the corresponding outcomes often result in a contradiction with actual meaning in real life, due to the IF number special size measurement. In other words, the contribution of the three elements (the degree of membership, non-membership and hesitation) to its size in the IF number is different from that in a regular real number. In view of this, Wang and Xin [25] made some improvements on Szmidt and Kacprzyk's method, but the fundamental problem has still not been solved.

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Currently, most of the existing intuitionistic MCDM models and methods with random backgrounds are based on expected utility theory, which assumes that the DMs are totally rational. But, in real life decision making processes, actual decision making behavior usually departs from the predictions of the expected utility theory, due to the ambiguity of the problems, individual cognitive limitations and lack of knowledge. As a descriptive model, the prospect theory, proposed by Kahneman and Tversky [26,27], has attracted a good deal of attention for its descriptive power, which well reflects the DMs' actual decision making behavior.

Some researchers have applied prospect theory to MCDM problems. Based on the central idea of prospect theory, Lahdelma and Salminen proposed SMAA-P [28], which is an extension of the SMAA method. They analyzed the type of importance weights and reference alternatives that would support each alternative for any given rank, which provided a good way to deal with different kinds of imprecise, uncertain or missing preferences. However, the authors in this paper fixed each alternative as the reference point, which might be quite different from DMs' actual reference points. Besides, Hu and Zhou [29] proposed a MCDM method for the risk decision making problem based on prospect theory. Wang et al. [30] proposed a fuzzy MCDM method with prospect theory, based on multiple criteria decision making problems in which the criteria weight was unknown and the criteria values of the alternative were in the form of trapezoidal fuzzy numbers. Hu et al. [31] proposed a MCDM method for risk decision making problems with linguistic evaluation information based on prospect theory, and Liu et al. [32] proposed a MCDM method based on prospect theory to deal with risk decision making problems with interval probability in which the attribute values were in the form of uncertain linguistic variables. The abovementioned research all assumes, excepting the first, that the reference point is an accurate value that can be determined by DMs. However, it is actually the status quo that a DM may be affected by many factors. Under the same condition, different DMs may have different reference points, and, for the same DM, when she/he confronts different conditions, the reference point may be different. Due to the uncertainty of the problems and the complexity of conditions, it is very difficult for a DM to determine a precise reference point.

From the above literature review, we know that the existing score functions and distance formulas of the IF set have their limitations. Besides, the existing MCDM methods based on prospect theory (except the SMAA-P method), only consider one accurate reference point, which is not easy to determine in practical situations.

As to the first problem, we propose a new score function based on DM's psychology of loss aversion. Meanwhile, a new distance measure of IF sets is defined based on the new proposed score function, transforming the IF number into a real number. Compared with the existing score functions and distance measures, those newly proposed can lead to more reasonable results in line with actual situations.

As to the second problem, we propose a method to produce multiple reference points in the form of interval numbers. First, we find the worst point and optimal point from the alternatives for each criterion. Then, we divide the distance between the worst and optimal points into multiple parts, and set each part as a reference interval equivalent to a reference point. Eventually, we calculate the prospect values of the alternatives based on multiple reference intervals. The outcomes can offer the types of reference intervals that support a certain alternative, and help the DM to find a satisfying solution by comparison with her/his own preference. The multiple reference intervals have some advantages over the single reference point. Firstly, when facing complex conditions, it is almost impossible for a DM to determine a precise reference point, while it is easier to determine a scope, which is much closer to DM's mental status when making decisions. Secondly, when the DM is quite confused about the reference point, the multiple references and corresponding outcomes can help the DM to find a reasonable solution, which has the most support of multiple references. Last but not least, multiple intervals, which are selected as the references, reveal the different risk attitudes of the DMs to some degree (the closer to the optimal point, the more risk preference of the DM; the closer to the worst point, the more loss aversion of the DM). Hence, the final outcomes can help the DM understand how different risk attitudes correspond to different choices.

The rest of this paper is organized as follows. In Section 2, we review some basic concepts of the IF set, interval numbers and prospect theory. Then, we put forward a new score function and distance formula of an IF set and make full comparisons with other methods in Section 3. In Section 4, the MCDM method based on prospect theory, under an intuitionistic random environment, is proposed. In Section 5, a numerical example is illustrated. Finally, some conclusions are given in Section 6.

2. Preliminaries

2.1. Intuitionistic fuzzy set

Definition 1 [2,33]. Let X be a finite universal set. Then, an IF set, $\tilde{A}$, in X is an object having the following form:

$$\tilde{A} = \{< x, u_{\tilde{A}}(X), v_{\tilde{A}}(x) > | x \in X\},$$

where $u_{\tilde{A}}, v_{\tilde{A}} : X \to [0,1]$ are functions satisfying $0 \leq u_{\tilde{A}}(x) + v_{\tilde{A}}(x) \leq 1$, $\forall x \in X$.

For each x, numbers $u_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ represent the degree of membership and degree of non-membership of the element, $x \in X$, to set $\tilde{A}$, respectively.

Let $\pi_{\tilde{A}} = 1 - u_{\tilde{A}}(x) - v_{\tilde{A}}(x)$ be the intuitionistic index of element x in set $\tilde{A}$ [34,35], which is the degree of indeterminacy membership of element x to set $\tilde{A}$. Obviously, $0 \leq \pi_{\tilde{A}}(x) \leq 1$. In practice, if X is a singleton, then IF set $\tilde{A} = \{< x, u_{\tilde{A}}(x), v_{\tilde{A}}(x) > |x \in X\}$ degenerates to an IF number, which can be denoted as $A =< u_A, v_A >$ or $A =< u_A, u_A, \pi_A >$.

As for real life problems, symbols u_A, v_A and π_A can denote the shares of supporters, dissenters and abstention groups for alternative, A, respectively.

Definition 2 [2,36]. Let $\tilde{A}$ and $\tilde{B}$ be two IF sets in set X, and $\beta \geq 0$, then:

1. $\tilde{A} + \tilde{B} = \Big\{< x, u_{\tilde{A}}(x) + u_{\tilde{B}}(x) - u_{\tilde{A}}(x).u_{\tilde{B}}(x), v_{\tilde{A}}(x).v_{\tilde{B}}(x) > |x \in X\Big\}$;
2. $\tilde{A}.\tilde{B} = \Big\{< x, u_{\tilde{A}}(x).u_{\tilde{B}}(x), v_{\tilde{A}}(x) + v_{\tilde{B}}(x) - v_{\tilde{A}}(x).v_{\tilde{B}}(x) > |x \in X\Big\}$;
3. $\tilde{A}^{\beta} = \Big\{< x, u_{\tilde{A}}(x)^{\beta}, 1 - (1 - v_{\tilde{A}}(x))^{\beta} > |x \in X\Big\}$;
4. $\beta\tilde{A} = \Big\{< x, 1 - (1 - u_{\tilde{A}}(x))^{\beta}, v_{\tilde{A}}(x)^{\beta} > |x \in X\Big\}$.

