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Online conflict-free coloring of intervals

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Abstract. In this paper, we study the problem of online conflict-free coloring of intervals on a line, where each newly inserted interval must be assigned a color upon insertion such that the coloring remains conflict-free, i.e. for each point p in the union of the current intervals, there must be an interval I with a unique color among all intervals covering p. We first present a simple algorithm which uses $\mathcal{O}(\sqrt{n})$ colors where n is the number of current intervals. Next, we propose an CF-coloring of intervals which uses $\mathcal{O}(\log^3 n)$ colors.

1. Introduction

Background. A cellular network consists of several base stations where each base station covers clients within a certain distance. In general, the coverage areas of base stations may overlap. This may lead to interference of signals for a client who is in the coverage area of more than one base station. Thus, one would like to assign frequencies to the base stations such that for each client within the coverage area of at least one base station, there is a base station with a unique frequency covering the client. The main goal is to do this using a few number of distinct frequencies. Recently, Even et al. [1] introduced *conflict-free colorings* to model this problem as defined next.

Let $\mathcal{S}$ be a set of n objects like points, and let $\mathcal{R}$ be a (possibly infinite) family of ranges like disks. For a range $r \in \mathcal{R}$, let $\mathcal{S}(r)$ be the subset of objects in $\mathcal{S}$ intersecting the range r. A *conflict-free coloring (CF-coloring) of $\mathcal{S}$ with respect to $\mathcal{R}$* is a coloring of $\mathcal{S}$ such that for any range $r \in \mathcal{R}$, for which $\mathcal{S}(r) \neq \emptyset$, there is an object $o \in \mathcal{S}(r)$ whose color is not used by any other object in $\mathcal{S}(r)$, i.e. its color is unique in $\mathcal{S}(r)$. It is obvious that a conflict-free coloring always exists: just color objects with different colors. However, one would like to find a coloring with a few number of colors. This is the conflict-free coloring problem. Notice that if we take $\mathcal{S}$ to be a set of disks - namely, the coverage area of each base station - and we take $\mathcal{R}$ to be the set of all points in $\mathbb{R}^2$ - namely, the clients - then we get exactly the frequency-assignment problem as discussed above. In this paper, we only consider the case where objects are intervals in $\mathbb{R}^1$, and ranges are points.

Related work. The offline variant of the problem, where all objects are given in advance, has attracted a lot of attention in the last decade. Even et al. [1] were the first to present a CF-coloring of points with respect to disks using $O(\log n)$ colors, which is tight in the worst case. Then, Har-Peled and Smorodinsky [2] extended those results by considering other range spaces.

The online version of the problems has been studied in [3]. When a point is inserted, a color is assigned to it and the color cannot be changed since then. The coloring should remain conflict-free at all times. Chen et al. [3] considered the CF-coloring of points on a line with respect to all intervals on the line. They presented both deterministic and

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randomized algorithms for this problem. The best deterministic algorithm uses $O(\log^2 n)$ colors, and the best randomized algorithm uses $O(\log n)$ colors with high probability. The best known lower bound for both randomized and deterministic algorithms, which also holds for the offline case, is $\Omega(\log n)$ colors [4,5]. For other interesting variants of the online CF-coloring problem, see [6].

Our results. In this paper, we pay our attention to the online CF-coloring of intervals with respect to points on a line. Here, intervals are arriving one by one and upon arrival of an interval, we should assign a color to this interval; this color can not be changed later. At any time the coloring must remain conflict-free, i.e. for each point p in the union of the current intervals, there must be an interval with a unique color among all intervals covering p.

To warm up, we start with the offline version of the CF-coloring of intervals with respect to points. We sketch a trivial algorithm that uses only 3 colors. Then, we explain that any online CF-coloring algorithm must use $\Omega(\log n)$ colors in the worst case. We then present two online CF-coloring algorithms: our first simple algorithm uses $\mathcal{O}(\sqrt{n})$ colors and our main algorithm uses $O(\log^3 n)$ colors where n is the number of the current intervals.

