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Using identical system synchronization with fractional adaptation law for identification of a hyper-chaotic system

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Abstract. Synchronization of two chaotic systems has been used in secure communications. In this paper, synchronization of two identical 4D Lü hyper-chaotic systems is used to identify the drive system. Parameters in both drive and response systems are unknown and the systems are synchronized by applying one state feedback controller. Since the goal here is to identify the parameters of the drive system, an adaptive method is used. The stability of the closed-loop system with the controller and convergence of parameters is studied using the Lyapunov theorem. In order to improve the speed of convergence in one parameter, a fractional adaptation law is used and the stability with the fractional law is shown. Finally, the results of both integer and fractional methods are compared.

1. Introduction

There is a phenomenon in nonlinear systems, which is called chaos. Chaos also exists in many real world problems. The most significant property of a chaotic system is high sensitivity to initial conditions. It is important to see that two responses of a chaotic system with initial values very close to each other diverge exponentially but still stay in a bounded region. This would cause some other properties of chaotic systems, which have rich frequencies, and many unstable periodic orbits. These properties of chaotic systems make it difficult to study them. The positive Lyapunov exponent is one way to show that a system is chaotic. If the chaotic system has more than one positive Lyapunov exponent, it would be called hyper-chaotic.

In real world applications, parameters of a chaotic system may be particularly or fully unknown. So, identification of the parameters of a chaotic system has been studied in many cases. Different methods are used for identification, such as the neural network state space model [1], adaptive control [2], modified recursive least square [3], robust control [4] etc. Most of these methods use an adaptive law which comes from the Lyapunov-based stability proof for the closed loop system. In most cases, lack of information about the system parameters makes it necessary to use an adaptation law. But, usually, these adaptation laws may not converge to true values and are just used to estimate a value for the parameters. In some cases, when the chaotic system has fractional nonlinear differential equations, identification has been made using optimization algorithms, such as Artificial Neural Networks (ANN) [5], Particle Swarm Optimization (PSO) [6], the output error approach [7] or differential evolution [8].

Synchronization of chaotic systems was introduced for the first time in the research undertaken by

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Pecora and Carroll [9] in 1990, and since then, it has been an interesting topic for researchers [10-13]. In synchronization of two systems, it is desired for the response system trajectories to follow the drive system. In many cases, parameters of the drive system are unknown, so, an adaptive method is used to estimate the parameters [14-18]. So, synchronization of two identical systems can be used to identify the parameters of the drive system. Recently, there have been many methods to synchronize two systems with unknown parameters. In [15,17], two non-identical chaotic systems are synchronized by applying the adaptive control law to each state of the response system. In [18], two identical chaotic systems are synchronized using a control law in only one of the states of the response system. In [19], two identical Lü hyper-chaotic systems with unknown parameters are synchronized with an adaptation law based on the Lyapunov stability theory. In [20], a new modified hyper-chaotic Lü system is synchronized with the use of the adaptive control law. In [19,20], the controllers are applied to all states, but the parameters convergence to the correct values is not shown. And, finally, in [21], two identical Lü hyper-chaotic systems with unknown parameters are synchronized by applying one state controller. Based on [21], in the present paper, some modifications are applied in cases of parameter identification in order to guarantee convergence to the correct values.

Fractional calculus has a history of about 300 years and more recently been recognized in the work done by Leibniz, Riemann, etc. Little attention was paid to it at that time, but, recently, there have been many more applications using fractional calculus [8,22-24]. It has been used to model some systems, e.g. viscoelastic systems, suspension systems etc. [23].

Fractional calculus has recently stepped into the control region and also chaos control [25-28]. It is shown that using fractional order controllers can have better results than using integer orders [29]. In designing a controller with fractional calculus, there is one more parameter which gives the designers more degrees of freedom to design a controller. This extra parameter is the order of differentiation that allows getting a better response from the controller, especially in the transient part of the solution.

In this paper a fractional order adaptation law is used to synchronize two integer order identical hyper-chaotic 4D Lü systems in order to identify the parameters of the drive system. In other words, the main idea here is to use chaos synchronization techniques to synchronize virtual computer-based dynamics with unknown parameters as the "Response System", with a real dynamical chaotic system as the "Drive System". To achieve this, at first, the controller designed in [21] for synchronizing two 4D hyper-chaotic systems is discussed again here. Then, the parameters of the response system are assumed to be unknown, and, using the Lyapunov stability theorem, an adaptive control algorithm is designed. The important point here is to use fractional order dynamics in adaptation laws to obtain better convergence and smaller oscillations in parameter estimation. Finally, the stability of the system with fractional order is discussed and it is proved that the system with the fractional order adaptation law remains stable.

2. Preliminaries and definitions

In fact, fractional calculus is a generalized version of integer order calculus. The integro-differential operator is shown by ${}_{t_0}D_t^{\alpha}$. Common formulations for fractional derivatives are as follows.

Definition 1. (Riemann-Liouville fractional derivative [30]) The Riemann-Liouville fractional derivative is defined as:

$$ {}_{t_0}^{\mathrm{RL}}D_t^{\alpha} f(t)=\begin{cases}\frac{1}{\Gamma(-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha-1}f(\tau)d\tau & \alpha<0\\ f(t) & \alpha=0\\ D^n\left[{}_{t_0}D_t^{\alpha-n}f(t)\right] & \alpha>0\end{cases} \tag{1}$$

where $n-1\leq\alpha<n$ and $\Gamma(.)$ is the standard gamma function, $\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt$.

Definition 2. (Caputo fractional derivative [30]) The Caputo fractional derivative is defined as:

$$ {}_{t_0}^{C}D_t^{\alpha} f(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau & n-1<\alpha<n\\ D^n f(t) & \alpha=n\end{cases} \tag{2}$$

The Caputo fractional derivative was almost used in engineering problems, because derivatives appeared on integer points, so, they could have physical implementation. But, in the Riemann-Liouville definition, derivatives appear in fractional points, and, in numerical solving, we must know the initial conditions in the fractional points of derivation, which may have not physical implementation.

Definition 3. A dynamic system in fractional calculus is defined as:

$$F\left(t,y(t),{}_{t_0}^{C}D_t^{\alpha_1}y(t),{}_{t_0}^{C}D_t^{\alpha_2}y(t),...,{}_{t_0}^{C}D_t^{\alpha_n}y(t)\right)=g(t), \tag{3}$$

where $\alpha_1<\alpha_2<...<\alpha_n$, $F(t,y_1,...,y_n)$ and $g(t)$ are real known functions. It can also be defined in the state space form as:

$$ {}_{t_0}^{C}D_t^{\alpha_i}x_i=f_i\left(t,x_1,x_2,...,x_n\right),$$

$$x_i(0)=X_{i0},\qquad i=1,2,...,n, \tag{4}$$

where $0<\alpha_i\leq 1$ for $i=1,2,...,n$.