Definition 3 [16]. Let $A =< u_A, v_A >$ be an IF number. Then, the score function of A is defined as follows:

$$\Delta(A) = u_A - v_A.$$

Obviously, $\Delta(A) \in [-1, 1]$.

Definition 4 [17]. Let $A =< u_A, v_A >$ be an IF number. Then, the accuracy function of A is defined as follows:

$$\sigma(A) = u_A + v_A = 1 - \pi_A,$$

where $\pi_A = 1 - u_A - v_A$.

Obviously, $\sigma(A) \in [0, 1]$, which represents the non-hesitation degree.

To rank two IF numbers, $A =< u_A, v_A >$ and $B =< u_B, v_B >$, their score functions and accuracy functions need to be compared, respectively. In general, the score function is more important than the accuracy function. Based on this principle, the ranking rules of IF numbers can be defined as follows:

1. If $\Delta(A) > \Delta(B)$, then, A is larger than B;
2. If $\Delta(A) = \Delta(B)$, then;
   (a) If $\sigma(A) = \sigma(B)$, then, A is equal to B;
   (b) If $\sigma(A) < \sigma(B)$, then, A is smaller than B;
   (c) If $\sigma(A) > \sigma(B)$, then, A is larger than B.

2.2. Interval numbers

Definition 5 [37]. Suppose A is an interval number, then its form is as follows:

$$A = [a_L, a_R] = \{a | a_L \leq a \leq a_R\},$$

where a_L and a_R are the left and right limit of interval A on the real line, R, respectively. If $a_L = a_R$, then, A degenerates to a real number.

Definition 6 [38]. Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be two interval numbers. Then, the Hamming distance between them is defined as follows:

$$d(A, B) = \frac{1}{2}(|a_L - b_L| + |a_R - b_R|). \tag{1}$$

If a_L, a_R and b_L all take the form of IF numbers, then the Hamming distance between A and B can be expressed as follows:

$$d(A, B) = \frac{1}{2}[d(a_L, b_L) + d(a_R, b_R)]. \tag{2}$$

Definition 7 [39]. Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be two interval numbers. Then, the comparing rules are defined as follows:

$$\text{If} \quad \frac{a_L + a_R}{2} > \frac{b_L + b_R}{2}, \text{ then,} \quad A > B,$$

$$\text{If} \quad \frac{a_L + a_R}{2} = \frac{b_L + b_R}{2}, \text{ then,} \quad A = B,$$

$$\text{If} \quad \frac{a_L + a_R}{2} < \frac{b_L + b_R}{2}, \text{ then,} \quad A < B.$$

2.3. Prospect theory

In prospect theory, the outcome is determined by the prospect value, which is a combination of value function and probability weight function [26], shown as follows:

$$V = \sum_{i=1}^{n} w(p_i) v(\Delta x_i), \tag{3}$$

where V is the prospect value; $w(p)$ is the probability weight function of the probability assessment; and $v(\Delta x)$ is the value function of the subjective feelings of the DM. Here, Δx_i is used to measure the difference between x_i and a certain reference point, x_0. If the outcome is larger than the reference point, then, we perceive the outcome as gain; otherwise, we perceive the outcome as loss.

In this paper, as there is no subtraction operator for the IF number, we define Δx_i as the distance between x_i and x_0, which is given as follows:

$$\Delta x_i = \begin{cases} d(x_i, x_0), & x_i \geq x_0 \\ -d(x_i, x_0), & x_i < x_0 \end{cases} \tag{4}$$

where x_i and x_0 take the form of IF numbers. Then we introduce the value function and the probability weight function as follows:

1. The value function.

   The value function proposed by Kahneman and Tversky [27] is as follows:

$$v(x) = \begin{cases} (x)^\alpha, & x \geq 0 \\ -\lambda(-x)^\beta, & x < 0 \end{cases} \tag{5}$$

   where x is the gain or the loss of the surface value, which is positive for the gain and negative for the loss; α and β are risk attitude coefficients satisfying $0 < \alpha, \beta < 1$; and λ is the loss aversion coefficient. The larger the parameters of α and β, the more the DMs are willing to seek risk. When $\alpha = \beta = 1$, DMs can be seen as risk neutral, and when $\lambda > 1$, it indicates that the DMs are more sensitive to loss. According to previous research, Kahneman and Tversky [27] considered that $\lambda = 2.25$ and $\alpha = \beta = 0.88$; Wu and Gonzalez [40] considered that $\lambda = 2.25$ and $\alpha = \beta = 0.52$; and Zeng [41] considered that $\lambda = 2.25$, $\alpha = 1.21$ and $\beta = 1.02$.

2. The probability weight function.

   Kahneman and Tversky [26] considered that the probability weight is the subjective judgment of the DM, based on probability p of the outcome. It is neither the probability nor the linear function of probability, but the corresponding weight on the probability. The probability weight function is shown as follows [27]:

$$\begin{cases} w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \\ w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}} \end{cases} \tag{6}$$

   where w^+ and w^- are the weighting function of the gain and the loss, respectively, γ is the risk gain attitude coefficient, and δ is the risk loss attitude coefficient. Scholars at home and abroad have undertakaen research into the coefficient of the weighting function. For example, Kahneman and Tversky [27] suggested $\gamma = 0.61$ and $\delta = 0.72$; Wu and Gonzales [40] suggested $\gamma = 0.74$ and $\delta = 0.74$; and Zeng [41] suggested $\gamma = 0.55$ and $\delta = 0.47$.

3. The score functions and distance formulas of IF set

3.1. The score functions of IF number

Researchers have proposed many score functions during the past 20 years, but most of them still have limitations. Wang et al. [23] made a full analysis of these score functions and proposed a prospect score function to conquer existing limitations. Based on this, in this paper, we propose a new score function based on DMs' degree of loss aversion, and the following examples will show its effectiveness and practicality.

Definition 8. Let $A =< u_A, v_A >$ be an IF number. Then, the new score function is defined as follows:

$$S(A) = u_A + \frac{1}{\lambda}(1 - u_A - v_A) = u_A + \frac{1}{\lambda}\pi_A \quad (\lambda > 1), \tag{7}$$

where $\pi_A = 1 - u_A - v_A$, and λ is the coefficient of loss aversion. Larger λ always means more loss aversion of the DM.

Definition 9. Suppose $A =< u_A, v_A >$ and $B =< u_B, v_B >$ are two IF numbers, then, we get that:

1. If $S(A) > S(B)$, then $A > B$;
2. If $S(A) = S(B)$, then $A = B$;
3. If $S(A) < S(B)$, then $A < B$.

Considering the influence of the abstention group on the score function, many researchers tend to divide the abstention group into three parts, as $u_A\pi_A$, $v_A\pi_A$ and $(1 - u_A - v_A)\pi_A$, to denote the shares of affirmation, dissent and abstention, respectively [18-20]. However, there is no empirical study to support them. Hence, the abstention group is absolutely uncertain, which means we cannot know how many shares of the abstention group tend to affirm, dissent or hesitate if we do not have extra information, such as a full analysis or some investigation into the abstention group. As a consequence, we hold that without any extra information, the abstention group would be absolutely uncertain. According to prospect theory [26-27], people tend to be more sensitive to loss than to gain, and they will be more loss averse when confronting gain. Hence, if the DM regards the abstention group as gain, with dissenters as the reference point, the coefficient assigned to π_A, which is determined by DM's degree of loss aversion, will be inevitably less than 1.