The paper is organized as follows: In Section 2 we present a simple algorithm that uses 3 colors for the offline version of the problem. Section 3 presents a CF-coloring of intervals in the online model using $O(\sqrt{n})$ colors. In Section 4, we present our CF-coloring of intervals. We depart with a few concluding remarks and open problems in Section 6.

2. From offline to online CF-coloring of intervals

We start with offline version of the problem where a set $\mathcal{I}$ of n intervals is given in advance. The main ingredients of our simple algorithm is (i) computing a set $\mathcal{I}' \subset \mathcal{I}$ such that each point on the line is covered by at most two intervals of $\mathcal{I}'$ and each interval in $\mathcal{I}/\mathcal{I}'$ is covered by the union of intervals in $\mathcal{I}'$, and (ii) CF-coloring of $\mathcal{I}'$ using 2 colors. Since the intervals in $\mathcal{I}/\mathcal{I}'$ are useless (in the sense that each point in the coverage area of set $\mathcal{I}$ is in the coverage area of set $\mathcal{I}'$), we color all of them with color 0 and avoid using color 0 in CF-coloring of set $\mathcal{I}'$. Next, we go into details of the two ingredients.

To construct $\mathcal{I}'$, we process the intervals in $\mathcal{I}$ in the increasing order of their left endpoints. The set $\mathcal{I}'$ is initially empty. Let I_{last} be the last interval inserted into $\mathcal{I}'$. Upon arrival of interval I, if one of the following conditions is satisfied, interval I is inserted into $\mathcal{I}'$ and recognized as the last interval (up to now) inserted to $\mathcal{I}'$, i.e. $I_{last} = I$:

- I does not overlap I_{last};
- I has the rightmost right endpoint among intervals overlapping I_{last} but not contained by I_{last}.

Clearly the set $\mathcal{I}'$ fulfillls the following properties: (i) $\bigcup_{I\in\mathcal{I}} I = \bigcup_{I\in\mathcal{I}'} I$, and (ii) each point on the line is inside at most two intervals of $\mathcal{I}'$. To color intervals in set $\mathcal{I}'$, we color the intervals by colors 1 and 2 alternatively in the increasing order of their left endpoints. As each point is in the coverage area of at most two intervals in $\mathcal{I}'$, this definitely is a conflict-free coloring of $\mathcal{I}'$ using just two colors. This coloring together with coloring of intervals in $\mathcal{I}/\mathcal{I}'$ with color 0 simply gives us a CF-coloring of set $\mathcal{I}$. Putting all this together, we get the following theorem.

Theorem 1. *For any set $\mathcal{I}$ of n intervals on $\mathbb{R}^1$, there is a CF-coloring with respect to points in the offline model using at most 3 colors.*

Although there is a CF-coloring of intervals w.r.t points using 3 colors in the offline model, it can be easily shown [4,5] that, in the online model, any such CF-coloring must use $\Omega(\log n)$ colors in the worst case. Consider $\mathcal{I} = \{I_1, \cdots, I_n\}$ where $I_j = [1, p_j]$ and $p_j = j$. Suppose the intervals in $\mathcal{I}$ arrive in the increasing order of their indices. In this senario, it is easy to see a CF-coloring of the intervals is equivalent to CF-coloring of points p_i, with respect to intervals which needs $\Omega(\log n)$ colors [1]. Indeed, if we assign the color of each interval I_r to its right endpoint (i.e. p_r), we can show that among points p_k in any interval $[i, j]$, one has a unique color.

3. Simple CF-coloring using $\mathcal{O}(\sqrt{n})$ colors

Suppose $\mathcal{I}$ is a set of n intervals arriving through time one by one and suppose for each point $p \in \mathbb{R}^1$, $\mathcal{I}(p)$ is the set of all intervals containing p at the current time. We denote colors by non-negative integer numbers and denote the color of interval I by $c(I)$. In this section, we present a simple algorithm which holds the unique-maximum invariant (UM invariant, for short): $mc(p) = \max_{I\in\mathcal{I}(p)} c(I)$ is unique in the multi set $\{c(I) : I \in \mathcal{I}(p)\}$ for all $p \in \mathbb{R}^1$. If this holds, the coloring is a CF-coloring. Indeed, the interval with the maximum color among intervals containing p has a unique color. Next, we explain the algorithm.