A linear dynamic system in state space form is like:

$$\begin{pmatrix} {}_{t_0}^{C}D_y^{\alpha_1} x_1 \\ {}_{t_0}^{C}D_y^{\alpha_2} x_2 \\ \vdots \\ {}_{t_0}^{C}D_y^{\alpha_n} x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \quad (5)$$

3. 4D Lü hyper-chaotic system

Systems with more than one (especially 2) positive Lyapunov exponents are known as hyper-chaotic systems in literature. This implies that their dynamics are expanded in several different directions simultaneously. In recent years, several hyper-chaotic systems were discovered in high-dimensional dynamics. For example, see the hyper-chaotic Rossler system [31], the hyper-chaotic Lorenz system [32], the hyper-chaotic Chua circuit [33] etc.

The 4D Lü hyper-chaotic dynamical system is based on the 3D original Lü system [34] by adding a state feedback. In 2006, Elabbasy, Agiza and El-Dessoky presented differential equations of the 4D Lü hyper-chaotic system as [19]:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = cx_2 - x_1x_3 + x_4 \\ \dot{x}_3 = x_1x_2 - bx_3 \\ \dot{x}_4 = x_3 - dx_4 \end{cases} \quad (6)$$

in which, the 4th state is a simple state feedback which is added to the 2nd state.

Both response and drive systems have characteristic equations, as the above, but the main difference is that all the states of the response system are followed by a controller:

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1) + u_1 \\ \dot{y}_2 = cy_2 - y_1y_3 + y_4 + u_2 \\ \dot{y}_3 = y_1y_2 - by_3 + u_3 \\ \dot{y}_4 = y_3 - dy_4 + u_4 \end{cases} \quad (7)$$

This system demonstrates a hyper-chaotic attractor with many different sets of parameters. In Figure 1, trajectories of the 4D Lü system with a set of parameters ($a = 20, b = 5, c = 10, d = 1.5$) are shown. These parameters made the system to be hyper-chaotic with Lyapunov exponents equal to $\{0.75, 0.03, -1.55, -15.73\}$.

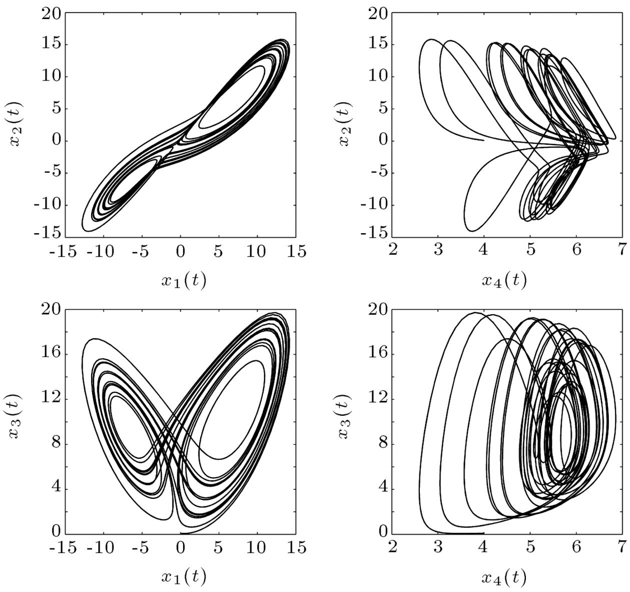

Figure 1. Trajectories of the 4D Lü hyper-chaotic system.

The Lyapunov dimension of the system with the above parameters is 3.34, which shows that the system is hyper-chaotic.

3.1. Synchronization

In [21], it was shown that the response system in Eq. (7) can be synchronized with the drive system in Eq. (6) using a single state feedback only on the 2nd state. The main theorem is stated here.

Theorem 1. For Eq. (6), suppose that B_2 and B_3 are the upper bounds of absolute values of state variables, x_2 and x_3, respectively. For the positive constant, $\lambda > dB_2^2/a(4bd-1) > 0$, the system in Eq. (7) with controllers $u_2 = -k_2(y_2 - x_2)$, $u_1 = u_3 = u_4 = 0$, can be synchronized to the system in Eq. (6), and the zero equilibrium point of the error dynamic system ($\mathbf{e} = \mathbf{y} - \mathbf{x}$) is globally asymptotically stable, where:

$$k_2 > \max(g_1, g_2, g_3),$$

$$g_1 = \min_{\lambda} \left(\frac{(\lambda a + B_3)^2}{4\lambda a} + c \right) > 0,$$

$$g_2 = \min_{\lambda} \left(\frac{b(\lambda a + B_3)^2}{4\lambda ab - B_2^2} + c \right) > 0,$$

$$g_3 = \min_{\lambda} \left(\frac{(\lambda a+B_3)^2(4bd-1)+4\lambda ab+2B_2(\lambda a+B_3)-B_2^2}{4\lambda a(4bd-1)-4dB_2^2} + c \right) > 0. \quad (8)$$

Proof. The error dynamics can be easily obtained by subtracting Eq. (6) from Eq. (7):

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + u_1 \\ \dot{e}_2 = ce_2 + e_4 - e_1e_3 - x_1e_3 - x_3e_1 + u_2 \\ \dot{e}_3 = -be_3 + e_1e_2 + x_2e_1 + x_1e_2 + u_3 \\ \dot{e}_4 = e_3 - de_4 + u_4 \end{cases} \tag{9}$$

A Lyapunov function is defined here as:

$$V_1(t) = \frac{1}{2}\left(\lambda e_1^2 + e_2^2 + e_3^2 + e_4^2\right). \tag{10}$$

Differentiating the Lyapunov function, with respect to time, yields:

$$\begin{aligned} \dot{V}_1(t) =& \lambda e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 \\ =& \lambda e_1\left(ae_2 - ae_1 + u_1\right) \\ &+ e_2\left(ce_2 + e_4 - e_1e_3 - x_1e_3 - x_3e_1\right) \\ &+ e_3\left(-be_3 + e_1e_2 + x_2e_1 + x_1e_2 + u_3\right) \\ &+ e_4\left(e_3 - de_4 + u_4\right) \\ =& -\lambda ae_1^2 + ce_2^2 - be_3^2 \\ &- de_4^2 + e_1e_2\left(\lambda a - x_3\right) + e_2e_4 + e_3e_4 \\ &+ e_1e_3(x_2) + \lambda e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4. \end{aligned} \tag{11}$$

Substituting $u_1 = u_3 = u_4 = 0$ and $u_2 = -k_2e_2$ in the above equation yields:

$$\begin{aligned} \dot{V}_1(t) =& -\lambda ae_1^2 + (c - k_2)e_2^2 - be_3^2 - de_4^2 \\ &+ e_1e_2(\lambda a - x_3) + e_2e_4 + e_3e_4 + e_1e_3(x_2) \\ &< -\mathbf{e}^{\mathbf{T}}\mathbf{P}\mathbf{e}, \end{aligned} \tag{12}$$

where:

$$\mathbf{P} = \begin{pmatrix} \lambda a & \frac{(-\lambda a + B_3)}{2} & \frac{-B_2}{2} & 0 \\ \frac{(-\lambda a + B_3)}{2} & k_2 - c & 0 & \frac{-1}{2} \\ \frac{-B_2}{2} & 0 & b & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-1}{2} & d \end{pmatrix},$$

$$e = \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \\ |e_4| \end{bmatrix}. \tag{13}$$

In [21], it was shown that if all the above conditions are satisfied, matrix **P** would be positive definite, so, $\dot{V}_1(t) < 0$, and the origin of the synchronization error space will be globally asymptotically stable. □

3.2. Adaptive synchronization

In real systems, some or all of the system parameters are unknown or maybe with some uncertainties. These unknown or uncertain parameters can completely destroy the procedure of synchronization. In this section, an adaptive synchronization method for two identical hyper-chaotic Lü systems is developed.