Referring to the loss aversion coefficient, λ, in prospect theory [27], we can also let λ be 2.25. Here, we make a full comparison of our method with other methods, as shown in Table 1.

From Examples 1-8, shown in Table 1, we can see that the outcomes acquired from the new score function

Table 1. Comparison of the results based on different methods.

	Examples	Hong and Choi's method [17]	Wang and Li's method [23]	The new score function ($\lambda = 2.25$)
1	$A =< 0.40, 0.10 >$ $B =< 0.50, 0.20 >$	$A < B$	$S_{\text{pt}}(A) = 0.3017$ $S_{\text{pt}}(B) = 0.4318$ $\Rightarrow A < B$	$S(A) = 0.6222$, $S(B) = 0.6333$ $\Rightarrow A < B$
2	$A =< 0.00, 0.20 >$ $B =< 0.00, 0.30 >$	$A > B$	$S_{\text{pt}}(A) = 0.4316$ $S_{\text{pt}}(B) = 0.5727$ $\Rightarrow A > B$	$S(A) = 0.3556$, $S(B) = 0.3111$ $\Rightarrow A > B$
3	$A =< 0.20, 0.40 >$ $B =< 0.30, 0.10 >$	$A < B$	$S_{\text{pt}}(A) = -0.4456$ $S_{\text{pt}}(B) = 0.2876$ $\Rightarrow A < B$	$S(A) = 0.3778$, $S(B) = 0.5667$ $\Rightarrow A < B$
4	$A =< 0.30, 0.40 >$ $B =< 0.20, 0.10 >$	$A < B$	$S_{\text{pt}}(A) = -0.1454$ $S_{\text{pt}}(B) = -0.1199$ $\Rightarrow A < B$	$S(A) = 0.4333$, $S(B) = 0.5111$ $\Rightarrow A < B$
5	$A =< 0.40, 0.20 >$ $B =< 0.20, 7/15 >$	$A > B$	$S_{\text{pt}}(A) = 0.3460$ $S_{\text{pt}}(B) = -0.5479$ $\Rightarrow A > B$	$S(A) = 0.5778$, $S(B) = 0.4074$ $\Rightarrow A > B$
6	$A =< 0.40, 0.20 >$ $B =< 0.60, 0.30 >$	$A < B$	$S_{\text{pt}}(A) = 0.0460$ $S_{\text{pt}}(B) = 0.3560$ $\Rightarrow A < B$	$S(A) = 0.5778$, $S(B) = 0.6444$ $\Rightarrow A < B$
7	$A =< 0.30, 0.60 >$ $B =< 0.30, 0.50 >$	$A < B$	$S_{\text{pt}}(A) = -0.3390$ $S_{\text{pt}}(B) = -0.2579$ $\Rightarrow A < B$	$S(A) = 0.3444$, $S(B) = 0.3889$ $\Rightarrow A < B$
8	$A =< 0.60, 0.04 >$ $B =< 0.68, 0.12 >$	$A < B$	$S_{\text{pt}}(A) = 0.6739$ $S_{\text{pt}}(B) = 0.6937$ $\Rightarrow A < B$	$S(A) = 0.7600$, $S(B) = 0.7689$ $\Rightarrow A < B$
9	$A =< 0.40, 0.10 >$ $B =< 0.60, 0.36 >$	$A > B$	$S_{\text{pt}}(A) = 0.1641$ $S_{\text{pt}}(B) = 0.2565$ $\Rightarrow A < B$	$S(A) = 0.6222$, $S(B) = 0.6178$ $\Rightarrow A > B$
10	$A =< 0.50, 0.50 >$ $B =< 0.30, 0.20 >$	$A < B$	$S_{\text{pt}}(A) = 0.0000$ $S_{\text{pt}}(B) = -0.1463$ $\Rightarrow A > B$	$S(A) = 0.5000$, $S(B) = 0.5222$ $\Rightarrow A < B$
11	$A =< 0.30, 0.00 >$ $B =< 0.60, 0.36 >$	$A > B$	$S_{\text{pt}}(A) = -0.0516$ $S_{\text{pt}}(B) = 0.2565$ $\Rightarrow A < B$	$S(A) = 0.6111$, $S(B) = 0.6178$ $\Rightarrow A < B$

are in accordance with results from the prospect score function, so the score function proposed in this paper can also solve the limitations of the existing functions, except the prospect score function of Wang and Li. Examples 9-10 (shown in Table 1) illustrate that the outcomes acquired by the new score function, with $\lambda = 2.25$, are closer to Hong and Choi's results than Wang and Li's prospect score function. Examples 9-11 demonstrate that when π_A is much larger than π_B, and $u_A - v_A$ is just a little larger than $u_B - v_B$, it is probable that alternative, A, will be inferior to alternative, B. In fact, whether alternative A will be inferior to alternative B or not is decided by DM's degree of loss aversion, namely, the value of λ. As example 9 shows, when $\lambda = 2.3 > 2.25$, $S(A) = 0.6174$ and $S(B) = 0.6174$ which indicates $A = B$. It is can be well explained by phenomena in real life. For example, when evaluating two alternatives, A and B, if alternative A has more advantages than alternative B, as well as more disadvantages, then the perceived values of alternative A and B may be the same for a certain DM. If we have to make a choice between alternatives A and B, we can modify the coefficient of loss aversion according to the real life situation.

According to the above analysis, we know that the new score function has several advantages over the prospect score function, which are shown as follows:

1. The form of the new score function is more simple, so its calculation is much easier;
2. In general ($\lambda = 2.25$), the results acquired by the new score function are closer to the results of Hong and Choi [17];
3. For some special DMs, whose sensitivity of loss is much stronger or weaker than the normal, or under some special surroundings in which the DM's degree of loss aversion is very different from that in normal situations, the outcomes acquired by the new score function can be modified by the parameter, λ,

while the prospect score function cannot make the corresponding modification.

