Scientia Iranica B (2014) 21(5), 1671–1682

Sharif University of Technology
Scientia Iranica
Transactions B: Mechanical Engineering
www.scientiairanica.com

Forward kinematic problem of three 4-DOF parallel mechanisms (4-$\underline{\text{P}}$RUR$_1$, 4-$\underline{\text{P}}$RUR$_2$ and 4-$\underline{\text{P}}$UU) with identical limb structures performing 3T1R motion pattern

P. Varshovi-Jaghargh[a], D. Naderi[a,*] and M. Tale-Masouleh[b]

a. *Department of Mechanical Engineering, Bu-Ali Sina University, Hamedan, P.O. Box 65175-4161, Iran.*
b. *Department of Mechatronics, Human and Robot Interaction Laboratory, Faculty of New Sciences and Technologies, University of Tehran, Tehran, Iran.*

Received 11 September 2012; received in revised form 7 December 2013; accepted 26 February 2014

KEYWORDS
Parallel mechanism;
3T1R motion pattern;
Schönflies motion;
Forward kinematic problem;
Three-dimensional Euclidean space.

Abstract. This paper investigates the forward kinematic problem of three 4-DOF parallel mechanisms performing three translations and one rotation motion, referred to as Schönflies motion. The kinematic arrangements of the mechanisms under study in this paper are such that two of them are classified as 4-$\underline{\text{P}}$RUR and one of them is a 4-$\underline{\text{P}}$UU. They are, respectively, special cases of 4-$\underline{\text{P}}$R$'$R$'$R$''$R$''$, 4-$\underline{\text{P}}$R$''$R$''$R$'$R$'$ and 4-$\underline{\text{P}}$R$''$R$'$R$'$R$''$ parallel mechanisms that have originated from the type synthesis of 4-DOF parallel mechanisms with identical limb structures. The forward kinematic problem is studied in three-dimensional Euclidean space, and a univariate expression describing the forward kinematic problem is obtained for each of the latter parallel mechanisms by the resultant method. The results obtained from this method show that a set of univariate expressions of degree (72, 64, 64, 8^2, 4, 2^8) describes the forward kinematic problem of 4-$\underline{\text{P}}$RUR$_1$ and 4-$\underline{\text{P}}$RUR$_2$ parallel mechanisms. Also, a quadratic univariate expression represents the forward kinematic problem of 4-$\underline{\text{P}}$UU parallel mechanisms. In addition, the system of equations corresponding to the forward kinematic problem is solved upon resorting to a homotopy continuation approach, which clarifies that the forward kinematic problems of 4-$\underline{\text{P}}$RUR$_1$, 4-$\underline{\text{P}}$RUR$_2$ and 4-$\underline{\text{P}}$UU parallel mechanisms admit up to 236, 236 and 2 finite solutions, real and complex.

1. Introduction

A Parallel Mechanism (PM) is a closed-loop kinematic chain mechanism, whose end-effector is linked to the base by several independent kinematic chains [1]. PMs have been used in a wide variety of applications, such as flight simulators [2,3], machine tools [4], industrial robots [5], twists [6], high-performance camera-orienting devices [7], wire robots [8], mining mechanisms [9], medical devices [10], micro-manipulators [11,12] and nano-manipulators [13]. In comparison with serial mechanisms, properly designed PMs generally have higher stiffness and accuracy [14]. However, there are some major deterrents to the widespread use of PMs in industrial applications, such the presence of extensive singularity configurations in their restricted workspace [15].

PMs usually used to have six DOF. The first

*. *Corresponding author. Tel.: +98 811 8257410; Fax: +98 811 8257400*
E-mail addresses: p.varshovi@.basu.ac.ir (P. Varshovi-Jaghargh); d_naderi@basu.ac.ir (D. Naderi); m.t.masouleh@ut.ac.ir (M. Tale-Masouleh)

application of a six-legged PM dates back to the 1950's, when a tire testing machine was developed by Gough [16], based on a PM. However, today, the prevalence of PM applications calls for the development of mechanisms with fewer numbers of DOF or limited-DOF [17]. In comparison with a 6-DOF PM, a limited-DOF PM has the advantages of a simple mechanical structure, low manufacturing cost, simple control algorithms, and a larger workspace and high-speed capability [18]. The study of limited-DOF PMs, such as 4-DOF PMs, has recently become a main focus among the robotics research community, because this type of PM has many industrial applications, such as Pick and Place operations [19].

The development of this type synthesis channels researchers to synthesize lower-mobility parallel mechanisms, since it was believed that parallel mechanisms with identical limb structures, i.e. topologically symmetrical, with 4- and 5-DOF, could not be built. In general, 4-DOF PMs are a class of PMs with reduced DOF, which, according to their mobility, fall into three motion patterns: (1) three translational and one rotational DOF (3T1R), (2) two translational and two rotational DOF (2T2R) and (3) three rotational and one translational DOF (3R1T) [14,20]. The 3T1R motion pattern, referred to as SCARA motion or Schönflies motion, consists of all translations, as well as rotations, about any axis in a given fixed direction [21]. In the late 1990's, researchers believed that general limited-DOF PMs could not be constructed with identical limb structures (A PM with identical limb structures consists of a mechanism where all the limbs follow the same imposed kinematic arrangement to realize the desired motion pattern. The kinematic limb structure or kinematic chain consists of the placement order and type of joint), as pointed out by Hunt [22] and Tsai [23]. Therefore, some 4-DOF PMs with non-identical limb structures have been reported. Hesselbach et al. in [24] introduced a 4-DOF PM with non-identical limb structures with two limbs for cutting convex glass panels. In [25], Rolland proposed two 4-DOF PMs, called Kanuk and Manta, both of which possess 3T1R DOF. Lenarčič et al. in [26] used a 4-DOF PM with one PS and three SPS limbs to simulate the shoulder of a humanoid. But, PMs with non-identical limb structures result in an asymmetrical workspace, which may complicate task planning. Hence, several researchers have made great efforts in designing 4-DOF PMs with identical limb structures, based on intuition and engineering skills. One of the earliest 4-DOF PM with identical limb structures is the Delta robot presented by Clavel in [27], which performed the so-called Schönflies motion. Company and Pierrot, in [28], presented a new PM whose platform performs 3T1R, and Pierrot et al., in [29], introduced a family of 4-DOF PMs with identical limb structures using the parallelogram concept. It should be noted that the first 4-DOF PM with four identical limb structures, performing a 3R1T motion pattern, was proposed by Zlatanov and Gosselin in [30], which comprises four RRRRR limbs; three intersecting and two parallel revolute joints.

All the aforementioned architecture was developed mainly based on engineering perception. Recently, several systematic approaches, such as screw theory [31], displacement group theory [32], single-opened-chain units [33], the virtual-chain approach [34] and the constraint-synthesis method [35,36], have been proposed for a type synthesis of PM in order to obtain all possible types of PM with a specific motion pattern. In [37], Fang and Tsai employed the screw theory and reciprocal screws for the structural synthesis of a given class of 4-DOF PMs with identical limb structures. In [14,20], Kong and Gosselin created a type synthesis of PM with special motion patterns and identical limb structures, based on screw theory and the virtual chain approach. Thus, the main concern in the analysis of PMs with identical limb structures was the type synthesis and this can be exemplified from the large number of papers published on this issue. However, there are still some gaps in their kinematic properties including, among others, the Forward Kinematic Problem (FKP). FKP, one of the challenging issues on the kinematics of PMs, pertains to finding the pose of the platform for a given set of actuated joints [38]. The position and orientation of the platform of a PM in space are, collectively termed, the pose.

In the past two decades, several algorithms and methods have been presented for solving the FKP of PMs. In [39], Gosselin and Merlet obtained a six degree polynomial for the FKP of the 3-RPR planar PM and proposed two simplified mechanisms in which the FKP of each one leads to a maximum of four real solutions. In [40,41], Husty and Schröker solved the FKP of general Gough-Stewart platforms using kinematics mapping introduced by Study [42], for the first time. This algorithm maps three dimensional motions to the seven-dimensional, quasi-elliptic space. The results reported in [40,41] reveal that the upper bound of the number of solutions for the Gough-Stewart platform is 40. In [43], Merlet revealed the problem of finding all the solutions of the FKP for every possible architecture of planar fully PMs. In [44], Tanev studied the FKP of a 4-DOF PM with one RRPR and two SPS limbs. Neural networks have been used for the FKP of PMs [45,46]. Lee and Shim in [47] proposed an algorithm for the FKP of a Stewart PM using the elimination theory, which is suitable for real time proposes. Richard et al., in [48], studied the FKP of a 3T1R 4-DOF PM, called Quadrupteron. They demonstrated that the FKP of Quadrupteron requires the solution of a univariate quadratic equation.