Consider the response system stated in Eq. (7) again with estimated parameters, a_r, b_r, c_r and d_r:

$$\begin{cases} \dot{y}_1 = a_r(y_2 - y_1) + u_1 \\ \dot{y}_2 = c_ry_2 - y_1y_3 + y_4 + u_2 \\ \dot{y}_3 = y_1y_2 - b_ry_3 + u_3 \\ \dot{y}_4 = y_3 - d_ry_4 + u_4 \end{cases} \tag{14}$$

The response system is a computer-based system that we wish to synchronize with the real drive system in Eq. (6), with unknown parameters.

The error dynamics equations can be derived again as:

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + e_a(y_2 - y_1) + u_1 \\ \dot{e}_2 = ce_2 + e_4 + e_1e_3 - y_1e_3 - y_3e_1 + e_cy_2 + u_2 \\ \dot{e}_3 = -be_3 - e_1e_2 + y_2e_1 + y_1e_2 - e_by_3 + u_3 \\ \dot{e}_4 = e_3 - de_4 - e_dy_4 + u_4 \end{cases} \tag{15}$$

In the above equation, $e_a = a_r - a$ is the parameter estimation error and e_b, e_c and e_d are defined similarly. Now, we can define the new Lyapunov function as:

$$V_2(t) = V_1^*(t) + \frac{1}{2}\left(e_a^2 + e_b^2 + e_c^2 + e_d^2\right) + \frac{1}{2}(\tilde{k}_2 - k_2)^2, \tag{16}$$

where $\tilde{k}_2$ is the estimate of the controller gain and $V_1^*(t)$ is in the form of Eq. (10). $\dot{V}_1^*(t)$ is computed as:

$$\begin{aligned} \dot{V}_1^*(t) =& \lambda e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 \\ =& \lambda e_1\left(ae_2 - ae_1 + e_ay_2 - e_ay_1 + u_1\right) \\ &+ e_2\left(ce_2 + e_4 + e_1e_3 - y_1e_3 - y_3e_1 + e_cy_2 + u_2\right) \\ &+ e_3\left(-be_3 - e_1e_2 + y_2e_1 + y_1e_2 - e_by_3 + u_3\right) \\ &+ e_4\left(e_3 - de_4 - e_dy_4 + u_4\right). \end{aligned} \tag{17}$$

Then, here the rate of change vs. time of the Lyapunov function is:

$$\dot{V}_2(t) = \dot{V}_1^*(t) + e_a\dot{e}_a + e_b\dot{e}_b + e_c\dot{e}_c + e_d\dot{e}_d + (\tilde{k}_2 - k_2)\dot{\tilde{k}}_2. \quad (18)$$

Assuming again $u_1 = u_3 = u_4 = 0$ and $u_2 = -\tilde{k}_2 e_2$, the above equation can be expanded as:

$$\begin{aligned}\dot{V}_2(t) = &- \lambda a e_1^2 + (c - k_2)e_2^2 - b e_3^2 - d e_4^2 \\ &+ e_1 e_2(\lambda a - x_3) + e_2 e_4 + e_3 e_4 + e_1 e_3(x_2) \\ &+ e_a(\lambda e_1 y_2 - \lambda e_1 y_1 + \dot{e}_a) + e_c(y_2 e_2 + \dot{e}_c) \\ &+ e_b(-e_3 y_3 + \dot{e}_b) + e_d(-y_4 e_4 + \dot{e}_d) \\ &+ (\tilde{k}_2 - k_2)(\dot{\tilde{k}}_2 - e_2^2). \end{aligned} \quad (19)$$

Thus:

$$\begin{aligned}\dot{V}_2(t) = &\dot{V}_1(t) + e_a(\lambda e_1 y_2 - \lambda e_1 y_1 + \dot{e}_a) \\ &+ e_c(y_2 e_2 + \dot{e}_c) + e_b(-e_3 y_3 + \dot{e}_b) \\ &+ e_d(-y_4 e_4 + \dot{e}_d) + (\tilde{k}_2 - k_2)(\dot{\tilde{k}}_2 - e_2^2), \end{aligned} \quad (20)$$

where $\dot{V}_1(t) < -\mathbf{e^T P e}$, as shown in Eq. (12). By substituting the adaptation laws as:

$$\begin{aligned} &\dot{e}_a = -\lambda e_1 y_2 + \lambda e_1 y_1, \\ &\dot{e}_b = y_3 e_3, \\ &\dot{e}_c = -y_2 e_2, \\ &\dot{e}_d = y_4 e_4, \\ &\dot{\tilde{k}}_2 = e_2^2, \end{aligned} \quad (21)$$

and applying in Eq. (20), we have:

$$\dot{V}_2(t) = \dot{V}_1(t) < -\mathbf{e^T P e} \Rightarrow \dot{V}_2(t) \leq 0. \quad (22)$$

Lemma 1. Consider all assumptions in Theorem 1. The zero equilibrium point of the error dynamic system in Eq. (15) is globally asymptotically stable by applying adaptation laws in Eq. (21) and using controller, $u_2 = -\tilde{k}(y_2 - x_2)$, $u_1 = u_3 = u_4 = 0$.

Proof. $\mathbf{P}$ is a symmetric positive definite matrix, so, it can be written in the form:

$$\mathbf{P} = \mathbf{S}^T\mathbf{S}. \quad (23)$$

So, we have:

$$\dot{V}_2(t) \leq -(\mathbf{Se})^T(\mathbf{Se}), \quad (24)$$

where $\mathbf{S}$ is a constant nonsingular matrix.

Assume the integral below:

$$I = \int_0^\infty -\dot{V}_2(t)dt = V_2(0) - V_2(\infty). \quad (25)$$

Since $\dot{V}_2$ is negative and $V_2(t)$ is always positive, $V_2(\infty)$ is bounded; it means that integral I is bounded too. According to Relation (24):

$$\int_0^\infty (\mathbf{Se})^T(\mathbf{Se})dt < I. \quad (26)$$

So, one can say:

$$(\mathbf{Se}) \in L_2. \quad (27)$$

Again, since $\dot{V}_2$ is negative semi definite, all of the state errors and parameter estimation errors will be bounded. Consequently, the time derivative of errors in Eq. (15) will be bounded and $\frac{d(\mathbf{Se})}{dt} = \mathbf{S}\dot{\mathbf{e}}$ will be bounded too.