3.2. The distance formulas of IF sets

Definition 10 [24]. Let $\tilde{A}=\{<x,u_{\tilde{A}}(x),v_{\tilde{A}}(x)>|x\in X\}$ and $\tilde{B}=\{<x,u_{\tilde{B}}(x),v_{\tilde{B}}(x)>|x\in X\}$ be two IF sets in set $X=\{x_1,x_2,,...,x_n\}$. Then, the Hamming distance between $\tilde{A}$ and $\tilde{B}$ is defined as follows:

$$d\left(\tilde{A},\tilde{B}\right)=\frac{1}{2}\sum_{i=1}^{n}\Big[|u_{\tilde{A}}(x_i)-u_{\tilde{B}}(x_i)|+|v_{\tilde{A}}(x_i)-v_{\tilde{B}}(x_i)|+|\pi_{\tilde{A}}(x_i)-\pi_{\tilde{B}}(x_i)|\Big], \tag{8}$$

where $\pi_{\tilde{A}}(x_i)=1-u_{\tilde{A}}(x_i)-v_{\tilde{A}}(x_i)$ and $\pi_{\tilde{B}}(x_i)=1-u_{\tilde{B}}(x_i)-v_{\tilde{B}}(x_i)$.

If X is a singleton, then $\tilde{A}=\{<x,u_{\tilde{A}}(x),v_{\tilde{A}}(x)>|x\in X\}$ and $\tilde{B}=\{<x,u_{\tilde{B}}(x),v_{\tilde{B}}(x)>|x\in X\}$ degenerate to two IF numbers, $A=<u_A,v_A>$ and $B=u_B,v_B>$, and the distance between them is as follows:

$$d(A,B)=|u_A-u_B|+|v_A-v_B|+|\pi_A-\pi_B|, \tag{9}$$

where $\pi_A=1-u_A-v_A$ and $\pi_B=1-u_B-v_B$.

Obviously, the distance formula of IF sets proposed by Szimidt and Kacprzyk [24] has good geometric property, but it cannot explain well the real meaning of some IF numbers. For example, let $A=<1,0>$, $B=<0.5,0>$ and $C=<0.5,0.5>$ be three IF numbers. Then, by Definition 10, we can obtain that the distance between A and B is equal to that between A and C, which is shown as follows:

$$d(A,B)=\frac{1}{2}\left[|1-0.5|+|0-0|+|0-0.5|\right]=0.5,$$

$$d(A,C)=\frac{1}{2}\left[|1-0.5|+|0-0.5|+|0-0|\right]=0.5.$$

However, according to the real meaning of the IF number, the ranking of A,B and C is $A>B>C$, namely, $d(A,B)<d(A,C)$, which is different from the above result. Consequently, in order to overcome this limitation, Wang and Xin [25] proposed a new distance measure of IF sets, which is expressed as follows:

$$d(\tilde{A},\tilde{B})=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{|u_{\tilde{A}}(x_i)-u_{\tilde{B}}(x_i)|+|v_{\tilde{A}}(x_i)-v_{\tilde{B}}(x_i)|}{4}+\frac{\max\left(|u_{\tilde{A}}(x_i)-u_{\tilde{B}}(x_i)|,|v_{\tilde{A}}(x_i)-v_{\tilde{B}}(x_i)|\right)}{2}\right]. \tag{10}$$

If X is a singleton, then, $\tilde{A}=\{<x,u_{\tilde{A}}(x),v_{\tilde{A}}(x)>|x\in X\}$ and $\tilde{B}=\{<x,u_{\tilde{B}}(x),v_{\tilde{B}}(x)>|x\in X\}$ degenerate to two IF numbers, $A=<u_A,v_A>$ and $B=<u_B,v_B>$, and the distance between them is as follows:

$$d(A,B)=\frac{|u_A-u_B|+|v_A-v_B|}{4}+\frac{\max\left(|u_A-u_B|,|v_A-v_B|\right)}{2}. \tag{11}$$

According to this distance formula, we can get $d(A,B)<d(A,C)$, and the concrete results are shown below:

$$d(A,B)=\left[\frac{|1-0.5|+|0-0|}{4}+\frac{\max\left(|1-0.5|,|0-0|\right)}{2}\right]=0.375,$$

$$d(A,C)=\left[\frac{|1-0.5|+|0-0.5|}{4}+\frac{\max\left(|1-0.5|,|0-0.5|\right)}{2}\right]=0.5.$$

For two IF numbers, $A=<u_A,v_A>$ and $B=<u_B,v_B>$, if there exist $u_A<u_B$ & $v_A<v_B$ or $u_A>u_B$ & $v_A>v_B$, the distance between A and B calculated by the existing distance formulas may be unreasonable. For example, let $A=<0.5,0.5>$, $B=<0.4,0.2>$ and $C=<0.6,0.3>$ be three IF numbers, then, the ranking of A,B and C is $A<B<C$ according to the real meaning of the IF number, indicating $d(A,B)<d(A,C)$, but the results obtained by Eqs. (9) and (11) are both $d(A,B)>d(A,C)$. Therefore, we can conclude that the existing distance formulas of IF numbers have their applicative condition: $u_A<u_B$ & $v_A<v_B$ or $u_A>u_B$ & $v_A>v_B$ cannot coexist for two IF numbers $A=<u_A,v_A>$ and $B=<u_B,v_B>$. However, this condition cannot always hold in real life situations, since the IF set $\tilde{A}$ in X consists of $\tilde{A}(x_i)=<x_i,u_{\tilde{A}}(x_i),v_{\tilde{A}}(x_i)>$ for all $x_i\in X$, which are actually IF numbers.

Definition 11 [25]. Let d: $IFSs(X)\times IFSs(X)\rightarrow[0,1]$ be a real function on X. If $d(\tilde{A},\tilde{B})$ satisfies the following properties:

1. $0\leq d(\tilde{A},\tilde{B})\leq 1$;
2. $d(\tilde{A},\tilde{B})=0$ if and only if $\tilde{A}=\tilde{B}$;
3. $d(\tilde{A},\tilde{B})=d(\tilde{B},\tilde{A})$;
4. If $\tilde{A}\subseteq\tilde{B}\subseteq\tilde{C},\tilde{A},\tilde{B},\tilde{C}\in IFSs(X)$, then $d(\tilde{A},\tilde{C})\geq d(\tilde{A},\tilde{B})$ and $d(\tilde{A},\tilde{C})\geq d(\tilde{B},\tilde{C})$; then $d(\tilde{A},\tilde{B})$ is a distance measure between IF sets, $\tilde{A}$ and $\tilde{B}$.

Definition 12 [2]. Let $\tilde{A}=\{<x,u_{\tilde{A}}(x),v_{\tilde{A}}(x)>|x\in X\}$ and $\tilde{B}=\{<x,u_{\tilde{B}}(x),v_{\tilde{B}}(x)>|x\in X\}$ be two IF sets in set $X = \{x_1, x_2, ..., x_n\}$. Then, we have:

$$\tilde{A} \subseteq \tilde{B} \quad \text{iff} \quad \left(\forall x_i \in X\right) \quad \left(u_{\tilde{A}}(x_i) < u_{\tilde{B}}(x_i) \right.$$

$$\left. \& \ v_{\tilde{A}}(x_i) > v_{\tilde{B}}(x_i)\right),$$

$$\tilde{A} = \tilde{B} \quad \text{iff} \quad \left(\forall x_i \in X\right) \quad \left(u_{\tilde{A}}(x_i) = u_{\tilde{B}}(x_i) \right.$$

$$\left. \& \ v_{\tilde{A}}(x_i) = v_{\tilde{B}}(x_i)\right).$$

In fact, IF numbers are also the kinds of numbers that can be compared with each other. When concerning the practical meaning of two IF numbers, $A =< 0.55, 0.15 >$ and $B =< 0.60, 0.22 >$, most DMs may feel that A is nearly equal to B. Hence, when the score functions of A and B are equal, we can say $A = B$ to the practical sense of the IF number. Similarly, when the score function of A is larger than that of B, we can say $A > B$.