Masouleh et al., in [49], investigated the FKP of 5-R<u>P</u>UR PM. Moreover, in [50], Tale Masouleh et al. proposed an algorithm based on the study parameters, referred to as the Linear Implicitization Algorithm (LIA), for obtaining, systematically, the Forward Kinematic Expression (FKE). The FKP of 5-DOF PMs (3R2T) with identical limb structures was investigated using this algorithm, i.e. LIA [38]. It should be noted that the FKE is a mathematical expression that pertains to finding the pose(s) for a given limb to reach given actuator coordinates.

To the best knowledge of the authors, until now, very few kinematic studies have been conducted on 4-DOF PMs with identical limb structures. This is probably due to their short history. Also, in an industrial context, 3T1R motion can cover a wide range of applications, including, among others, Pick-and-Place operations. In addition, the analytical solution of FKP in the context of PMs, due to its mathematical complexity, initiated much research, both in mathematics and mechanics. Accordingly, in this paper, the FKP of two 4-<u>P</u>RUR PMs with different geometric structures, and one 4-<u>P</u>UU PM performing a 3T1R motion pattern, are investigated using two approaches. In the first approach, the resultant method is used toward obtaining the FKE for each limb, and the FKP for the mechanism as a whole. The FKE of each limb should be free of passive variables, i.e., non-actuated joint coordinates. Accordingly, more emphasis is placed on the kinematic modeling of one limb. Moreover, the results obtained with the latter approach for the FKP are verified using Bertini software based on homotopy continuation. Finally, the real solutions obtained from the two approaches are compared with each other and their configurations are depicted in the CAD environment.

The remainder of this paper is organized as follows: The architecture and the general kinematic properties of three 4-DOF PMs that originated from the type synthesis performed in [14] are first outlined. Then, the FKP analyses of these PMs are fully investigated with the aim of obtaining a univariate expression for each case. For the sake of comparison, for different case studies, the FKP analysis is carried out by resorting to a homotopy continuation method. Finally, the paper concludes with some remarks and analysis on the results obtained for the FKP.

2. Geometric architecture and kinematic modelling

Figures 1(a), 2(a) and 3(a) provide, respectively, a schematic representation of 4-<u>P</u>R′R′R″R″, 4-<u>P</u>R″R′R′R″ and 4-<u>P</u>R″R″R′R′ PMs. Here, and throughout this paper, R, P and U joints stand, re-

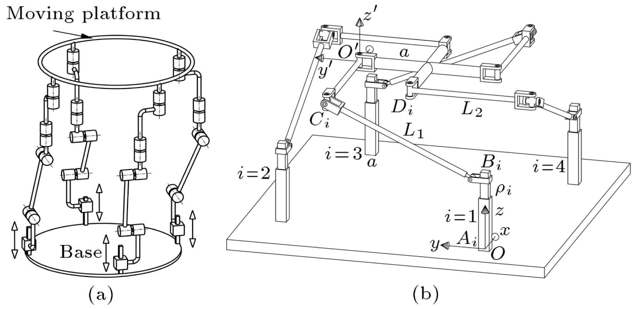


Figure 1. (a) Schematic representation of a 4-<u>P</u>R′R′R″R″ PM [14]. (b) CAD model of a 4-<u>P</u>RUR$_1$ PM.

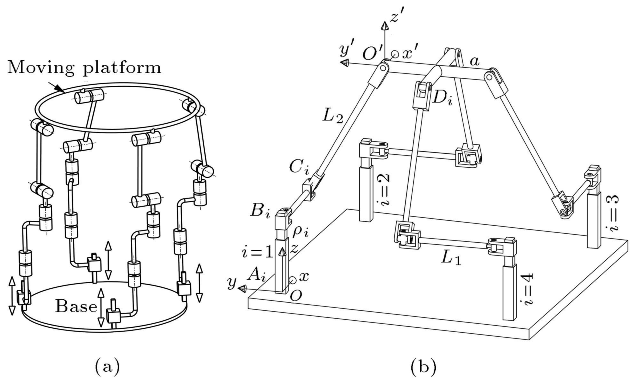


Figure 2. (a) Schematic representation of a 4-<u>P</u>R″R″R′R′ PM [14]. (b) CAD model of a 4-<u>P</u>RUR$_2$ PM.

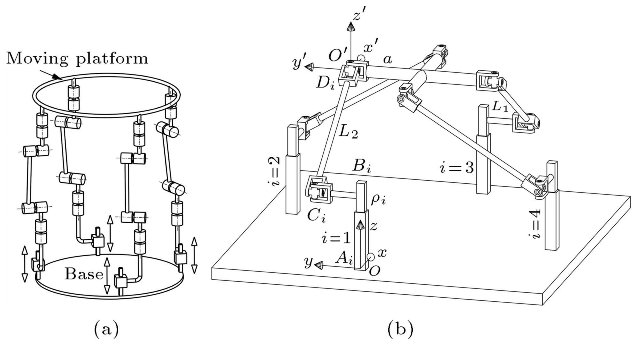


Figure 3. (a) Schematic representation of a 4-<u>P</u>R″R′R′R″ PM [14]. (b) CAD model of a 4-<u>P</u>UU PM.

spectively, for revolute, prismatic and universal joints, where the underlined one is actuated. Moreover, joints with the same superscript have parallel axes. The mentioned PMs have been originated from the type synthesis performed for the PMs having 3T1R as the motion pattern [14]. Figures 1(b), 2(b) and 3(b) depict the CAD model of 4-<u>P</u> RUR$_1$, 4-<u>P</u>RUR$_2$ and 4-<u>P</u>UU PMs that are, respectively, special cases of 4-<u>P</u>R′R′R″R″, 4-<u>P</u>R″R′R′R″ and 4-<u>P</u>R″R″R′R′ 4-DOF PMs performing a 3T1R motion pattern. The input of the mechanism is provided by the four linear prismatic actuators fixed at the base. In addition, four passive revolute joints are in each limb.

Referring to Figures 1(b), 2(b) and 3(b), a fixed reference frame, O_{xyz}, is attached to the base of the mechanism with **i**, **j** and **k** as its unit vectors, and a moving reference frame (mobile frame), $O'_{x'y'z'}$, is attached to the moving platform. In this paper, the superscript $'$ stands for a vector representation in the mobile frame. The three mentioned PMs provide all three translations DOF plus one independent rotation DOF of the end-effector, namely, x, y, z and θ, that are known as the pose (position and orientation) of the platform. Indeed, they are limited-DOF PMs, which cannot rotate about any axis that is parallel to x- and y-axes. In the latter notation, x, y and z represent the translational DOF, with respect to the fixed frame O, illustrated in Figures 1(b), 2(b) and 3(b). θ (rotation from the fixed frame, O_{xyz}, to the moving frame, $O'_{x'y'z'}$) stands for the orientation DOF around z-axes (the vertical axis). In the ith limb, the extension of the actuated prismatic joint is measured, with respect to the reference point, A_i, located on the base, by the joint coordinate, ρ_i, which is the signed distance between point A_i and reference point B_i attached to the prismatic joint.

Figures 4, 5 and 6 represent, respectively, the schematic of PRUR$_1$, PRUR$_2$ and PUU limbs. The vectors, $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ and $\mathbf{e_{\rho i}}$, are defined as the unit vectors in the directions of the revolute joints parallel to the x-axis, the revolute joint parallel to the y-axis, the revolute joint parallel to the $\mathbf{e_1} \times \mathbf{e_2}$ axis and the prismatic joint, respectively. Therefore, the vector connecting point A_i to point B_i can be written as $\boldsymbol{\rho_i} = \rho_i \mathbf{e_{\rho i}} (A_iB_i \perp \mathbf{e_1}, \mathbf{e_2})$. Vector $\mathbf{r_i}$ (a constant vector) is defined as the position vector of point A_i in the fixed reference frame. Similarly, vector $\mathbf{d'_i}$ is the vector connecting point O' of the mobile platform to

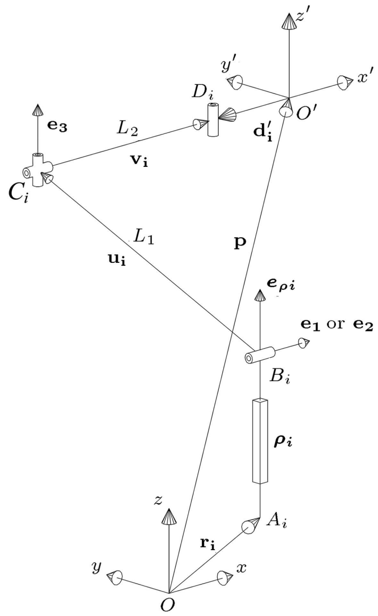


Figure 4. Schematic representation of a PRUR$_1$ limb.