Now, using the Barbalat lemma, we may have:

$$\lim_{t\to\infty} \mathbf{Se}(t) = 0. \quad (28)$$

According to the fact that $\mathbf{S}$ is nonsingular, it can be said:

$$\lim_{t\to\infty} \mathbf{e}(t) = 0. \quad (29)$$

So, the proof is completed. □

Now, if we solve equation $\dot{V}_2 = 0$ according to Lemma 1, the only admissible answer, when time converges to infinity, is the inevitable solution, which is $\mathbf{e}(\infty) = 0$. Applying this solution to the system in Eq. (15) and knowing that $\dot{\mathbf{e}}(\infty) = 0$, we will have:

$$e_a(\infty) = e_b(\infty) = e_c(\infty) = e_d(\infty) = 0. \quad (30)$$

So, using LaSalle's invariant principle [35], the origin of the system of Eqs. (15) and (21), together, will be asymptotically stable.

The above statements can be collected in the following theorem.

Theorem 2. Consider all assumptions in Theorem 1. Eq. (7) with controllers $u_1 = u_3 = u_4 = 0$ and $u_2 = -\tilde{k}(y_2 - x_2)$ can be synchronized to Eq. (6), and the zero equilibrium point of error dynamic system in Eq. (15) is globally asymptotically stable by applying adaptation laws in Eq. (21), and the convergence of the parameters is guaranteed.

Proof. The whole procedure is stated from the beginning of the current subsection. □

According to Theorem 2, a computer-based system in Eq. (14) with unknown parameters, a_r, b_r, c_r and d_r, can be synchronized with the real system in Eq. (6) by measuring all the states and using a single state feedback only. Theorem 2 shows that the errors go to zero asymptotically and the parameters also converge to true values.

4. Chen chaotic system

In this section, we take Chen chaotic system as our drive system. Its differential equations are [36]:

$$\begin{cases}\dot{x}_1 = a(x_2 - x_1)\\ \dot{x}_2 = bx_2 - cx_1 - x_1x_3\\ \dot{x}_3 = x_1x_2 - dx_3\end{cases} \tag{31}$$

where the parameters are $a = 35$, $b = 28$, $c = 7$ and $d = 3$. Also, we take the response system to be the same as the drive one.

$$\begin{cases}\dot{y}_1 = a(y_2 - y_1) + u_1\\ \dot{y}_2 = by_2 - cy_1 - y_1y_3 + u_2\\ \dot{y}_3 = y_1y_2 - dy_3 + u_3\end{cases} \tag{32}$$

4.1. Synchronization

Theorem 3. For system in Eq. (31), as in Theorem 1, suppose that B_2 and B_3 are the upper bounds of absolute values of state variables, x_2 and x_3, respectively. For the positive constant, $\lambda > B_2^2/4ad > 0$, Eq. (32) with controllers, $u_2 = -k_2(y_2 - x_2)$ and $u_1 = u_3 = 0$, can be synchronized to Eq. (31) and the zero equilibrium point of error dynamic system ($\mathbf{e} = \mathbf{y} - \mathbf{x}$) is globally asymptotically stable, where:

$$k_2 > \max(g_1, g_2),$$

$$g_1 = \min_{\lambda}\left(\frac{(\lambda a - c + B_3)^2}{4\lambda a} + b\right) > 0,$$

$$g_2 = \min_{\lambda}\left(\frac{d(\lambda a - c + B_3)^2}{4\lambda ad - B_2^2} + b\right) > 0. \tag{33}$$

Proof. The procedure is completely like the proof of theorem 1. □

4.2. Adaptive synchronization

Consider the response system stated in Eq. (32) again, with estimated parameters, a_r, b_r, c_r and d_r:

$$\begin{cases}\dot{y}_1 = a_r(y_2 - y_1) + u_1\\ \dot{y}_2 = b_ry_2 - c_ry_1 - y_1y_3 + u_2\\ \dot{y}_3 = y_1y_2 - d_ry_3 + u_3\end{cases} \tag{34}$$

Theorem 4. Consider all assumptions in Theorem 3. The system in Eq. (32) with controllers $u_1 = u_3 = 0$ and $u_2 = -\tilde{k}(y_2 - x_2)$ can be synchronized to Eq. (31), and the zero equilibrium point of error dynamic system is globally asymptotically stable by applying adaptation laws in Eq. (35). Thus, the convergence of the parameters is guaranteed.

$$\dot{e}_a = -\lambda e_1y_2 + \lambda e_1y_1,$$

$$\dot{e}_b = -y_2e_2,$$

$$\dot{e}_c = y_1e_2,$$

$$\dot{e}_d = y_3e_3,$$

$$\dot{\tilde{k}}_2 = e_2^2. \tag{35}$$

Proof. The whole procedure is completely like the proof of Theorem 2. □

5. Fractional adaptation

The 4D Lü system is hyper-chaotic; therefore the speed of parameter identification is very important in synchronization. So, we can change the adaptation laws' dynamics with fractional order equations to obtain better identification and faster state synchronization.

To achieve this goal, adaptation laws are assumed to be:

$$D^{\alpha_1}e_a = -\lambda e_1y_2 + \lambda e_1y_1,$$

$$D^{\alpha_3}e_b = y_3e_3,$$

$$D^{\alpha_2}e_c = -y_2e_2,$$

$$D^{\alpha_4}e_d = y_4e_4, \tag{36}$$

where $0 < \alpha_i \leq 1$ and D^{α} denotes the Caputo derivative from $t_0 = 0$. Because of the similarity of fractional systems with a derivative order of less than one to damped systems, these new adaptation laws would be stable and can estimate parameters faster and with fewer fluctuations.

Also, here, for achieving better convergence in single-state feedback gain, the dynamics of controller gain can be assumed to be fractional as:

$$D^{\alpha}\tilde{k}_2 = e_2^2, \tag{37}$$

where $0 < \alpha \leq 1$ too.

As we will see in the next section, convergence in parameters a, b and c are faster than d and it seems to be over-damped. But, parameter d has some fluctuations, and, then, error in the 4th state takes more time to go to zero. This problem can be solved easily by applying less degrees of differentiation in the adaptation law for this parameter.

Also for the Chen chaotic system, we take the following as adaptation laws:

$$D^{\alpha_1}e_a = -\lambda e_1y_2 + \lambda e_1y_1,$$

$$D^{\alpha_2}e_b = -y_2e_2,$$

$$D^{\alpha_3}e_c = y_1e_2,$$

$$D^{\alpha_4}e_d = y_3e_3,$$

$$D^{\alpha_5}\tilde{k}_2 = e_2^2, \tag{38}$$

in which we use $\alpha_4 = 0.7$ for parameter d, and other adaptation laws remain in integer order.