Consequently, we modify Definition 12 as Definition 13, which can better reflect the practical meaning of the IF number.

Definition 13. Let $\tilde{A} = \{< x, u_{\tilde{A}}(x), v_{\tilde{A}}(x) > | x \in X\}$ and $\tilde{B} = \{< x, u_{\tilde{B}}(x), v_{\tilde{B}}(x) > | x \in X\}$ be two IF sets, in set $X = \{x_1, x_2, ..., x_n\}$. Then, we have:

$$\tilde{A} \subseteq \tilde{B} \ \text{iff} \ \left(\forall x_i \in X\right) \ \ S\left(\tilde{A}(x_i)\right) \leq S\left(\tilde{B}(x_i)\right),$$

$$\tilde{A} = \tilde{B} \ \text{iff} \ \left(\forall x_i \in X\right) \ \ S\left(\tilde{A}(x_i)\right) = S\left(\tilde{B}(x_i)\right).$$

Here, $S\left(\tilde{A}(x_i)\right)$ and $S\left(\tilde{B}(x_i)\right)$ are the score functions of the IF number, $\tilde{A}(x_i)$ and $\tilde{B}(x_i)$, respectively.

Based on the new score function and Definition 13 proposed in this paper, we put forward a novel distance measure of IF sets, which can make up for the deficiency of the existing ones.

Definition 14. Let $\tilde{A} = \{< x, u_{\tilde{A}}(x), v_{\tilde{A}}(x) > | x \in X\}$ and $\tilde{B} = \{< x, u_{\tilde{B}}(x), v_{\tilde{B}}(x) > | x \in X\}$ be two IF sets in set $X = \{x_1, x_2, ..., x_n\}$, and denote that:

$$d(\tilde{A}, \tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\left|S\left(\tilde{A}(x_i)\right) - S\left(\tilde{B}(x_i)\right)\right|$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left|\left(u_{\tilde{A}}(x_i) + \frac{1}{\lambda}\pi_{\tilde{A}}(x_i)\right) - \left(u_{\tilde{B}}(x_i) + \frac{1}{\lambda}\pi_{\tilde{B}}(x_i)\right)\right|, \tag{12}$$

where $\pi_{\tilde{A}}(x_i) = 1 - u_{\tilde{A}}(x_i) - v_{\tilde{A}}(x_i)$ and $S\left(\tilde{A}(x_i)\right)$ and $S\left(\tilde{B}(x_i)\right)$ are the score functions of the IF number, $\tilde{A}(x_i)$ and $\tilde{B}(x_i)$, respectively.

When X is a singleton, $\tilde{A} = \{< x, u_{\tilde{A}}(x), v_{\tilde{A}}(x) > | x \in X\}$ and $\tilde{B} = \{< x, u_{\tilde{B}}(x), v_{\tilde{B}}(x) > | x \in X\}$ degenerate to two IF numbers, $A =< u_A, v_A >$ and $B = u_B, v_B >$. Then, the distance between them is as follows:

$$d(A, B) = |S(A) - S(B)|$$
$$= \left|\left(u_A + \frac{1}{\lambda}\pi_A\right) - \left(u_B + \frac{1}{\lambda}\pi_B\right)\right|, \tag{13}$$

where $\pi_A = 1 - u_A - v_A$, and $S(A)$ and $S(B)$ are the score functions of IF numbers, A and B, respectively.

Theorem 1. $d(\tilde{A}, \tilde{B})$ defined in Definition 14 is the distance measure of IF sets, $\tilde{A}$ and $\tilde{B}$, in the set $X = \{x_1, x_2, ..., x_n\}$.

Proof. According to Definition 13, it is obvious that $d(\tilde{A}, \tilde{B})$ satisfies properties (2)-(4) in Definition 11. So, we only need to prove $d(\tilde{A}, \tilde{B})$ satisfies property (1).

The new score function for any IF number, $\tilde{A}(x_i)$, is: $S\left(\tilde{A}(x_i)\right) = u_{\tilde{A}}(x_i) + \frac{1}{\lambda}\pi_{\tilde{A}}(x_i)$ ($\lambda > 1$). Since $\lambda > 1$, we can get $1 - \frac{1}{\lambda} > 0$, $-\frac{1}{\lambda} < 0$.

Hence, the larger the $u_{\tilde{A}}(x_i)$, the larger the $S\left(\tilde{A}(x_i)\right)$, while the larger the $v_{\tilde{A}}(x_i)$, the smaller the $S\left(\tilde{A}(x_i)\right)$.

Besides, since $0 \leq u_{\tilde{A}}(x_i) \leq 1$, $0 \leq v_{\tilde{A}}(x_i) \leq 1$ and $0 \leq u_{\tilde{A}}(x_i) + v_{\tilde{A}}(x_i) \leq 1$, we can get $\max\left(S(\tilde{A}(x_i))\right) = 1(u_{\tilde{A}}(x_i)=1 v_{\tilde{A}}(x_i) = 0)$, $\min\left(S(\tilde{A}(x_i))\right) = 0(u_{\tilde{A}}(x_i) = 0,\ v_{\tilde{A}}(x_i) = 1)$, namely, $S\left(\tilde{A}(x_i)\right) \in [0, 1]$.

Since $d(\tilde{A}, \tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\left|S(\tilde{A}(x_i)) - S(\tilde{B}(x_i))\right|$, where $S\left(\tilde{A}(x_i)\right) \in [0, 1]$, $S\left(\tilde{B}(x_i)\right) \in [0, 1]$, it is obvious that $\left|S\left(\tilde{A}(x_i)\right) - S\left(\tilde{B}(x_i)\right)\right| \in [0, 1]$. As a consequence, $0 \leq d(\tilde{A}, \tilde{B}) \leq 1$, which means that $d(\tilde{A}, \tilde{B})$ satisfies property (1).

Therefore, $d(\tilde{A}, \tilde{B})$ is the distance measure of two IF sets, $\tilde{A}$ and $\tilde{B}$, in set X. The proof is completed.