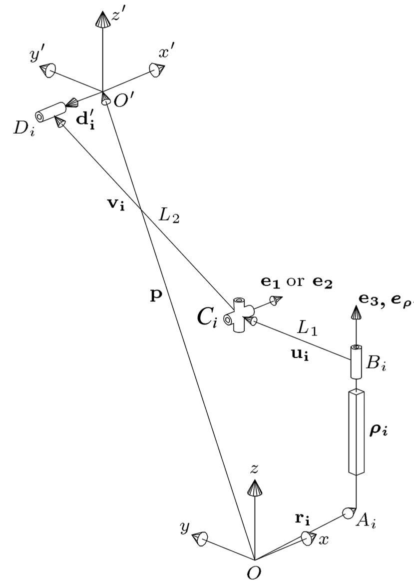


Figure 5. Schematic representation of a PRUR$_2$ limb.

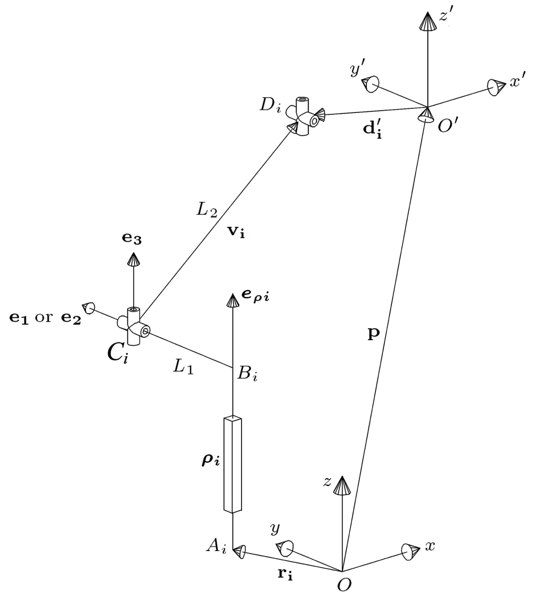


Figure 6. Schematic representation of a PUU limb.

a reference point D_i on the axis of the last revolute joint of the ith limb. Vectors $\mathbf{u_i}$ and $\mathbf{v_i}$ are the vectors connecting point C_i to point D_i and point D_i to point E_i, respectively, and the magnitude of vectors $\mathbf{u_i}$ and $\mathbf{v_i}$ are, respectively, L_1 and L_2. Finally, the position of the platform is represented by vector $\mathbf{p} = \begin{bmatrix} x, & y, & z \end{bmatrix}^T$, connecting point O to point O', and the orientation of the moving frame, with respect to the fixed frame, is given by a rotation matrix, $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

For a given value of angle θ, matrix $\mathbf{Q}$ is readily computed, and vector $\mathbf{d}'_i$ is then obtained as:

$$\mathbf{d}_i = \mathbf{Q}\mathbf{d}'_i. \quad (2)$$

2.1. Kinematic modelling of 4-PRUR$_1$ PM

Figure 4 provides a schematic representation of a PRUR$_1$ limb. From the type synthesis presented in [14], the geometric characteristics associated with the components of each limb of a 4-PRUR$_1$ PM are as follows: The four revolute joints attached to the moving platform (the last R joints in each of the limbs) have parallel axes, the four prismatic joints attached to the base have parallel axes, $\mathbf{e}_{\rho i}||z$, the first two revolute joints of each limb have parallel axes, and the last two revolute joints of each limb have parallel axes, $\mathbf{e_2}||z$. It should be noted that the second and third revolute joints in each limb are built with intersecting and perpendicular axes and are, thus, assimilated to a U joint. In addition, the direction of the P joint is parallel to the last two R joints and perpendicular to the axis of its adjacent R joint.

$$\mathbf{e_1} = \mathbf{Q}\mathbf{e}'_1 = \begin{bmatrix}-\sin\theta & \cos\theta & 0\end{bmatrix}^T, \quad (3)$$

$$\mathbf{e_2} = \mathbf{Q}\mathbf{e}'_2 = \begin{bmatrix}\cos\theta & \sin\theta & 0\end{bmatrix}^T. \quad (4)$$

2.2. Kinematic modelling of 4-PRUR$_2$ PM

In this case, as observed in Figure 5, the prismatic actuator is parallel to the z-axes, $\mathbf{e}_{\rho i}||z$. The first and second revolute joints of each limb have parallel axes that are perpendicular to the $x-y$ plane, $\mathbf{e_3}||z$. The third and fourth revolute joints of each limb have parallel axes in which they are parallel to the $x-y$ plane, $\mathbf{e_1}0 \perp z$ and $\mathbf{e_2} \perp z$. Similar to 4-PRUR$_1$ PM, the second and third revolute joints of each limb can be replaced by a universal joint (point C_i).

2.3. Kinematic modelling of a 4-PUU PM

Figure 6 represents, schematically, a limb of a 4-PUU PM. It is worth noticing that this kinematic arrangement is used for the so-called Quadrupteron [51], belonging to the n-petron orthogonal PMs [52], which has some remarkable kinematic properties. The direction of each prismatic actuator, the first and fourth R joints in each limb parallel to the z-axis, and the directions of the second and third R joints in each limb are perpendicular to the z-axis ($\mathbf{v_i} \perp \mathbf{e_1}$ or $\mathbf{v_i} \perp \mathbf{e_2}$). The first and second joints, and the third and fourth joints, have perpendicular axes and each two joints form a U joint (Figure 6).

3. Forward Kinematic Problem (FKP)

The FKP pertains to finding the pose of the moving platform for a given set of actuated joints [38]. The study of the FKP of the PM requires a suitable mathematical framework in order to describe both translations and rotations in a most general way [50]. In this investigation, the FKP is studied in three-dimensional Euclidean space. It should be noted that the FKP is solved in polynomial form, when it is made equivalent to determining the roots of a univariate polynomial equation [40,53]. In what follows, first, the FKE for each limb of the PM under study is obtained using the coordinates of the joints and the kinematic constraints. The FKE should be free of the passive joint coordinates, i.e. joints whose positions are not known from the outset. Then, a univariate expression describing the FKP of the PM is obtained using the FKEs and upon resorting to the so-called resultant method. Resultant is an alternative approach to the problem of elimination. In summary, if f and g are two polynomials with a positive degree, written as Eqs. (5) and (6), then, the resultant of f and g, denoted by Res (f, g), is the $(l+m)\times(l+m)$ determinant [54]:

$$f = a_0x^l + ... + a_l, \qquad a_0 \neq 0, \quad l > 0, \quad (5)$$

$$g = b_0x^m + ... + b_m, \qquad b_0 \neq 0, \quad m > 0, \quad (6)$$

$$\mathrm{Res}(f,g) = \det\begin{pmatrix}
a_0 & & & & b_0 & & & \\
a_1 & a_0 & & & b_1 & b_0 & & \\
a_2 & a_1 & \ddots & & b_2 & b_1 & \ddots & \\
\vdots & a_2 & \ddots & a_0 & \vdots & b_2 & \ddots & b_0 \\
a_l & \vdots & \ddots & a_1 & b_m & \vdots & \ddots & b_1 \\
 & a_l & \vdots & a_2 & & b_m & \vdots & b_2 \\
 & & \ddots & \vdots & & & \ddots & \vdots \\
 & & & a_l & & & & b_m
\end{pmatrix}, \quad (7)$$

where the blank spaces are filled with zeros and a_i and b_i are constant coefficients or polynomials. Thus, Res (f, g, x) stands for the resultant of f and g by eliminating the variable, x.