6. Numerical simulations

In this section, numerical simulations are presented to show the effectiveness of the fractional method discussed in the previous section. The PECE algorithm [37] is used to solve differential Eqs. (6), (14), (36) and (37), assuming a Caputo derivative with a time step of size 0.001. This method is like fourth-order Runge-Kutta for integer order equations. For fractional solution only, in Eq. (36), $\alpha_4 = 0.55$ is taken and all others are equal to 1, which means they are integer order differential equations. $B_2 = B_3 = 20$ and $\lambda = 2.0345$ are taken, and to solve Eq. (6), the parameters are taken as $a = 20, b = 5, c = 10, d = 1.5$ and the initial conditions are set as $\mathbf{x}_0 = \begin{bmatrix}0.1, & 0.1, & 0.1, & 0.1\end{bmatrix}^T$. Also, the initial conditions for Eqs. (14), (36) and (37) are, respectively, $\mathbf{y_0} = \begin{bmatrix}-9.9, & -4.9, & 5.1, & 10.1\end{bmatrix}^T$, $a_r(0) = 25$, $b_r(0) = 10$, $c_r(0) = 15$, $d_r(0) = 6$ and $\tilde{k}_2(0) = 30$.

Figure 2 shows the estimation of parameters for a_r, b_r, c_r, and Figure 3 shows the estimation of parameter d_r of the 4D Lü hyper-chaotic system. It is obvious that all of them are converged to their true value by both methods. But, according to these two figures, parameter d_r fluctuates more than others in the integer order method. So, the fractional order method is used for modifying the equation of this parameter and d_r converges smoothly to the final value. This method almost does not affect the other estimations. Also, Figure 4 shows the convergency of the last parameter when a different order of fractional derivative is used.

Figure 5 shows the error of synchronization for the first, second and third state of the response system and Figure 6 shows the error of synchronization for the fourth state of the 4D Lü hyper-chaotic system. It shows that the difference between the fractional and integer order method is negligible for the first three states. Also, these three errors approach zero fast and

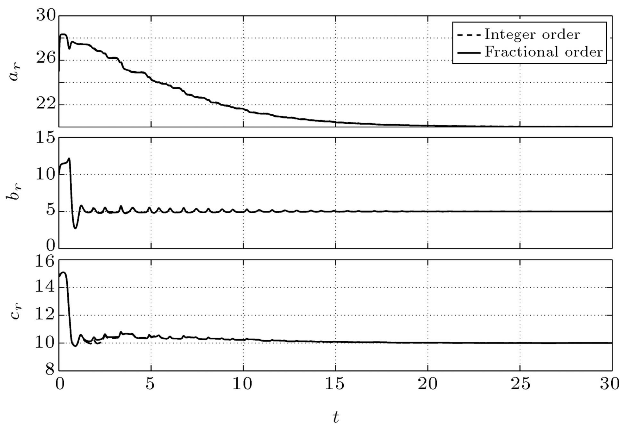

Figure 2. Three parameter estimation a_r, b_r and c_r for the 4D Lü hyper-chaotic system.

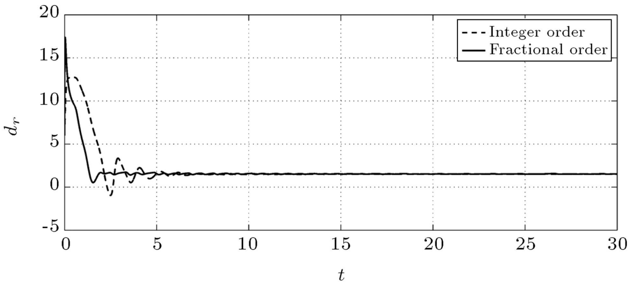

Figure 3. Parameter estimation d_r for the 4D Lü hyper-chaotic system.

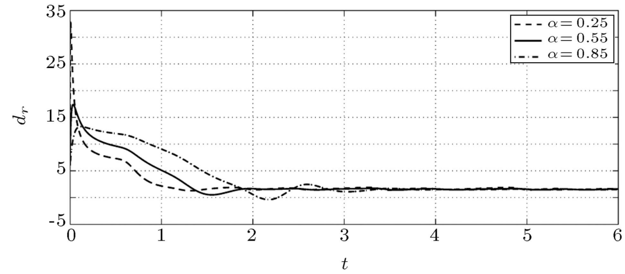

Figure 4. Error in parameter estimation d_r with different fractional orders for the 4D Lü hyper-chaotic system.

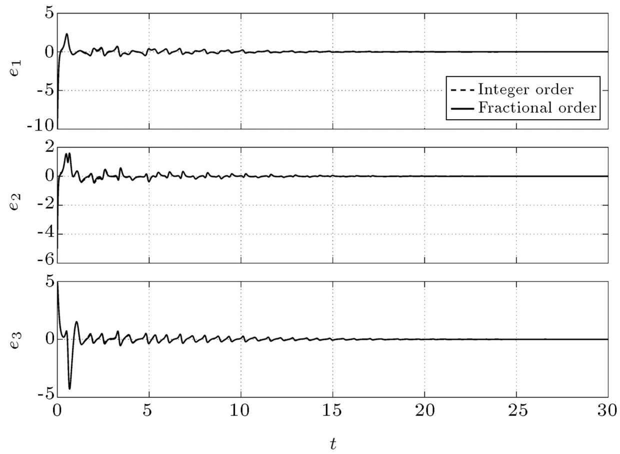

Figure 5. Synchronization error of the first three states for the 4D Lü hyper-chaotic system.

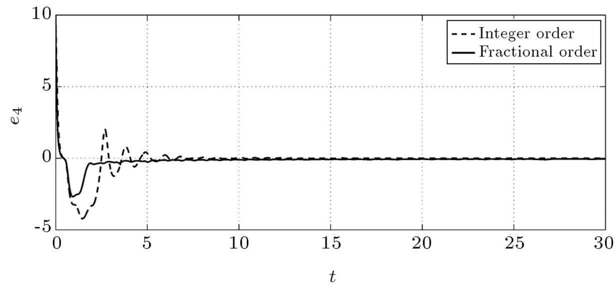

Figure 6. Synchronization error of the fourth state for the 4D Lü hyper-chaotic system.

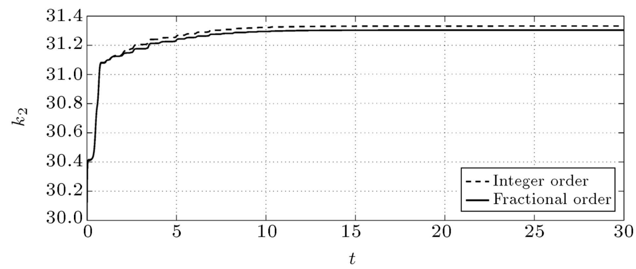

Figure 7. Control gain calculation for the 4D Lü hyper-chaotic system.