Researchers have made many studies of the distance measure of IF numbers, but their proposed methods do not always make sense. The actual reason is that IF numbers cannot be compared with each other as real numbers. As a result, for three IF numbers, A, B and C, $A > B > C$ does not always mean that $d(A, C) \geq d(A, B)$, according to traditional distance formulas. In this paper, we develop a new distance formula from the perspective of perceived values of the IF number, namely, the score function, which can well satisfy $d(A,C) \geq d(A,B)$, if $A > B > C$ for three IF numbers, A, B and C. Although our proposed method does not have good geometric properties, it conforms better to practical situations.

4. Intuitionistic random MCDM method based on prospect theory

4.1. Intuitionistic random MCDM problems with multiple reference intervals

Suppose there exists an alternative set, $A = \{A_1, A_2, ..., A_m\}$, consisting of m alternatives, from which the best alternative is to be selected, and that each alternative is assessed on n criteria, denoted by $c = \{c_1, c_2, ..., c_n\}$. $X_{ij}(i = 1, 2, .., m; j = 1, 2, ..., n)$ is the evaluation of the alternative, with respect to $c_j(j = 1, 2, ..., n)$, in the form of IF number, $< u_{ij}, v_{ij} >$, where u_{ij} and v_{ij} are the degree of membership and the degree of non-membership of the alternative, A_i, with respect to the criterion, c_j, for the fuzzy concept "excellence", respectively, and it reveals the degree of satisfaction or dissatisfaction of the DM to criterion c_j for alternative A_i. Let $\theta_j = \{\theta_{j1}, \theta_{j2}, ..., \theta_{jt}\}$ be the possible status which belongs to the criterion, c_j, and $p_j^k(k = 1, 2, ..., t)$ be the corresponding possibility, where $t \geq 1$ and $0 \leq p_j^k \leq 1$. Suppose $\omega = (w_1, w_2, ..., w_n)^T$ is the weight vector of criteria that has been completely known, where $\omega_j \in [0, 1](j = 1, 2, ..., n)$ $\sum_{j=1}^n \omega_j = 1$.

Due to the complexity of decision making problems, it is quite difficult for DM to determine a single accurate reference point for each criterion. Thus, in this paper, the decision making reference points of different criteria are expressed in the form of intervals. Meanwhile, in order to simplify the problem, we only choose four intervals. In fact, we can use three or five intervals instead, according to real situations, but the intervals should not be less than three, because if the DM has no idea about the reference point completely, we cannot determine which alternative has more support.

Definition 15. Let $x_j^{\min}$ and $x_j^{\max}$ be the worst and optimal values from the criteria value space, with respect to criterion, c_j, respectively. Then, we divide the distance between $x_j^{\min}$ and $x_j^{\max}$ into four equal pieces, which are shown as follows:

$$x_j^{c_0} = \frac{x_j^{\min} + x_j^{\max}}{2}, \tag{14}$$

$$x_j^{c_1} = \frac{x_j^{\min} + x_j^{c_0}}{2}, \tag{15}$$

$$x_j^{c_2} = \frac{x_j^{c_0} + x_j^{\max}}{2}. \tag{16}$$

Thus, we can obtain four intervals as the reference points, denoted by $x_j^{01}, x_j^{02}, x_j^{03}, x_j^{04}$, where $x_j^{01} = [x_j^{\min}, \quad x_j^{c_1}]$, $x_j^{02} = [x_j^{c_1}, \quad x_j^{c_0}]$, $x_j^{03} = [x_j^{c_0}, \quad x_j^{c_2}]$ and $x_j^{04} = [x_j^{c_2}, \quad x_j^{\max}]$.

In general, when the reference point chosen by DM is closer to the optimal point, it indicates the DM is more risk seeking; otherwise, it means the DM is more loss-aversion.

4.2. Procedures of the decision making method

Step 1: Determine the evaluation values of alternative $A_i(i = 1, 2, ..., m)$, with respect to criterion $c_j(j = 1, 2, ..., n)$, in the form of $< u_{ij}, v_{ij} >$.

Step 2: Determine the reference points, and obtain four reference intervals through Eqs. (14)-(16).

Step 3: Calculate the prospect value, Z_{ijk}^l, of alternative $A_i(i = 1, 2, ..., m)$, with respect to criterion $c_j(j = 1, 2, ..., n)$, in the kth status, based on the four reference intervals above.

$$Z_{ijk}^l = V(\Delta x_{ijk}),$$
$$l = 01, 02, 03, 04, \quad k = 1, 2, ..., t, \tag{17}$$

where $V(x)$ is the value function defined in Eq. (5), and Δx_{ijk} is the distance between x_{ijk} and reference interval x_j^l, which can be calculated through Eq. (4).

Step 4: Calculate the prospect value of criterion $c_j(j = 1, 2, ..., n)$ for alternative $A_i(i = 1, 2, ..., m)$.

$$Z_{ij}^l = \sum_{k=1}^t w_{ij}^k Z_{ijk}^l,$$
$$l = 01, 02, 03, 04, \quad k = 1, 2, ..., t, \tag{18}$$

where w_{ij}^k is the probability weight function, which can be calculated by Eq. (6).

Step 5: Calculate the weighted prospect value of alternative $A_i(i = 1, 2, ..., m)$.

$$Z_i^l = \sum_{j=1}^n w_j Z_{ij}^l \quad (l = 01, 02, 03, 04), \tag{19}$$

where $w_j(j = 1, 2, ..., n)$ is the weight of criterion $c_j(j = 1, 2, ..., n)$.

Step 6: Rank the alternatives based on the different reference intervals. The larger the weighted prospect function value, the better the outcome.

5. A numerical example

5.1. Background introduction

A company, X, wants to invest in new products, and there are three alternatives to be considered, denoted by $A_i(i = 1, 2, 3)$. Three criteria of profitability (c_1), social benefits (c_2) and pollution loss (c_3) will be used to evaluate the alternatives, respectively, and the corresponding weight is $w = (0.38, 0.32, 0.30)^T$. The evaluation values take the form of IF number. $< u_{ij}, v_{ij} >$, which reveals the degree of satisfaction or dissatisfaction of the DM to criterion, c_j, for alternative, A_i. As the evaluation information may be different under different economic environments in the future, different evaluation information is given with discrete probability distribution, respectively, under possible economic environments in the future, which are shown in Tables 2-4.

5.2. The steps of decision making

Step 1: Calculate the reference intervals through Eqs. (14)-(16); the results are shown in Table 5.

Step 2: Calculate the prospect value Z^l_{ijk} of alternative $A_i(i = 1, 2, ..., m)$, with respect to criterion $c_j(j = 1, 2, ..., n)$, in the kth status, respectively, based on the four reference intervals above. Here, we let $\alpha = \beta = 0.88$ and $\theta = 2.25$ according to the research by Kahneman and Tversk [27]. The results are given in Tables 6-8.

Step 3: Transform the probabilities into probability weight function values.

If $Z^l_{ij1} > 0$ $(l = 01, 02, 03, 04)$, we can get $\omega^+_1 = 0.318$; otherwise, $\omega^-_1 = 0.329$, with $\gamma = 0.61$, $\delta = 0.72$, according to Kahneman and Tversky [27]. Similarly,

Table 2. The criteria values of alternatives under good economic environment.