3.1. Forward kinematic problem of 4-PRUR$_1$

The following relations hold for the coordinate of points A_i and D_i for all the mechanisms under study in this paper:

$$\begin{bmatrix}x_{Ai} & y_{Ai} & z_{Ai}\end{bmatrix}^T = \mathbf{r_{Ai}} \quad \text{for} \quad i = 1,2,3,4, \quad (8)$$

$$\begin{bmatrix}x_{Di} & y_{Di} & z_{Di}\end{bmatrix}^T = \mathbf{p} + \mathbf{Q}\mathbf{d}'_i. \quad (9)$$

In the above, $\mathbf{r_{Ai}}$ is the vector connecting O to A_i. Since D_i is attached to the platform, its coordinates can be written directly in terms of the platform pose for the three PMs. Moreover, $\boldsymbol{d}'_i$ is the position vector of point D_i in the moving frame.

With reference to Figure 4, the following equations, arising from the kinematic constraint of the ith limb, can be written as:

$$z_{Bi} - z_{Ai} - \rho_i = 0, \tag{10}$$

$$B_iC_i = (y_{Ci} - y_{Bi})^2 + (z_{Ci} - z_{Bi})^2 - L_1^2 = 0 \quad \text{for} \quad i = 1, 3, \tag{11}$$

$$B_iC_i = (x_{Ci} - x_{Bi})^2 + (z_{Ci} - z_{Bi})^2 - L_1^2 = 0 \quad \text{for} \quad i = 2, 4, \tag{12}$$

$$C_iD_i = (x_{Di} - x_{Ci})^2 + (y_{Di} - y_{Ci})^2 - L_2^2 = 0. \tag{13}$$

In the above, the last four equations represent, respectively, the magnitude of vectors, $\boldsymbol{\rho_i}$ and $\mathbf{u_i}$, of the 1st and 3rd limbs, $\mathbf{u_i}$ of the 2nd and 4th limbs, and $\mathbf{v_i}$. According to the FKP analysis, the above system of equations should be solved in terms of the pose of the platform with respect to input data, which are the lengths of the prismatic actuators, ρ_i, and the design parameters. To this end, the coordinates of all passive joints (B_i, C_i and D_i) should be eliminated from Eqs. (11)-(13).

The coordinates of D_i can be obtained from Eq. (9), with respect to the pose of the platform. The vector, $\mathbf{p}$, of Eq. (9) is the position of the moving frame in a fixed frame that is represented by vector $\mathbf{p} = \begin{bmatrix} x, & y, & z \end{bmatrix}^T$. Moreover, $\mathbf{d'_i}$ of Eq. (9) is the position vector of point D_i in the moving frame that can be written as $\mathbf{d'_1} = \begin{bmatrix} 0, & 0, & 0 \end{bmatrix} T$, $\mathbf{d'_2} = \begin{bmatrix} a, & -a, & 0 \end{bmatrix}^T$, $\mathbf{d'_3} = \begin{bmatrix} 0, & -2a, & 0 \end{bmatrix}^T$ and $\mathbf{d'_i} = \begin{bmatrix} -a, & -a, & 0 \end{bmatrix}^T$. One of the kinematic constraints applied to all limbs can be expressed as $\mathbf{v_i} \perp \mathbf{e_3}$, which leads to $z_{Ci} = z_{Di}$. Another constraint of the first and third (second and fourth) limb is $\mathbf{u_i} \perp \mathbf{e_1} (\mathbf{u_i} \perp \mathbf{e_2})$, which results in $x_{Ci} = x_{Bi} = x_{Ai} (y_{Ci} = y_{Bi} = y_{Ai})$. In addition, there are two other constraints in the form of $z_{Bi} = z_{Ai} + \rho_i$ and $y_{Bi} = y_{Ai}$, which result from the geometry of all three mechanisms.

The following equations can be written for the first limb, $i = 1$, by substituting $z_{C1} = z$, $y_{B1} = y_{A1}$, $z_{B1} - z_{A1} + \rho_1$ and $z_{A1} = 0$ (derived from the above kinematic constraints) into Eq. (11) and replacing $x_{D1} = x$, $y_{D1} = y$ and $x_{C1} = x_{A1}$ (derived from the above kinematic constraints) into the Eq. (13):

$$B_1C_1 = z^2 - 2z\rho_1 - L_1^2 + \rho_1^2 + y_{A1}^2 - 2y_{A1}y_{C1} + y_{C1}^2, \tag{14}$$

$$C_1D_1 = x^2 - 2xx_{A1} + y^2 - 2yy_{C1} - L_2^2 + x_{A1}^2 + y_{C1}^2. \tag{15}$$

Coordinate y_{C1} is unknown in Eqs. (14) and (15), which can be eliminated by a resultant method, and the FKE of the first limb, Eq. (16), can be obtained in the terms of platform pose, input variable, ρ_1, design parameters, L_1 and L_2, and coordinates of point $A_1(x_{A1}, y_{A1})$ attached to the base:

$$\begin{aligned} F_1 =& \mathrm{Res}(B_1C_1, C_1D_1, y_{C1}) = x^4 + 2x^2y^2 - 2x^2z^2 \\ &+ y^4 + 2y^2z^2 + z^4 + 4\rho_1x^2z - 4\rho_1y^2z - 4\rho_1z^3 \\ &- 2L_2^2x^2 - 2\rho_1^2x^2 + 2L_1^2x^2 - 2L_2^2y^2 + 2\rho_1^2y^2 \\ &- 2L_1^2y^2 - 2L_1^2z^2 + 6\rho_1^2z^2 + 2L_2^2z^2 + 4\rho_1L_1^2z \\ &- 4\rho_1L_2^2z - 4\rho_1^3z + \rho_1^4 + L_2^4 - 2L_1^2L_2^2 + 2\rho_1^2L_2^2 \\ &+ L_1^4 - 2L_1^2\rho_1^2. \end{aligned} \tag{16}$$

As observed from the above equation, angle θ does not appear in the equations of the 1st limb, because the origin of the moving frame is attached to point D_1, and $\mathbf{d'_1} = \begin{bmatrix} 0, & 0, & 0 \end{bmatrix}^T$. For the second limb, the following equations can be written from Eqs. (12) and (13):

$$B_2C_2 = z^2 - 2z\rho_2 - L_1^2 + \rho_2^2 + x_{A2}^2 - 2x_{A2}x_{C2} + x_{C2}^2, \tag{17}$$

$$\begin{aligned} C_2D_2 =& x^2 + y^2 + 4s_\theta x - 2x_{C2}x + 4c_\theta x - 2ac_\theta y \\ &- 2y_{A2}y + 2as_\theta y - 2a^2c_\theta s_\theta - 2as_\theta y_{A2} + a^2 \\ &+ 2ac_\theta y_{A2} - L_2^2 + 8c_\theta s_\theta - 4c_\theta x_{C2} \\ &- 4s_\theta x_{C2} + x_{C2}^2 + y_{A2}^2 + 4, \end{aligned} \tag{18}$$

where, in the above equations, $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. After obtaining the equations of B_iC_i and C_iD_i, the FKE of the second to fourth limb, called F_i, $i = 2, 3, 4$, in terms of platform pose, input variable, ρ_i, design parameters, L_1, L_2, a, x_{Ai}, and y_{Ai}, can be obtained as follows:

$$F_2 = \mathrm{Res}(B_2C_2, C_2D_2, x_{C2}), \tag{19}$$

$$F_3 = \mathrm{Res}(B_3C_3, C_3D_3, y_{C3}), \tag{20}$$

$$F_4 = \mathrm{Res}(B_4C_4, C_4D_4, x_{C4}). \tag{21}$$

Finally, the following operations are performed to obtain a univariate expression for the FKP:

$$F_{12}(y, z, t) = \mathrm{Res}(F_1, F_2, x), \tag{22}$$

$$F_{13}(y, z, t) = \mathrm{Res}(F_1, F_3, x), \tag{23}$$

$$F_{14}(y, z, t) = \mathrm{Res}(F_1, F_4, x), \tag{24}$$

$$F_{1213}(z,t) = \mathrm{Res}(F_{12}, F_{13}, y), \tag{25}$$

$$F_{1214}(z,t) = \mathrm{Res}(F_{12}, F_{14}, y), \tag{26}$$

$$F_k(t) = \mathrm{Res}(F_{1213}, F_{1214}, z), \tag{27}$$

$$F_k(t) = \left(\Pi_{i=1}^{n}\left(\sum_{j=1}^{m}(\kappa_j t^j)^k\right)\right), \tag{28}$$