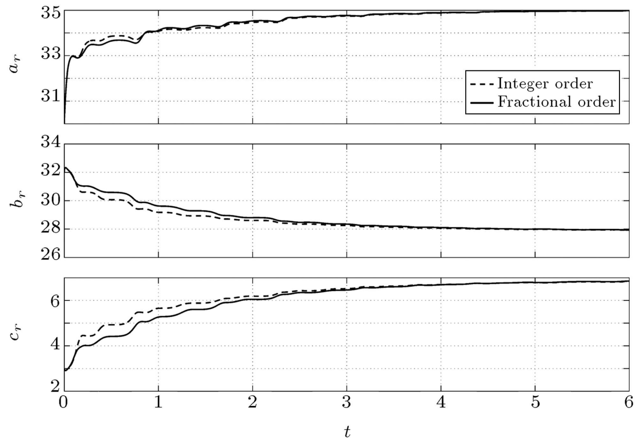

Figure 8. Three parameter estimation a_r, b_r and c_r for the Chen chaotic system.

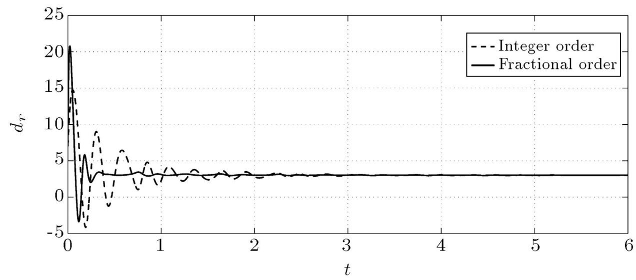

Figure 9. Parameter estimation d_r for the Chen chaotic system.

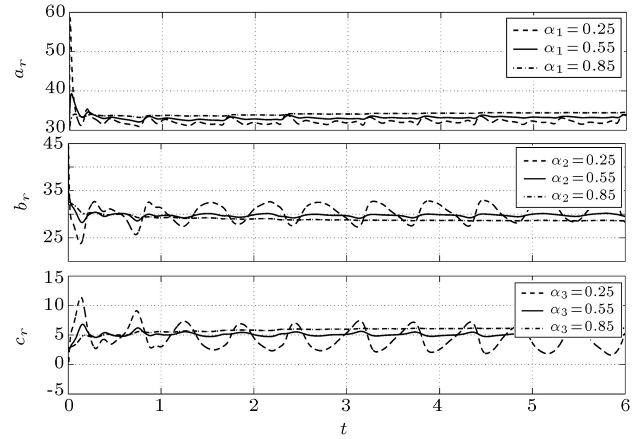

Figure 10. Error in parameters estimation a_r, b_r and c_r with different fractional orders for the Chen chaotic system.

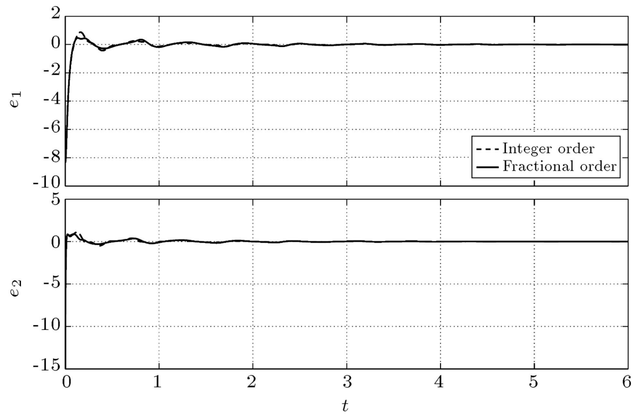

Figure 11. Synchronization error of the first and second states for the Chen chaotic system.

almost in a proper manner in comparison to the last state in the integer method. The reason why the last state error fluctuates is because parameter d_r fluctuates in this method of solution. But, the fractional method causes both estimations and, consequently, synchronization error would converge more smoothly.

Figure 7 shows the control gain estimation for both integer order and fractional order methods for the 4D Lü hyper-chaotic system.

In numerical simulation of the Chen chaotic system, in Eq. (38), $\alpha_4 = 0.7$ is taken and all others are equal to 1, which means they are integer order differential equations. $B_2 = 30$, $B_3 = 50$ and $\lambda = 3$ are taken, and to solve Eq. (31), the parameters are assumed as $a = 35, b = 28, c = 7, d = 3$ and the initial conditions are set to $\mathbf{x_0} = \begin{bmatrix}1, & 5, & 20\end{bmatrix}^T$. Also, the initial conditions for Eqs. (34) and (38) are, respectively, $\mathbf{y_0} = \begin{bmatrix}-7, & -10, & 35\end{bmatrix}^T$, $a_r(0) = 30$, $b_r(0) = 32$, $c_r(0) = 3$, $d_r(0) = 7$ and $\tilde{k}_2(0) = 211$.

Figure 8 shows the estimation of parameters for a_r, b_r, c_r and Figure 9 shows the estimation of parameter d_r of the Chen chaotic system. As seen, all parameters have converged to their actual value. Also, the estimation of d_r has less fluctuations in the fractional order method and has settled sooner. Actually, settling time using the fractional order method is almost one third of the integer order method. Moreover, the fractional order method does not affect other estimations.

In Figure 10, estimation of parameters a_r, b_r and c_r for the Chen chaotic system is shown. The difference is that for each parameter, one of the adaptation laws in Eq. (38) is obtained as fractional to examine the effect of fractional order in other parameters. As can be seen, while α_i is getting closer to unity, the estimation becomes better.

Figures 11 and 12 show synchronization errors for the states of the Chen chaotic system. In both

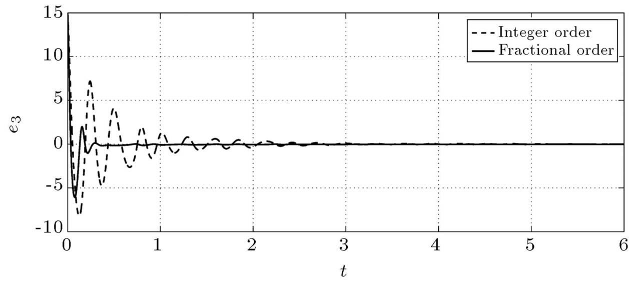

Figure 12. Synchronization error of the third state for the Chen chaotic system.

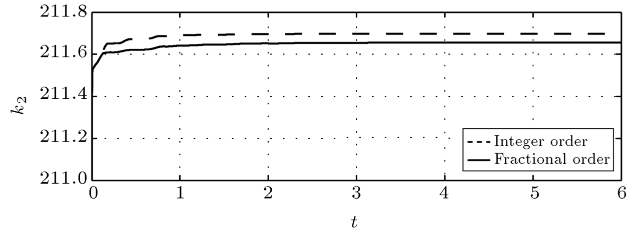

Figure 13. Control gain calculation for the Chen chaotic system.

responses, the error has become zero, but, in the fractional order, according to the influence of d_r on the third state, it has settled sooner with fewer fluctuations. Also, it is obvious in Figure 11 that the fractional order method does not affect the response.

Figure 13 shows the control gain estimation for both integer order and fractional order methods for the Chen chaotic system.