	Under good economy environment (30%)		
	Profitability (c_1)	**Social benefits (c_2)**	**Pollution loss (c_3)**
A_1	$< 0.80, 0.10 >$	$< 0.80, 0.16 >$	$< 0.55, 0.40 >$
A_2	$< 0.70, 0.23 >$	$< 0.66, 0.26 >$	$< 0.70, 0.10 >$
A_3	$< 0.80, 0.18 >$	$< 0.66, 0.32 >$	$< 0.66, 0.18 >$

Table 3. The criteria values of alternatives under normal economic environment.

	Under normal economy environment (50%)		
	Profitability (c_1)	**Social benefits (c_2)**	**Pollution loss (c_3)**
A_1	$< 0.70, 0.17 >$	$< 0.70, 0.20 >$	$< 0.40, 0.30 >$
A_2	$< 0.65, 0.30 >$	$< 0.60, 0.35 >$	$< 0.64, 0.17 >$
A_3	$< 0.72, 0.20 >$	$< 0.62, 0.35 >$	$< 0.60, 0.20 >$

Table 4. The criteria values of alternatives under poor economic environment.

	Under poor economy environment (20%)		
	Profitability (c_1)	**Social benefits (c_2)**	**Pollution loss (c_3)**
A_1	$< 0.63, 0.22 >$	$< 0.60, 0.24 >$	$< 0.40, 0.40 >$
A_2	$< 0.50, 0.30 >$	$< 0.58, 0.36 >$	$< 0.60, 0.16 >$
A_3	$< 0.61, 0.28 >$	$< 0.50, 0.50 >$	$< 0.50, 0.33 >$

Table 5. The reference intervals of each criterion.

	Profitability (c_1)	**Social benefits (c_2)**	**Pollution loss (c_3)**
x^{01}	$[< 0.50, 0.30 >, < 0.60, 0.22 >]$	$[< 0.50, 0.50 >, < 0.60, 0.37 >]$	$[< 0.40, 0.40 >, < 0.50, 0.28 >]$
x^{02}	$[< 0.60, 0.22 >, < 0.68, 0.17 >]$	$[< 0.60, 0.37 >, < 0.68, 0.28 >]$	$[< 0.50, 0.28 >, < 0.58, 0.20 >]$
x^{03}	$[< 0.68, 0.17 >, < 0.76, 0.13 >]$	$[< 0.68, 0.28 >, < 0.75, 0.21 >]$	$[< 0.58, 0.20 >, < 0.64, 0.14 >]$
x^{04}	$[< 0.76, 0.13 >, < 0.80, 0.10 >]$	$[< 0.75, 0.21 >, < 0.80, 0.16 >]$	$[< 0.64, 0.14 >, < 0.70, 0.10 >]$

Table 6. The prospect values of alternative A_1.

	Profitability (c_1)	**Social benefits (c_2)**	**Pollution loss (c_3)**
x^{01}	0.2986, 30%;	0.3068, 30%;	0.0772, 30%
	0.1585, 50%,	0.2295, 50%;	-0.1737, 50%;
	0.0868, 20%;	0.1484, 20%;	-0.1737, 20%;
x^{02}	0.1673, 30%;	0.2018, 30%;	-2.2045, 30%;
	0.0646, 50%;	0.0617, 50%;	-0.3082, 50%;
	-0.1103, 20%;	0.0617, 20%;	-0.4210, 20%
x^{03}	0.0923, 30%;	0.1143, 30%;	-0.3877, 30%;
	-0.1062, 50%;	0.0523, 50%;	-0.4840, 50%;
	-0.2467, 20%;	-0.1938, 20%;	-0.5910, 20%;
x^{04}	-0.5883, 30%;	0.0389, 30%;	-0.5245, 30%
	-0.7083, 50%;	-0.1564, 50%;	-0.6173, 50%
	-0.8482, 20%;	-0.3525, 20%;	-0.7211, 20%

Table 7. The prospect values of alternative A_2.

	Profitability (c_1)	**Social benefits (c_2)**	**Pollution loss (c_3)**
x^{01}	0.1280, 30%;	0.1760, 30%;	0.2906, 30%
	0.0660, 50%,	0.0909, 50%;	0.2223, 50%;
	-0.1485, 20%;	0.0800, 20%;	0.2030, 20%;
x^{02}	0.0501, 30%;	0.0617, 30%;	0.1896, 30%;
	-0.1357, 50%;	-0.1389, 50%;	0.1162, 50%;
	-0.3596, 20%;	-0.1580, 20%;	0.0950, 20%;
x^{03}	-0.1517, 30%;	-0.1243, 30%;	0.1096, 30%;
	-0.3111, 50%;	-0.3240, 50%;	0.0457, 50%;
	-0.5191, 20%;	-0.3638, 20%;	-0.1028, 20%;
x^{04}	-0.7697, 30%;	-0.2893, 30%;	0.0397, 30%
	-0.9032, 50%;	-0.4745, 50%;	-0.1292, 50%
	-1.0876, 20%;	-0.5124, 20%;	-0.1357, 20%

we can get $\omega_2^+ = 0.421$, $\omega_2^- = 0.464$, $\omega_3^+ = 0.261$ and $\omega_3^- = 0.254$.

Step 4: Calculate the prospect value, Z_{ij}^l, of criterion, $c_j(j = 1, 2, ..., n)$, for alternative, $A_i(i = 1, 2, ..., m)$, based on the four reference intervals ($l = 01, 02, 03, 04$); the results are shown in Table 9.

Step 5: Calculate the weighted prospect value, $Z_i^l(i = 1, 2, ..., m)$, of alternative, $A_i(i = 1, 2, ..., m)$, based on the four reference intervals ($l = 01, 02, 03, 04$); the results are shown in Table 10.

Step 6: Rank the alternatives based on the four reference intervals, and the results are given in Table 11.

Table 8. The prospect values of alternative A_3.

	Profitability (c_1)	**Social benefits (c_2)**	**Pollution loss (c_3)**
x^{01}	0.2151, 30%;	0.1459, 30%;	0.2295, 30%
	0.1560, 50%,	0.1043, 50%;	0.1834, 50%;
	0.0663, 20%;	-0.1799, 20%;	0.0772, 20%;
x^{02}	0.1267, 30%;	0.0617, 30%;	0.1241, 30%;
	0.0617, 50%;	-0.1389, 50%;	0.0730, 50%;
	-0.1743, 20%;	-0.4375, 20%;	-0.0868, 20%;
x^{03}	0.0472, 30%;	-0.2896, 30%;	0.0457, 30%;
	-0.1062, 50%;	-0.2954, 50%;	-0.1028, 50%;
	-0.3460, 20%;	-0.6241, 20%;	-0.3793, 20%;
x^{04}	-0.5883, 30%;	-0.3584, 30%;	-0.1095, 30%
	-0.7134, 50%;	-0.4475, 50%;	-0.2288, 50%
	-0.9335, 20%;	-0.7636, 20%;	-0.5164, 20%

Table 9. The prospect values of alternatives with respect to each criterion.