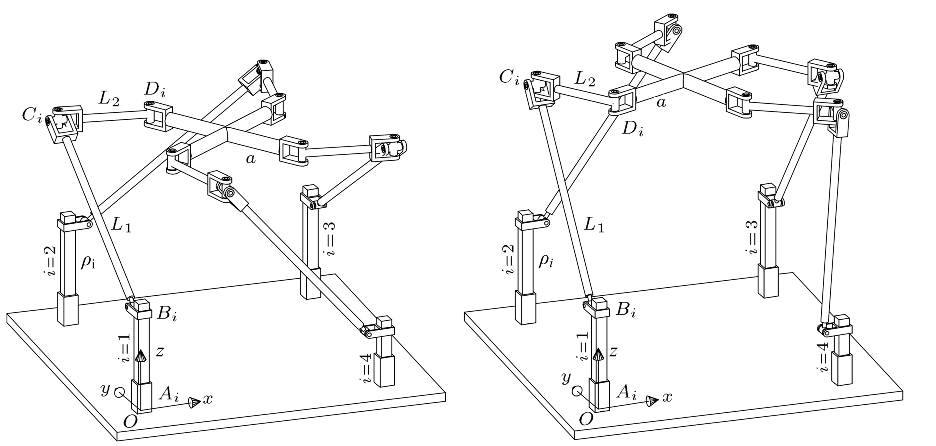
$F_k(t)$ is the univariate expression based on the variable t, where $t = \tan(\theta/2)$, $\sin\theta = 2t/(1+t^2)$, and $\cos\theta = (1-t^2)/(1+t^2)$ is the tan-half substitution. In Eq. (28), κ_j is the constant coefficient, depending on the design parameters. Numerous random examples have been solved for the 4-$\underline{\mathrm{P}}$RUR$_1$ PM using the resultant method, and it should be noted that the univariate expression of a 4-$\underline{\mathrm{P}}$RUR$_1$ PM always consists of 6 separated polynomials with respect to t, of degree 72, 64, 64, 8^2, 4 and 2^8, so that these polynomials contain the FKP answer. Polynomials of degree 8^2 and 2^8, respectively, mean a polynomial of degree 8 to the power of 2 and a polynomial of degree 2 to the power of 8.

It should be explained that, in all examples, there is one set of forward kinematic solutions for every answer obtained from the polynomial of the univariate expression to the power of one. Also, there are m sets of solution for every answer obtained from a polynomial of degree n to the power of m. Thus, there is one solution of the FKP for every answer obtained from the polynomial of degree 72, 64, 64 and 4. However, two sets of solutions are obtained from each answer of the polynomial of degree 8, and eight sets of solutions are resulted from each answer of the polynomial of degree 2. Indeed, the total degree of polynomials of degree 8 to the power of 2 and degree 2 to the power of 8 are, respectively, $8 \times 2 = 16$ and $2 \times 8 = 16$. It means that each polynomial has 16 sets of forward kinematic solutions. Finally, the total degree of these univariate polynomials, in each example, is 236 ($72 + 64 + 64 + 8 \times 2 + 4 + 2 \times 8 = 236$), which demonstrated that a general 4-$\underline{\mathrm{P}}$RUR$_1$ PM admits up to 236 solutions, real and complex. Moreover, the FKP of a 4-$\underline{\mathrm{P}}$RUR$_1$ PM is investigated using Bertini software, which obtains the numerical solution of the system of polynomial equations using the homotopy continuation approach [55]. In this procedure, all expressions and kinematic equations obtained, in terms of the platform poses (x, y, z and θ) and coordinates of passive variables (x_{Ci}, y_{Ci} and z_{Ci}), are as input equations (16 equations and 16 unknowns). In each example, 236 solutions, real and complex, are obtained using Bertini software. The results show that all 236 solutions, real and complex, obtained from the two approaches perfectly match each other. In addition, all the solutions were placed in the inverse kinematic equations and the FKP result is confirmed.

As an example, 10 real and 226 complex solutions have been obtained for the values given in Table 1 for this PM using two approaches. It should be noted that, in this case, two of the real finite solutions result from the polynomial of degree 72, two of them result from the first polynomial of degree 64, two of them result from the second polynomial of degree 64, and four of them result from the polynomial of degree 8. For example, two configurations of 4-$\underline{\mathrm{P}}$RUR$_1$ PM, in this case, are given in Table 2 and Figure 7.

Table 1. The design parameters, prismatic elongations and coordinates of A_i for the 4-$\underline{\mathrm{P}}$RUR$_1$ PM under study.

A_1	A_2	A_3	A_4	a	L_1	L_2	ρ_1	ρ_2	ρ_3	ρ_4
(0,0,0)	(1,5,0)	(6,4,0)	(5,-1,0)	2	5	2	3	3	3	2



(a) First configuration (b) Second configuration

Figure 7. Schematic representation of two configurations for the FKP of a 4-$\underline{\mathrm{P}}$RUR$_1$ PM represented in Table 2.

Table 2. Two solutions, among ten, obtained for the FKP of 4-$\underline{\text{P}}$RUR$_1$ using the two approaches.

	4-$\underline{\text{P}}$RUR$_1$	
	First configuration	**Second configuration**
x	1.949913171	1.108969655
y	3.702150841	1.348854999
z	5.793363063	6.990033699
θ	31.67116557	109.0003277
C_1	(0.0000, 4.1469, 5.7933)	(0.0000, 3.0132, 6.9900)
C_2	(5.1469, 5.0000, 5.7933)	(4.0132, 5.0000, 6.9900)
C_3	(6.0000, -0.1469, 5.7933)	(6.0000, 0.9867, 6.9900)
C_4	(1.7426, -1.0000, 5.7933)	(5.3155, -1.0000, 6.9900)
D_1	(1.9499, 3.7021, 5.7933)	(1.1089, 1.3488, 6.9900)
D_2	(4.7021, 3.0501, 5.7933)	(2.3488, 3.8910, 6.9900)
D_3	(4.0500, 0.2978, 5.7933)	(4.8910, 2.6511, 6.9900)
D_4	(1.2978, 0.9499, 5.7933)	(3.6511, 0.1089, 6.9900)

3.2. Forward kinematic problem of a 4-$\underline{P}RUR_2$ PM

Using the same reasoning as above, from Figure 5, one has the following for the 4-PRUR2 PM:

$$z_{Bi} - z_{Ai} - \rho_i = 0, \tag{29}$$

$$(x_{Ci} - x_{Bi})^2 + (y_{Ci} - y_{Bi})^2 - L_1^2 = 0, \tag{30}$$

$$(x_{Di} - x_{Ci})^2 + (y_{Di} - y_{Ci})^2 + (z_{Di} - z_{Ci})^2 - L_2^2 = 0. \tag{31}$$

Since the FKP is of concern, the values of the prismatic actuators, i.e. the coordinates of points B_i, are known. The coordinates of points D_i can be obtained from Eq. (9), in terms of platform pose. But, the coordinates of points C_i are not known and should be eliminated from Eqs. (29)-(31) based on the constraints of each limb. The first constraint of the limb is $\mathbf{v_i} \perp \mathbf{e_3}$ which leads to $z_{Ci} = z_{Bi}$. The direction of $\mathbf{v_i}(C_iD_i)$ always makes an angle, θ, with respect 1to the x-axis. Therefore, the second constraint is written as follows for each limb:

$$x_{C1} = x_{D1} - \eta_1 s_\theta, \qquad y_{C1} = y_{D1} + \eta_1 c_\theta, \tag{32}$$

$$x_{C2} = x_{D2} + \eta_2 c_\theta, \qquad y_{C2} = y_{D2} + \eta_2 s_\theta, \tag{33}$$

$$x_{C3} = x_{D3} + \eta_3 s_\theta, \qquad y_{C3} = y_{D3} - \eta_3 c_\theta, \tag{34}$$

$$x_{C4} = x_{D4} - \eta_4 c_\theta, \qquad y_{C4} = y_{D4} - \eta_4 s_\theta, \tag{35}$$

where, in this equation, η_i is the image of $\mathbf{v_i}$ on the $x-y$ plane, as depicted in Figure 8. Eqs. (30) and (31) are written for the first limb after substituting the coordinates of points C_i (obtained in the previous paragraph):

$$B_1C_1 = x^2 + y^2 - 2\eta_1 c_\theta x - 2\eta_1 s_\theta y + \eta_1^2 - L_1^2, \tag{36}$$