7. Stability analysis

As mentioned in the previous section, the adaptation law for parameter d is replaced by the following:

$$D^\alpha e_d = y_4 e_4, \tag{39}$$

where D^α denotes the Caputo derivative with $t_0 = 0$. The Caputo derivative definition plays an important role in stability analysis because they have appeared inside the integral (see Eq. (2)) and we used this property. All other adaptation laws were kept unchanged. We can now rearrange the Lyapunov function in Eq. (20) as the following:

$$\begin{aligned}\dot{V}_2(t) =&\dot{V}_1(t) + e_a\left(\lambda e_1 y_2 - \lambda e_1 y_1 + \dot{e}_a\right)\\ &+ e_c(y_2 e_2 + \dot{e}_c) + e_b(-e_3 y_3 + \dot{e}_b)\\ &+ (\tilde{k}_2 - k_2)(\dot{\tilde{k}}_2 - e_2^2) + e_d(-y_4 e_4\\ &+ D^\alpha e_d - D^\alpha e_d + \dot{e}_d) = \ldots\\ &+ e_d(-y_4 e_4 + D^\alpha e_d) + e_d(\dot{e}_d - D^\alpha e_d).\end{aligned} \tag{40}$$

Substituting adaptation laws in the above equation yields:

$$\begin{aligned}\dot{V}_2(t) =&\dot{V}_1(t) + e_d(\dot{e}_d - D^\alpha e_d) < -\mathbf{e}^{\mathbf{T}}\mathbf{P}\mathbf{e}\\ &+ e_d(\dot{e}_d - D^\alpha e_d).\end{aligned} \tag{41}$$

It is enough to show that the last term is negative and/or behaves in an appropriate manner.

Lemma 2. Consider:

$$w(\tau) = \frac{1}{\Gamma(1-\beta)(t-\tau)^\beta} - \frac{1}{\Gamma(1-\alpha)(t-\tau)^\alpha},$$

where $0 < \alpha < \beta < 1$. Function $w(\tau)$ is always negative and descending when $t > 0$, $0 \le \tau \le t$ and $\beta \to 1$.

Proof. We can rewrite $w(\tau)$ as:

$$w(\tau) = \frac{\Gamma(1-\alpha)(t-\tau)^\alpha - \Gamma(1-\beta)(t-\tau)^\beta}{\Gamma(1-\alpha)(t-\tau)^\alpha \Gamma(1-\beta)(t-\tau)^\beta}. \tag{42}$$

The denominator of $w(\tau)$ is always positive, so, we must prove only that the numerator is negative. Let us define $n(\tau)$ as:

$$\begin{aligned}n(\tau) \triangleq&\operatorname{num}(w(\tau)) = \Gamma(1-\alpha)(t-\tau)^\alpha\\ &- \Gamma(1-\beta)(t-\tau)^\beta \Rightarrow n(0) = \Gamma(1-\alpha)t^\alpha\\ &- \Gamma(1-\beta)t^\beta.\end{aligned} \tag{43}$$

It is obvious that for any positive t there exists a β near 1 where $n(0)$ is negative. Besides, we have:

$$\begin{aligned}&\lim_{\substack{\beta\to 1\\ t>0}} n(0) = \lim_{\substack{\beta\to 1\\ t>0}} \Gamma(1-\alpha)t^\alpha - \Gamma(1-\beta)t^\beta = -\infty\\ &\qquad \Rightarrow \lim_{\substack{\beta\to 1\\ t>0}} w(0) < 0.\end{aligned} \tag{44}$$

Again, computing a derivative of $n(\tau)$ with respect to τ, we have:

$$\frac{d}{d\tau}n(\tau) = \Gamma(1-\alpha)\alpha(t-\tau)^{\alpha-1} - \Gamma(1-\beta)\beta(t-\tau)^{\beta-1}, \tag{45}$$

which is negative when $0 \le \tau \le t$ and β is near 1. Now, we want to find the zero point of the above function with respect to τ:

$$\begin{aligned}&\frac{d}{d\tau}n(\tau) = 0\\ &\quad \Rightarrow \Gamma(1-\alpha)\alpha(t-\tau)^{\alpha-1} = \Gamma(1-\beta)\beta(t-\tau)^{\beta-1}\\ &\quad \Rightarrow \frac{\Gamma(1-\alpha)\alpha}{\Gamma(1-\beta)\beta} = (t-\tau)^{\beta-\alpha}\\ &\quad \Rightarrow \tau^* = t - \left(\frac{\Gamma(1-\alpha)\alpha}{\Gamma(1-\beta)\beta}\right)^{1/\beta-\alpha}.\end{aligned} \tag{46}$$

We can easily show $\tau^* \to t$, when $\beta \to 1$, so, $\frac{d}{d\tau}n(\tau)$ has no zero point in the interval $(0,t)$. From Eqs. (42) to (46), we see:

1. $w(\tau)$ is a function with a negative starting point. $w(0) < 0$;
2. The denominator of $w(\tau)$ is always positive and ascending; numerator $n(\tau)$ has a negative derivative in all domains $(0,t)$.

So, $w(\tau)$ is a negative and descending function all over the domain. □

Proposition. The term $e_d(\dot{e}_d - D^\alpha e_d)$ in Eq. (41) always behaves such that the asymptotic stability of the controlled system is guaranteed.

Discussion. Let us replace $\dot{e}_d$ with $D^\beta e_d$, where β is tending to 1. So, we may have:

$$e_d({}_0D_t^\beta e_d -{}_0 D_t^\alpha e_d) = e_d(t)$$
$$\times\left\{\frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\dot{e}_d(\tau)d\tau}{(t-\tau)^\beta} - \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{e}_d(\tau)d\tau}{(t-\tau)^\alpha}\right\}$$
$$= e_d(t)\left\{\int_0^t \dot{e}_d(\tau)d\tau\left(\frac{1}{\Gamma(1-\beta)(t-\tau)^\beta}\right.\right.$$
$$\left.\left. - \frac{1}{\Gamma(1-\alpha)(t-\tau)^\alpha}\right)\right\}. \tag{47}$$

Defining:

$$w(\tau) = \frac{1}{\Gamma(1-\beta)(t-\tau)^\beta} - \frac{1}{\Gamma(1-\alpha)(t-\tau)^\alpha},$$

we have:

$$e_d({}_0D_t^\beta e_d -{}_0 D_t^\alpha e_d) = e_d(t)\int_0^t w(\tau)\dot{e}_d(\tau)d\tau, \tag{48}$$

where $w(\tau)$ is a weight function. From Lemma 2, we have seen $w(\tau)$ is always a negative descending function when $\beta \to 1$. We define A_i and I_i as below:

$$I_i = \int_{\theta_{i-1}}^{\theta_i} A_i(t,\tau)w(\tau)d\tau, \quad i = 1,...,N$$
$$A_i(\tau',\tau) = e_d(\tau')\dot{e}_d(\tau), \quad \tau \in [\theta_{i-1},\theta_i], \tag{49}$$

where $\theta_0 = 0$ and $\theta_N = t$. Also, the sign of $A_i(\tau,\tau)$ does not change in interval $[\theta_{i-1},\theta_i]$, so, N is the number of changing signs occurred in $A_i(\tau,\tau)$ before both e_d and $\dot{e}_d$ tend to zero. Now, Eq. (48) can be rewritten again as:

$$I = e_d(\dot{e}_d - D^\alpha e_d)$$
$$= e_d(t)\int_0^t w(\tau)\dot{e}_d(\tau)d\tau = \sum_{i=1}^N I_i. \tag{50}$$

e_d and $\dot{e}_d$ are continuous, and only one of $e_d(t)$ or $\dot{e}_d(t)$ changes signs when $A(\tau,\tau)$ changes sign.