		Profitability (c_1)	**Social benefits (c_2)**	**Pollution loss (c_3)**
A_1	x^{01}	0.1843	0.2329	-0.1018
	x^{02}	0.0524	0.1303	-0.3172
	x^{03}	-0.0826	0.0091	-0.5022
	x^{04}	-0.7376	-0.1497	-0.6421
A_2	x^{01}	0.0308	0.1151	0.2390
	x^{02}	-0.1384	-0.0850	0.1340
	x^{03}	-0.3261	-0.2836	0.0280
	x^{04}	-0.9486	-0.4455	-0.0818
A_3	x^{01}	0.1514	0.0446	0.1703
	x^{02}	0.0220	-0.1560	0.0481
	x^{03}	-0.1222	-0.3909	-0.1295
	x^{04}	-0.7617	-0.5195	-0.2734

Table 10. The weighted prospect values.

	A_1	A_2	A_2
x^{01}	0.1140	0.1202	0.1229
x^{02}	-0.0336	-0.0396	-0.0271
x^{03}	-0.1791	-0.2063	-0.2104
x^{04}	-0.5208	-0.5276	-0.5377

Meanwhile, the ranking result calculated by expected utility theory is: $A_2 \prec A_3 \prec A_1$.

5.3. A comparison with SMAA-P method

In the SMAA-P method, the reference points are fixed for each alternative. In the following, we will take each alternative as the reference point to analyze

Table 11. The ranking result.

x^{01}	x^{02}	x^{03}	x^{04}
$A_1 \prec A_2 \prec A_3$	$A_2 \prec A_1 \prec A_3$	$A_3 \prec A_2 \prec A_1$	$A_3 \prec A_2 \prec A_1$

Table 12. The weighted prospect values by SMAA-P.

Refrence point	Alternatives		
	A_1	A_2	A_3
A_1	0	-0.1490	-0.1077
A_2	-0.0784	0	-0.0533
A_3	-0.0560	-0.0522	0

Table 13. The ranking result by SMAA-P.

Reference point	A_1	A_2	A_3
Ranking result	$A_2 \prec A_3 \prec A_1$	$A_1 \prec A_3 \prec A_2$	$A_1 \prec A_2 \prec A_3$

the above problem, and in order to show that our proposed method is superior to SMAA-P in dealing with reference points, we assume that the criteria value, criteria weight and loss aversion coefficient are all known.

As the calculation process of SMAA-P method is similar to that of our method, we just give the final result directly. Here, as the criteria value, criteria weight and loss aversion coefficient have been determined, the only uncertainty variable is the reference points, which have been fixed as each alternative. So, our result is much easier, and there is no need to calculate the probabilities for each rank. The result is shown in Table 12.

Then, we can obtain the final ranking result, as Table 13 shows. From Table 13, we can see that taking different alternatives as the reference points may result in different final results, but, we cannot get more information. Therefore, compared with our method in dealing with the reference point, the SMAA-P method has its limitations.

Firstly, it is not suitable to handle problems with few alternatives, as the example above, as it will make it difficult to select the best alternative.

Secondly, from the final result, we cannot obtain the relationship between the results and the DMs' risk attitudes. This is because taking which alternative as the reference point has nothing to do with the DMs' real feelings.

Last but not least, the SMAA-P method can tell the DMs which alternative is better when they are not sure about their preference, but it cannot display how the preference (risk attitude) affects the results.

The above limitations of the SMAA-P method can be solved by our method, and we will discuss them in detail in the following part.

5.4. Discussion

In the example above, the rank of the alternatives by our method is different when based on different reference intervals. For example, alternative A_1 is inferior to others when taking x^{01} and x^{02} as the reference intervals, but it becomes superior to others when taking x^{03} and x^{04} as the reference intervals. On the contrary, alternative A_3 is superior to others when taking x^{01} and x^{02} as the reference intervals, while it becomes inferior to others when taking x^{03} and x^{04} as the reference intervals. In traditional MCDM problems, the researchers always take the mean value, worst value or optimal value as the reference point. In fact, different DMs will have different reference alternatives based on the context and their own personalities, which can be affected by the change of DMs' risk attitudes. When a DM is more risk seeking under a certain circumstance, the reference point he/she chooses will be closer to the optimal value; otherwise, it will be closer to the worst value. Hence, the results obtained in Table 11 indicate that if the DM is willing to take on more risks, alternative A_1 is the best choice, otherwise, alternative A_3 is better, which conforms better to real decision making situations.

From Tables 2-4, we know that for criterion profitability (c_1) and social benefits (c_2), alternative A_1 has evident advantages over others, but, as to criterion pollution loss (c_3), alternatives A_2 and A_3 have evident advantages over alternative A_1, namely, alternative A_1 is an extreme alternative, which has both best and worst values on some criteria. However, compared with alternative A_1, alternative A_3 does not have extreme values on each criterion. As a result, if the DM is not willing to take on more risks on pollution loss (c_3), he/she will tend to invest in alternative A_3. But, the choice can only be alternative when based on the expected utility theory, which cannot well reflect the actual decision making psychology of different DMs with different risk attitudes.

The multiple reference intervals applied in this paper can help DMs understand how the references affect the final result and acquire deeper information about the results, telling the DMs to make a satisfying decision according to their own preferences. This cannot be achieved by the SMAA-P method.

The MCDM method proposed in this paper can be applied when DMs cannot determine the reference point on each criterion, or can only determine a rough range of the reference point, which is a common problem in actual decision making situations. Our method of dealing with reference points can be used in the SMAA-P method, thus, helping to conquer its limitations proposed in Section 5.3.

6. Conclusions

In this paper, we develop a method to deal with MCDM problems under both stochastic and IF uncertainties, based on prospect theory, for a more accurate description of DMs' behaviors.

Considering that traditional prospect theory has limitations in dealing with reference points, we propose multiple reference intervals, extending the reference points to reference intervals and providing multiple choices, which can help DMs find satisfying solutions. Meanwhile, the multiple reference intervals can help DMs find more information about the final decision, and tell them which alternative is better under different situations. Although the SMAA-P method also supplies multiple reference points, it cannot achieve the function as in our method.

As for the IF number, we define a new score function based on the common loss aversion psychology of people. We illustrate its effectiveness and practicality through the representative examples and compare the results with other score functions. In order to overcome the fundamental deficiency of existing distance measures of the IF set, which will result in a contradiction with our understanding of the actual meaning of the IF number, we put forward a novel distance measure of the IF set based on the new proposed score function. The main feature of the method is that we transform the IF number into a real number based on its practical meaning in life, regardless of its form, which can better describe actual situations. For example, two patients may have different symptoms, but they may be diagnosed with the same disease.

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