Algorithm. We maintain the maximum color used so far in a variable, say m; at the beginning $m = 1$. We assign a color upon arrival of I, as follows. Let $S_I \subset \{1, \cdots, m\}$ be a set of *forbidden colors* for I in the sense that if we assign one of them to I, then the UM invariant does not hold any more. If $S_I = \{1, \cdots, m\}$, we increase m by 1 and assign it to I. Otherwise, one arbitrary unforbidden color from the set $\{1, \cdots, m\}$ is assigned to I.

Analysis. Imagine we know all n intervals in advance. Let $p_1, p_2, \cdots, p_m$ be the list of distinct interval endpoints, sorted from left to right ($m \leq 2n$ as some endpoints may coincide). Consider the partitioning of $\mathbb{R}^1$ into the elementary intervals $(\infty : p_1), [p_1 : p_1], (p_1 : p_2), [p_2 : p_2], \cdots, (p_{m1} : p_m), [p_m : p_m], (p_m : +\infty)$. The list of elementary intervals consists of open intervals between two consecutive endpoints p_i and p_{i+1}, alternated with closed intervals consisting of a single endpoint. The reason that we treat the points p_i themselves as intervals is, of course, that the set of intervals covering p_i is not necessarily the same at the set of intervals covering p where p can be any point close to p_i. For an elementary interval e, let $mc(e)$ be the maximum color covering e. At the beginning when no interval has arrived, $mc(e) = 0$ for all (at most) $4n+1$ elementary intervals. It is clear that $mc(e)$ is not decreasing through time for each elementary interval e. If after getting all intervals, $mc \leq \sqrt{n/2}$, we are done. Otherwise, consider the first time an interval, say σ_i, gets color $\sqrt{n/2} + i$. Assigning $\sqrt{n/2} + i$ to σ_i implies that assigning any color less than $\sqrt{n/2} + 1$ would not hold the UM invariant. Therefore, there must be $\sqrt{n/2}$ elementary intervals, denoted by the set $E(\sigma_i)$, whose colors are $\{1, 2, \cdots, \sqrt{n/2}\}$. After assigning $\sqrt{n/2} + i$ to σ_i, the maximum color of all intervals in $E(\sigma_i)$ is increased to $\sqrt{n/2} + i$ which is greater than $\sqrt{n/2}$. Therefore, sets $E(\sigma_1), E(\sigma_2), \cdots$ are disjoint. Since the total number of these sets is $m - \sqrt{n/2}$ and their union is a subset of all elementary intervals, we have $(m - \sqrt{n/2})\sqrt{n/2} \leq 4n + 1$ which simply implies $m = O(\sqrt{n})$.

Theorem 2. *There is an online CF-coloring algorithm for a set of n intervals in the online model that uses $O(\sqrt{n})$ colors.*

4. CF-coloring using $\mathcal{O}(\log^3 n)$ colors

We first start with a special case where we know that all intervals cover a specific point x. For this special case, we present a CF-coloring using $O(\log n)$ color while holding the UM invariant. Then, we exploit this as the main ingredient of our CF-coloring algorithm for the general case.

The special case. If the new interval I is a subset of the union of the current intervals, we assign 0 to I. Otherwise, I extends the coverage area from left, right, or both sides. Based on this fact, we categorize intervals into three sets S_ℓ, S_r, and $S_{r,\ell}$, respectively. we use three disjoint groups of colors - each group contains at most $O(\log n)$ colors - for CF-coloring of S_ℓ, S_r, and $S_{r,\ell}$. CF-coloring of these three sets is simply reduced to the problem of online CF-coloring of points with respect to intervals (as explained in Section 2) where the arrival time of points is in either the increased or decreased order of their coordinates. This is known to be CF-colorable using $O(\log n)$ colors [3]. Since we use three disjoint sets of colors for sets S_ℓ, S_r, and $S_{r,\ell}$, the whole coloring remains CF-coloring.