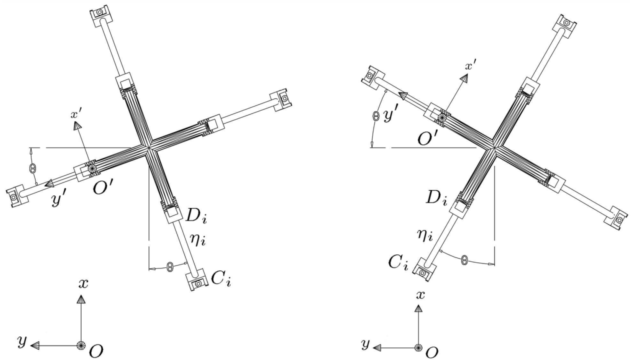


Figure 8. The image of $\mathbf{v_i}$ (C_iD_i) on the $x-y$ plane.

$$C_1D_1 = z^2 - 2\rho_1 z - L_2^2 + \eta_1^2 + \rho_1^2. \tag{37}$$

The FKE of the first limb is obtained by applying the resultant method to the above equations:

$$\begin{aligned} F_1 =& \text{Res}(B_1C_1, C_1D_1, \eta_1) = x^4 + 2x^2y^2 + 4c_\theta^2 x^2 z^2 \\ & - 2x^2z^2 + 8c_\theta s_\theta xyz^2 + y^4 + 4s_\theta^2 y^2 z^2 - 2y^2z^2 \\ & + z^4 + 4\rho_1 x^2 z - 8c_\theta^2 \rho_1 x^2 z - 16 c_\theta s_\theta \rho_1 xyz \\ & - 8s_\theta^2 \rho_1 y^2 z + 4\rho_1 y^2 z - 4\rho_1 z^3 - 2\rho_1^2 x^2 + 2L_2^2 x^2 \\ & + 4c_\theta^2 \rho_1^2 x^2 - 4c_\theta^2 L_2^2 x^2 - 2L_1^2 x^2 - 8c_\theta s_\theta L_2^2 xy \\ & + 8c_\theta s_\theta \rho_1^2 xy - 4s_\theta^2 L_2^2 y^2 + 4s_\theta^2 \rho_1^2 y^2 + 2L_2^2 y^2 \\ & - 2L_1^2 y^2 - 2\rho_1^2 y^2 + 6\rho_1^2 z^2 + 2L_1^2 z^2 - 2L_2^2 z^2 \\ & + 4\rho_1 L_2^2 z - 4\rho_1^3 z - 4\rho_1 L_1^2 z - 2L_1^2 L_2^2 + 2\rho_1^2 L_1^2 \\ & + L_1^4 + \rho_1^4 - 2L_2^2 \rho_1^2 + L_2^4. \end{aligned} \tag{38}$$

Moreover, the tan-half substitution, $t = \tan(\theta/2)$, is used and, finally, a univariate expression is obtained, with respect to t.

$$F_k(t) = \left(\Pi_{i=1}^{n} \left(\sum_{j=1}^{m} (\vartheta_j t^j)^k \right) \right). \tag{39}$$

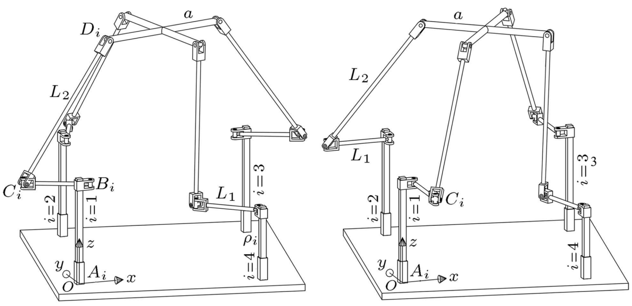
In the above equation, ϑ_j is a constant coefficient that consists of design parameters. Numerous random examples were solved and the following result is obtained: The univariate expression of a 4-$\underline{\text{P}}$RUR$_2$ PM consists of polynomials in degrees of 72, 64, 64, 8^2, 4 and 2^8 with respect to t. These polynomials contain, always, the answer of FKP. In addition, the FKP is solved by Bertini, and 236 solutions, real and complex, are obtained. The results obtained from two approaches demonstrate that the FKP of the 4-$\underline{\text{P}}$RUR$_2$ PM, the same as the FKP of the 4-$\underline{\text{P}}$RUR$_1$ PM, has up to 236 real solutions. Moreover, all the solutions, real and

Table 3. The design parameters, prismatic elongations and coordinates of A_i for the 4-PRUR$_2$ PM under study.

A_1	A_2	A_3	A_4	a	L_1	L_2	ρ_1	ρ_2	ρ_3	ρ_4
(0,0,0)	(1,5,0)	(6,4,0)	(5,-1,0)	2	2	5	3	3	3	2

Table 4. Two solutions, among eighteen, obtained for the FKP of 4-PRUR$_2$ using the resultant method.

	4-PRUR$_2$	
	First configuration	**Second configuration**
x	1.062255611	1.79767717
y	1.504877102	0.40174475
z	6.995919809	6.99591980
θ	104.3332580	143.0468769
C_1	(- 1.8496, 0.7608, 3.0000)	(-0.0090, -1.9999, 3.0000)
C_2	(1.7608, 6.8496, 3,0000)	(-0.9999, 5.0090, 3.0000)
C_3	(7.8496, 3.2391, 3.0000)	(6.0090, 5.9999, 3.0000)
C_4	(3.4451, 0.2579, 2.0000)	(4.4368, 0.9190, 2.0000)
D_1	(1.0622, 1.5048, 6.9959)	(1.7976, 0.4017, 6.9959)
D_2	(2.5048, 3.9377, 6.9959)	(1.4017, 3.2023, 6.9959)
D_3	(4.9377 2.4951 6.9959	(4.2023 3.5982 6.9959)
D_4	(3.4951, 0.0622, 6.9959)	(4.5982, 0.7976, 6.9959)



(a) First configuration (b) Second configuration

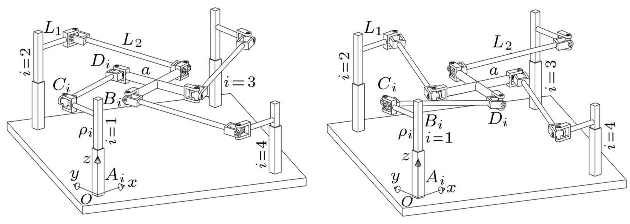
Figure 9. Schematic representation of two configurations for the FKP of a 4-PRUR$_2$ PM represented in Table 4.

complex, were placed in the inverse kinematic equations and the accuracy of the results has been proved.

For example, for the values given in Table 3, the FKP of the 4-PRUR$_2$4 PM has 236 solutions (18 real and 218 complex solutions). It can be noted that, in this example, two of the real solutions result from the polynomial of degree 72, four of them result from the first polynomial of degree 64, four of them result from the second polynomial of degree 64, four of them result from the polynomial of degree 8, two of them result from the polynomial of degree 4 and two of them result from the polynomial of degree 2. As an example, two configurations of the 4-PRUR2 PM under study are given in Table 4 and Figure 9.