Proposition. Consider the sign of $A_i(\tau,\tau)$ in interval $[\theta_{i-1},\theta_i]$:

- If it has a positive sign, it means that e_d and $\dot{e}_d$ have the same sign and the system goes to instability. So, at the end of the interval at $\tau = \theta_i$ only $\dot{e}_d$ can change sign.
- If it has a negative sign, it means e_d and $\dot{e}_d$ have the opposite sign. So, e_d tends to zero and changes sign at the end of the interval at $\tau = \theta_i$.

Now, we discuss the number of intervals, N:

- If $N = 1$, it is clear that $I = I_1$ and no change in sign of e_d or $\dot{e}_d$ is occurred.

$$I = I_1 = e_d(t)\left\{\int_0^t \dot{e}_d(\tau)w(\tau)d\tau\right\}$$
$$= \int_0^t e_d(t)\dot{e}_d(\tau)w(\tau)d\tau$$
$$= \int_0^t A_1(t,\tau)w(\tau)d,\tau$$
$$\text{if} \quad A_1(t,\tau) > 0 \Rightarrow I < 0 \Rightarrow \dot{V}_2(t) < -\mathbf{e^TPe} + \mathbf{I} < 0$$
$$\text{if} \quad A_1(t,\tau) < 0,$$
$$\Rightarrow \begin{cases} \text{if } \mathbf{e^TPe} > \mathbf{I} \Rightarrow \dot{V}_2(t) < 0 \\ \text{if } \mathbf{e^TPe} < \mathbf{I} \Rightarrow \mathbf{e^TPe} \text{ is bounded} \end{cases} \tag{51}$$

It is obvious that if e_d and $\dot{e}_d$ have opposite signs, $|e_d|$ tends to zero, so, the upper bound of $\mathbf{e^TPe}$ tends to zero and the system is stable.

- If $N = 2$, it is clear that $I = I_1 + I_2$ and only one change in sign of e_d or $\dot{e}_d$ is occurred.

I. Assume $A_1(\tau,\tau) < 0$, so $A_2(\tau,\tau) > 0$. From the previous proposition, we know the sign of $\dot{e}_d(\tau)$ cannot change at $\tau = \theta_1$, then, the sign of $e_d(\tau)$ is changed at $\tau = \theta_1$. For $t < \theta_1$, $A_1(t,\tau) < 0$ is always negative, so, $e_d(\tau)$ tends to zero at the end of this interval. At the second interval, when $\theta_1 < t < \theta_2$, we may have:

$$\text{for} \quad t > \theta_1 : \dot{V}_2(t) < -\mathbf{e^TPe} + I_1 + I_2 = -\mathbf{e^TPe}$$
$$+ \int_0^{\theta_1} A_1(t,\tau)w(\tau)d\tau + \int_{\theta_1}^{\theta_2} A_2(t,\tau)w(\tau)d\tau$$
$$\forall t < \theta_1 : A_1(t,\tau) < 0$$
$$\Rightarrow \left.\begin{array}{l} \forall t > \theta_1 : A_1(t,\tau) > 0 \\ \forall t > \theta_1 : A_2(t,\tau) > 0 \end{array}\right\}$$
$$\Rightarrow I_1, I_2 < 0 \Rightarrow \dot{V}_2(t) < 0. \tag{52}$$

II. Assume $A_1(\tau,\tau) > 0$, so, $A_2(\tau,\tau) < 0$. From the previous proposition, we know the sign of $\dot{e}_d(\tau)$ changes at $\tau = \theta_1$. In the first interval, we have:

$$\text{for} \quad t < \theta_1 : \dot{V}_2(t) < -\mathbf{e}^{\mathbf{T}}\mathbf{Pe} + I_1 = -\mathbf{e}^{\mathbf{T}}\mathbf{Pe} + \int_0^{\theta_1} A_1(t,\tau)w(\tau)d\tau < 0. \quad (53)$$

So, the system is stable. In the second interval:

$$\text{for} \quad t > \theta_1 : \dot{V}_2(t) < -\mathbf{e}^{\mathbf{T}}\mathbf{Pe} + I_1 + I_2 = -\mathbf{e}^{\mathbf{T}}\mathbf{Pe} + \int_0^{\theta_1} A_1(t,\tau)w(\tau)d\tau + \int_{\theta_1}^{\theta_2} A_2(t,\tau)w(\tau)d\tau$$

$$\forall\, t < \theta_1 : A_1(t,\tau) > 0$$

$$\Rightarrow \left.\begin{matrix} \forall\, t > \theta_1 : A_1(t,\tau) > 0 \\ \forall\, t > \theta_1 : A_2(t,\tau) < 0 \end{matrix}\right\}$$

$$\Rightarrow I_1 < 0, I_2 > 0 \quad (54)$$

We can rewrite Eq. (54) like Eq. (51) and then:

$$\begin{cases} \text{if} \ \ \mathbf{e}^{\mathbf{T}}\mathbf{Pe} - I_1 > I_2 \Rightarrow \dot{V}_2(t) < 0 \\ \text{if} \ \ \mathbf{e}^{\mathbf{T}}\mathbf{Pe} - I_1 < I_2 \Rightarrow \mathbf{e}^T\mathbf{Pe} < I_2 \\ \quad \Rightarrow \mathbf{e}^T\mathbf{Pe} \ \ \text{is bounded} \end{cases} \quad (55)$$

Again, it is obvious that if e_d and $\dot{e}_d$ have opposite signs, $|e_d|$ tends to zero, so, the upper bound of $\mathbf{e}^T\mathbf{Pe}$ tends to zero and the system is stable.

This analysis can be repeated for $N \geq 3$. □

8. Conclusions

This paper has shown that identification of chaotic or hyper-chaotic systems can be done based on the synchronization of two identical systems. Two systems are synchronized by applying one state feedback controller. Adaptation laws used to find unknown parameters came from the Lyapunov stability theorem. By applying fewer degrees of differentiation in some of the adaptation laws (usually parameters with more ripples), less fluctuations in convergence of the parameter occur, as the results have shown in the numerical simulations. Finally, a discussion about the analytical proof of the stability of the controlled system using the fractional adaptation law is presented.

As can be seen, all simulations and analyses have been undertaken assuming Caputo definition for fractional differentiation. In the Riemann- Liouville definition, differentiation takes place after integration, so, some analyses and discussions may be done for other types of fractional-order derivatives in future work.

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