Lemma 1. *For a special case where all intervals cover a specific point, there is an online CF-coloring of intervals using $O(\log n)$ colors.*

We now turn our attention to the general case. Suppose that we partition intervals received so far into sets $S_1, \cdots, S_m$ in such a way that for every $1 \leq i \leq m$ all the intervals in S_i have a common point, say x_i. Also suppose points in $\mathcal{X} = \{x_1, \cdots, x_m\}$ have been CF-colored by the online algorithm given in [3] using $O(\log^2 m)$ colors (note that in this coloring, the color of each point, x_i, $c(x_i)$, is a pair (a, b) and the maximum color inside each interval is unique where the order of colors is defined as follows: $(a_1, b_1) < (a_2, b_2) \leftrightarrow (a_1 < b_1) \vee (a_1 = b_1 \wedge a_2 < b_2)$.) At the beginning when we receive the first interval I_1, the set S_1 is set to be $\{I_1\}$ and x_1 is set to be any arbitrary point inside I_1. Next, we go into details of the algorithm upon arrival of I_j, the jth interval.

Coloring algorithm. If I_j is a subset of $\bigcup_{i=1}^{j-1} I_i$, we assign 0 to I_j, and it is considered as a useless interval. Otherwise, we distinguish two cases: (i) There is a point of $\mathcal{X}$ inside I_j, (ii) There is not such a point. In case (i), we select the point with the maximum color (which is unique) among points of $\mathcal{X}$ inside I_j, say x_r, and insert I_j into S_r. We assign the color $(c(x_r), c_{S_r}(I_j))$ to I_j where $c_{S_r}(I_j)$ is the color assigned to I_j when inserted into S_r as explained in the special case. In case (ii) we select a point, say x_{m+1}, inside I_j which is not covered by any other I_i $(1 \leq i \leq j-1)$. Point x_{m+1} exists, as I_j is not a subset of $\bigcup_{i=1}^{j-1} I_i$. We add x_{m+1} into $\mathcal{X}$ and assign a color to it such that the coloring of $\mathcal{X}$ remains conflict free with respect to intervals. We also create the set $S_{m+1} = \{I_j\}$ and we assign a color to I_j in set S_{m+1} as explained in the special case.

The number of colors. Each color is a pair of colors whose first entry is $O(\log^2 j)$ (note $m \leq j$) and whose second entry is $O(\log j)$ (by Lemma 1). Therefore, after getting n intervals, we use at most $O(\log^3 n)$ colors.

In the following lemma, we show the algorithm produces a CF-coloring.

Lemma 2. *After getting I_j, the coloring remains conflict free.*

Proof. Suppose I_j is inserted into S_r. For the sake of contradiction, assume the coloring is not conflict free after insertion of I_j. Therefore, there is a point p inside I_j such that $mc(p) = (c(x_r), c_{S_r}(I_j))$ and there is another interval I_i ($i \leq j$) covering p with $c(I_i) = c(I_j)$. Suppose I_i is inserted into S_k and its color is $(c(x_k), c_{S_k}(I_i))$. If $x_k = x_r$, the coloring of S_r is not conflict free which is a contradiction. Therefore, we have $x_k \neq x_r$. Without loss of generality, we assume $x_k < x_r$. Due to the UM invariant, there is a point x inside $[x_k, x_r]$ whose color is unique and maximum. x differs from x_r and x_k as $c(x_k) = c(x_r)$. Since x is covered by the union of I_i and I_j and it was created before receiving I_i and I_j (note that when a new point is inserted into $\mathcal{X}$, at that time no interval contains it), one of these two intervals must be assigned to x_h which is a contradiction.

Putting all this together, we get our main result.

Theorem 3. *There is an online CF-coloring algorithm for a set of n intervals in the online model that uses $O(\log^3 n)$ colors.*

5. Conclusion

We studied the problem of online conflict-free coloring of intervals on a line, where each newly inserted interval must be assigned a color upon insertion such that the coloring remains conflict-free. We first presented a simple algorithm which uses $\mathcal{O}(\sqrt{n})$ colors where n is the number of current intervals. Next, we proposed an CF-coloring of intervals which uses $\mathcal{O}(\log^3 n)$ colors. It is interesting to see whether there is a CF-coloring algorithm using $o(\log^3 n)$ colors.

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