3.3. Forward kinematic problem of 4-PUU

With reference to Figure 6, the following can be written for the ith limb of a 4-PUU PM:



(a) First configuration (b) Second configuration

Figure 10. Schematic representation of two configurations for the FKP of a 4-PUU PM represented in Table 4.

$$z_{Bi} - z_{Ai} - \rho_i = 0, \tag{40}$$

$$(x_{Di} - x_{Ci})^2 + (y_{Di} - y_{Ci})^2 + (z_{Di} - z_{Ci})^2 - L_2^2 = 0. \tag{41}$$

The coordinates of D_i can be obtained from Eq. (9), with respect to the pose of the platform. Thus, the FKE of the first limb can be written as follows:

$$F_1 = x^2 + y^2 + z^2 - 2L_1 y - 2\rho_1 z + L_1^2 + \rho_1^2 - L_2^2. \tag{42}$$

Similarly, the FKE of the second limb is obtained as follows:

$$\begin{aligned} F_2 =& x^2 + y^2 + z^2 - 4c_\theta x + 4s_\theta x - 2x_{A2}x - 2L_1 x \\ &- 4c_\theta y + 4s_\theta y - 2y_{A2}y - 2\rho_2 z + L_1^2 - 4L_1 c_\theta \\ &- 4L_1 s_\theta + 2L_1 x_{A2} - L_2^2 - 4c_\theta x_{A2} + 4c_\theta y_{A2} \\ &+ \rho_2^2 - 4s_\theta x_{A2} - 4s_\theta y_{A2} + x_{A2}^2 + y_{A2}^2 + 8. \end{aligned} \tag{43}$$

Finally, the univariate expression describing the FKP of a 4-PUU PM is obtained as a univariate quadratic expression by applying the tan-half substitution, $t = \tan(\theta/2)$, and using Eqs. (22) to (27). Therefore, a second degree polynomial is constantly describing the FKP of 4-PUU PM, which is consistent with the result obtained in [51]. Moreover, the number of solutions of the FKP, real and complex, obtained by Bertini is equal to 2. Numerous random examples have been solved for the 4-PUU PM using the resultant method and the homotopy continuation approach, and this result is confirmed by them. In addition, the results of each example are confirmed by placing solutions of FKP in the inverse kinematic problem equations. For example, for the values given in Table 5, the FKP of the 4-PUU PM have 2 real solutions (Table 6), both of whose configurations are represented in Figure 10.

Table 5. The design parameters, prismatic elongations and coordinates of A_i for the 4-PUU PM under study.

A_1	A_2	A_3	A_4	a	L_1	L_2	ρ_1	ρ_2	ρ_3	ρ_4
(0,0,0)	(3,7,0)	(10,4,0)	(7,-3,0)	2	2	5	3	3	3	2

Table 6. The only two solutions obtained for the FKP of 4-PUU using the resultant method.

	4-PUU	
	First configuration	**Second configuration**
x	4.574999996	4.575000000
y	3.954322134	0.04567786726
z	2.499999958	2.499999997
θ	12.26889913	167.7311009
D_1	(4.5749, 3.9543, 2.4999)	(4.5750, 0.0456, 2.4999)
D_2	(6.9543, 2.4250, 2.4999)	(3.0456, 2.4250, 2.4999)
D_3	(5.4250, 0.0456, 2.4999)	(5.4250, 3.9543, 2.4999)
D_4	(3.0456, 1.5749, 2.4999)	(6.9543, 1.5749, 2.4999)

4. Conclusions

This paper investigated the FKP of three kinds of 4-DOF PMs, 4-PRUR$_1$, 4-PRUR$_2$ and 4-PUU, with identical limb structures performing a 3T1R motion pattern, or the so-called Schönflies motion, using the resultant method and the homotopy continuation approach. From this study, it follows that a set of univariate expressions of degree (72, 64, 64, 8^2, 4, 2^8) describe the FKP of 4-PRUR$_1$ and 4-PRUR$_2$ PMs, which demonstrated that the FKP of both PMs have up to 236 solutions. Also, a second degree univariate polynomial represents the solutions for the FKP of a 4-PUU PM. Moreover, upon resorting to Bertini software for solving polynomial systems using the homotopy continuation approach, it was confirmed that the FKP of a 4-PRUR$_1$, 4-PRUR$_2$ and 4-PUU admit 236, 236 and 2 finite solutions, real and complex. The results obtained from the two approaches reveal that the solutions to the FKP can be in different sets of univariate expression obtained using the resultant method. Ongoing work includes the design of an optimum 4-DOF PMs.

References

1. Merlet, J.-P., *Parallel Robots*, Springer, Sophia-Antipolice, France (2006).
2. Stewart, D. "A platform with six degrees of freedom", *Proceedings of the Institution of Mechanical Engineers*, **180**(1), pp. 371-386 (1965).
3. Pouliot, N.A., Gosselin, C.M. and Nahon, M.A. "Motion simulation capabilities of three-degree-of-freedom flight simulators", *Journal of Aircraft*, **35**(1), pp. 9-17 (1998).
4. Huang, T., Whitehouse, D. and Wang, J. "The local dexterity, optimal architecture and design criteria of parallel machine tools", *CIRP Annals-Manufacturing Technology*, **47**(1), pp. 346-350 (1998).
5. Cleary, K. and Brooks, T. "Kinematic analysis of a novel 6-DOF parallel manipulator", *Proceedings of the IEEE International Conference of Robotics and Automation*, pp. 708-713 (1993).
6. Agrawal, S.K., Desmier, G. and Li, S. "Fabrication and analysis of a novel 3 DOF parallel wrist mechanism", *Journal of Mechanical Design*, **117**(2), pp. 343-245 (1995).
7. Gosselin, C.M. and St-Pierre, É. "Development and experimentation of a fast 3-DOF camera-orienting device", *The International Journal of Robotics Research*, **16**(5), pp. 619-630 (1997).
8. Albus, J., Bostelman, R. and Dagalakis, N. "The NIST robocrane", *Journal of Robotic Systems*, **10**(5), pp. 709-724 (1993).
9. Arai, T., Stoughton, R., Homma, K., Adachi, H., Nakamura, T. and Nakashima, K. "Development of a parallel link manipulator", *Fifth International Conference on Advanced Robotics, Robots in Unstructured Environments'*, Pisa, Italy, pp. 839-844 (1991).
10. Brandt, G., Zimolong, A., Carrat, L., Merloz, P., Staudte, H.-W., Lavallee, S., Radermacher, K. and Rau, G. "CRIGOS: A compact robot for image-guided orthopedic surgery", *IEEE Transactions on Information Technology in Biomedicine*, **3**(4), pp. 252-260 (1999).
11. Jensen, K.A., Lusk, C.P. and Howell, L.L. "An XYZ micromanipulator with three translational degrees of freedom", *Robotica*, **24**(3), pp. 305-314 (2006).
12. Liu, X.-J., Wang, J., Gao, F. and Wang, L.-P. "On the design of 6-DOF parallel micro-motion manipulators", *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems*, pp. 343-348 (2001).
13. Li, Y.-M. and Xu, Q. "Design and analysis of a new 3-DOF compliant parallel positioning platform for nanomanipulation", *5th IEEE Conference on Nanotechnology*, pp. 861-864 (2005).
14. Kong, X. and Gosselin, C.M., *Type Synthesis of Parallel Mechanisms*, Springer Berlin Heidelberg, Berlin (2007).
15. Siciliano, B. and Khatib, O., *Springer Handbook of Robotics*, Springer Berlin Heidelberg, Berlin (2008).
16. Gough, V. and Whitehall, S. "Universal tyre test machine", *Proc. FISITA 9th Int. Technical Congress*, pp. 117-137 (1962).
17. Rezaei, A., Akbarzadeh, A. and Akbarzadeh-T, M.-R. "An investigation on stiffness of a 3-PSP spatial parallel mechanism with flexible moving platform using

invariant form", *Mechanism and Machine Theory*, **51**, pp. 195-216 (2012).

18. Joshi, S.A. and Tsai, L.-W. "Jacobian analysis of limited-DOF parallel manipulators", *Journal of Mechanical Design*, **124**(2), pp. 254-258 (2002).

19. Briot, S. and Bonev, I.A. "Pantopteron-4: A new 3T1R decoupled parallel manipulator for pick-and-place applications", *Mechanism and Machine Theory*, **45**(5), pp. 707-721 (2010).

20. Kong, X. and Gosselin, C.M. "Type synthesis of 3T1R 4-DOF parallel manipulators based on screw theory", *IEEE Transactions on Robotics and Automation*, **20**(2), pp. 181-190 (2004).

21. Hervé, J. and Sparacino, F. "Structural synthesis of parallel robots generating spatial translation", *Proc. 5th Int. Conf. Advanced Robotics*, pp. 808-813 (1991).

22. Hunt, K. "Structural kinematics of in-parallel-actuated robot-arms", *ASME* (1983).

23. Tsai, L.-W. "Systematic enumeration of parallel manipulators", in *Parallel Kinematic Machines*, pp. 33-49, Springer, London, UK (1999).

24. Hesselbach, J., Plitea, N., Frindt, M. and Kusiek, A. "A new parallel mechanism to use for cutting convex glass panels", in *Advances in Robot Kinematics: Analysis and Control*, pp. 165-174, Springer, Netherlands (1998).

25. Rolland, L. "The manta and the kanuk: Novel 4-dof parallel mechanisms for industrial handling", *ASME Dynamic Systems and Control Division, IMECE'99 Conference*, pp. 831-844 (1999).

26. Lenarčič, J., Stanišić, M. and Parenti-Castelli, V. "A 4-DoF parallel mechanism simulating the movement of the human Sternum-Clavicle-Scapula complex", in *Advances in Robot Kinematics*, pp. 325-332, Springer, United States of America (2000).

27. Clavel, R. "Delta, a fast robot with parallel geometry", *Proceedings of the International Symposium on Industrial Robots*, pp. 91-100 (1988).

28. Company, O. and Pierrot, F. "A new 3T-1R parallel robot", *Presented at the ICAR'99*, Tokyo, Japan (1999).

29. Pierrot, F., Marquet, F. and Gil, T. "H4 parallel robot: Modeling, design and preliminary experiments", *Proceedings of the IEEE International Conference on Robotics and Automation*, pp. 3256-3261 (2001).

30. Zlatanov, D. and Gosselin, C.M. "A family of new parallel architectures with four degrees of freedom", *Computational Kinematics*, pp. 57-66 (2001).

31. Kong, X. and Gosselin, C.M. "Generation of parallel manipulators with three translational degrees of freedom using screw theory", *Presented at the CCToMM Symposium on Mechanisms, Machines and Mechatronics*, Montreal, Canada (2001).

32. Hervé, J. "The Lie group of rigid body displacements, a fundamental tool for mechanism design", *Mechanism and Machine Theory*, **34**(5), pp. 719-730 (1999).

33. Jin, Q., Yang, T.L., Liu, A.X., Shen, H.P. and Yao, F.H. "Structure synthesis of a class of 5-DOF parallel mechanism mechanisms based on single opened-chain units", *Design Engineering Technical Conferences & Computers and Information in Engineering Conference*, Pittsburgh (2001).

34. Kong, X. and Gosselin, C.M. "Type synthesis of 4-DOF SP-equivalent parallel manipulators: A virtual chain approach", *Mechanism and Machine Theory*, **41**(11), pp. 1306-1319 (2006).

35. Huang, Z. and Li, Q. "General methodology for type synthesis of symmetrical lower-mobility parallel manipulators and several novel manipulators", *The International Journal of Robotics Research*, **21**(2), pp. 131-145 (2002).

36. Huang, Z. and Li, Q. "Type synthesis of symmetrical lower-mobility parallel mechanisms using the constraint-synthesis method", *The International Journal of Robotics Research*, **22**(1), pp. 59-79 (2003).

37. Fang, Y. and Tsai, L.-W. "Structure synthesis of a class of 4-DoF and 5-DoF parallel manipulators with identical limb structures", *The International Journal of Robotics Research*, **21**(9), pp. 799-810 (2002).

38. Masouleh, M.T., Walter, D., Husty, M. and Gosselin, C. "Forward kinematics of the symmetric 5-DOF parallel mechanisms (3R2T) using the linear implicitization algorithm", *13th World Congress in Mechanism and Machine Science*, Guanajuato, Mexico (2011).

39. Gosselin, C.M. and Merlet, J.-P. "The direct kinematics of planar parallel manipulators: Special architectures and number of solutions", *Mechanism and Machine Theory*, **29**(8), pp. 1083-1097 (1994).

40. Husty, M.L. "An algorithm for solving the direct kinematics of general Stewart-Gough platforms", *Mechanism and Machine Theory*, **31**(4), pp. 365-379 (1996).

41. Husty, M.L. and Schröcker, H.-P. "Algebraic geometry and kinematics", in *Nonlinear Computational Geometrye*, pp. 85-107, Springer, New York, United States of America (2010).

42. Study, E. "von den Bewegungen und Umlegungen", *Mathematische Annalen*, **39**(4), pp. 441-565 (1891).

43. Merlet, J.-P. "Direct kinematics of planar parallel manipulators", *Proceedings of the IEEE International Conference on Robotics and Automation*, pp. 3744-3749, Minneapolis, Minnesota (1996).

44. Tanev, T. "Forward displacement analysis of a three-legged four-degree-of-freedom parallel manipulator", in *Advances in Robot Kinematics: Analysis and Control*, pp. 147-154, Springer, United States of America (1998).

45. Lim, K.-b. "Forward kinematics solution of Stewart platform using neural networks", *Neurocomputing*, **16**(4), pp. 333-349 (1997).

46. Parikh, P.J. and Lam, S.S. "A hybrid strategy to solve the forward kinematics problem in parallel manipulators", *IEEE Transactions on Robotics*, **21**(1), pp. 18-25 (2005).

47. Lee, T.-Y. and Shim, J.-K. "Forward kinematics of the general 6-6 Stewart platform using algebraic elimination", *Mechanism and Machine Theory*, **36**(9), pp. 1073-1085 (2001).

48. Richard, P.-L., Gosselin, C.M. and Kong, X. "Kinematic analysis and prototyping of a partially decoupled 4-DOF 3T1R parallel manipulator", *J. Mech. Des.*, **129**, p. 611 (2007).

49. Masouleh, M.T., Gosselin, C., Saadatzi, M.H., Kong, X. and Taghirad, H.D. "Kinematic analysis of 5-RPUR (3T2R) parallel mechanisms", *Meccanica*, **46**(1), pp. 131-146 (2011).

50. Masouleh, M.T., Gosselin, C., Husty, M. and Walter, D.R. "Forward kinematic problem of 5-RPUR parallel mechanisms (3T2R) with identical limb structures", *Mechanism and Machine Theory*, **46**(7), pp. 945-959 (2011).

51. Richard, P.-L., Gosselin, C.M. and Kong, X. "Kinematic analysis and prototyping of a partially decoupled 4-DOF 3T1R parallel manipulator", *Journal of Mechanical Design*, **129**(6), pp. 611-616 (2007).

52. Gosselin, C.M., Masouleh, M.T., Duchaine, V., Richard, P.-L., Foucault, S. and Kong, X. "Parallel mechanisms of the multipteron family: Kinematic architectures and benchmarking", *IEEE International Conference on Robotics and Automation*, pp. 555-560 (2007).

53. Innocenti, C. "Forward kinematics in polynomial form of the general Stewart platform", *Journal of Mechanical Design*, **123**(2), pp. 254-260 (2001).

54. Cox, D.A. Little, J. and Oshea, D. "Using algebraic geometry", **185**, 2nd Edn., Springer, New York, United States of America (2005).

55. Bates, D.J., Hauenstein, J.D., Sommese, A.J. and Wampler, C.W., *Bertini: Software for Numerical Algebraic Geometry* (2006).

Biographies

Payam Varshovi-Jaghargh received an MS degree in Solid Mechanical Engineering, in 2007, from Bu-Ali Sina University, Iran, where he is currently pursuing his PhD degree studies in Solid Mechanical Engineering. He received a scholarship from the Ministry of Science, Research and Technology at the Robotics Department of Hamedan University of Technology, Iran, in 2008.

His educational experiences include teaching in the department of robotics engineering at Hamedan University of Technology and in the department of mechanical engineering at Bu-Ali Sina University. His research interests include path planning of mobile robots, forward and inverse kinematic and dynamic problems of parallel mechanisms, geometric algebra and seven-dimensional kinematic space.

He is author and/or co-author of more than 10 papers published in international conferences or journals.

Davood Naderi received BS, MS and PhD degrees in Mechanical Engineering from Sharif University of Technology, Tehran, Iran, and is currently faculty member of the Mechanical Engineering Department at Bu Ali Sina University, Hamedan, Iran. His major areas of interest are kinematics, dynamics, and robotics.

Mehdi Tale Masouleh received BS, MS and PhD degrees in Mechanical Engineering (Robotic) from Laval University, Québec, Canada, in 2006, 2007 and 2010, respectively, and a Postdoctoral Fellow in the Robotic Laboratory of the university. He is currently a faculty member in the Faculty of New Sciences and Technology at the University of Tehran, Iran, and director of the Human-Robot Interaction Laboratory. His research interests include kinematics, the dynamic and design of serial and parallel robotic systems, humanoid, mobile robots and optimization techniques (interval analysis and convex optimization) for robotic applications. He is supervising several undergraduate and graduate student theses and has published several papers in different fields of robotic mechanical systems. He has also been involved in the following industrial projects: development of a 3-DOF decoupled parallel robot, a haptic device for dental education simulation, the development of a FPGA-based mobile robot called MRTQ (a counter part for epuck) and a 6-DOF pneumatically actuated parallel robot.