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On determination of mode shapes of linear continuous systems

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Abstract. Determination of mode shapes of a vibrating system is an important step in vibration analysis. A procedure to determine the mode shapes of linear continuous systems, based on the concept of *normal equation*, is presented. This procedure is very effective, especially when implemented by computers. This method can be applied easily to multi-step structures, such as stepped beams and bars. To demonstrate the application of the proposed method, some examples are solved.

1. Introduction

Determination of mode shapes of a linear continuous system is an important step in vibration analysis [1,2]. For a simple structure, such as single-step beams and bars, this is an easy task. But, for multi-step structures, such as stepped beams with many segments, this should be carried out using a computer. The presentation of an algorithm suitable for implementation by a computer is the aim of the present paper.

Analysis of the longitudinal vibration of stepped rods with non-classical boundary conditions was investigated by Hsueh [3]. Li [4] used the transfer matrix method and the solutions of one step non-uniform bars to analyze the stepped beams. Yavari et al. [5] presented a solution to the bending problem of beams under singular loading conditions with different discontinuities using a generalized function theory. By selecting suitable expressions, Li et al. [6] reduced the equation of motion of a non-uniform bar to an equation having an analytical solution. Koplow et al. [7] obtained an analytical solution for the response of a discontinuous beam. Their analysis was validated by experimental tests. Bashash et al. [8] presented a new method for forced vibration analysis of beams with multiple discontinuities in the cross section. A new analytical method is developed by Yang. [9] for transient vibration analysis of stepped systems. Using a transfer function method and a residue formula for inverse Laplace transform, this method can give the exact transient response of stepped systems with multiple discontinuities. Using a differential transformation method, Suddoung et al. [10] considered the free vibration analysis of stepped beams made from functionally graded materials.

The determination of mode shapes is a basic step in vibration analysis. To calculate the mode shapes, the temporal part of the solution is assumed as harmonic. With this assumption, the partial differential equation of motion is reduced to a differential eigenvalue problem as [1]:

$$L[u] = \lambda m u, \tag{1}$$

where u is deflection of the system, $L[.]$ is the homogenous linear differential operator of order $2p$, λ is eigenvalue and m is mass density. The boundary conditions are:

$$B_i[u] = 0, \qquad i = 1, 2, ..., p, \tag{2}$$

where $B_i[.]$ is the homogenous linear differential operator of order p evaluated at boundaries.

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For example, in longitudinal vibrations of a fixed-free bar, with length, l, Young's modulus, E, and cross-sectional area, A, the above operators are [1]:

$$L[u] = \frac{d}{dx}\left(EA\frac{d}{dx}(u)\right),$$

$$B_1[u] = u|_{x=0},$$

$$B_2[u] = EA\frac{d(u)}{dx}|_{x=l}. \tag{3}$$

It is assumed that the rod is fixed in $x = 0$ and is free in $x = l$.

The solution of Eq. (1) is found as:

$$u(x) = c_1\varphi_1(x) + ... + c_p\varphi_p(x). \tag{4}$$

There are p constants $c_j(j = 1, ..., p)$ determined by using p boundary conditions (2). It is noted that $\varphi_j(x)$, $j = 1, ..., p$ is also a function of eigenvalue, λ. With application of boundary conditions (Eq. (2)), a homogenous system of linear algebraic equations are found as:

$$\begin{bmatrix} A_{11} & . & . & A_{1p} \\ . & & & . \\ . & & & . \\ A_{p1} & . & . & A_{pp} \end{bmatrix} \begin{bmatrix} c_1 \\ . \\ . \\ c_p \end{bmatrix} = \begin{bmatrix} 0 \\ . \\ . \\ 0 \end{bmatrix}, \tag{5}$$

or:

$$[A]_{p\times p}\{c_j\}_{p\times 1} = \{0\}_{p\times 1}.$$

The components of the matrix $[A]$ are found as:

$$[A]_{ij} = B_i[\varphi_j(x)]. \tag{6}$$

To determine the natural frequency of the system, the determinant of $[A]$ is equated to zero. This is a well-known result. Our contribution is in finding the mode shapes. To find the mode shapes, one variable (say c_1) is chosen and then, other variables are solved with respect to this variable. With this procedure, the mode shape is found as:

$$u(x) = c_1\Phi(x). \tag{7}$$

In the next section, we propose a simple and systematic method to compute the mode shape of a linear continuous system. This procedure is very effective, especially when implemented by computer. Some examples are solved using the proposed method. The usefulness of the method is more obvious when dealing with stepped structures with many segments.

2. The proposed method

Now, the problem is how one may solve variables (say) $c_2...c_n$ in terms of c_1. In vibrations and structural dynamics textbooks, a procedure is applied as follows: first, the variable, c_2, is solved with respect to $c_1, c_3, ..., c_p$ and, then, substituted in the second equation. So, the second equation is written only in terms of $c_1, c_3, ..., c_p$. Similarly, variable c_3 in the second equation is solved, with respect to $c_1, c_4, ..., c_p$, and the procedure is continued until all variables, $c_2, c_3, ..., c_p$, are written in terms of c_1. The results are then substituted into Eq. (4) to find the mode shape (i.e. Eq. (7)). For problems with large p (e.g. stepped beams with many segments), this is a lengthy procedure and not suitable for implementation by computer.

Here, a systematic procedure, based on the concept of *normal equation*, is proposed to carry out the above mentioned computations, easily. The approach is especially appropriate for computer programming.

Suppose that coefficients $c_2, c_3, ..., c_p$ should be solved with respect to c_1. To do this, the terms in Eq. (5) that contain c_1 are moved to the right hand side of the equation. So, Eq. (5) reads as:

$$\begin{bmatrix} A_{12} & . & . & A_{1p} \\ . & & & . \\ . & & & . \\ A_{p2} & . & . & A_{pp} \end{bmatrix} \begin{bmatrix} c_2 \\ . \\ . \\ c_p \end{bmatrix} = -c_1 \begin{bmatrix} A_{11} \\ . \\ . \\ A_{p1} \end{bmatrix}, \tag{8}$$

or:

$$[A]_{p\times(p-1)}\{c_j\}_{(p-1)\times 1} = \{B\}_{p\times 1},$$

where:

$$[B] = -c_1 \begin{bmatrix} A_{11} \\ . \\ . \\ A_{p1} \end{bmatrix}.$$

Now, Eq. (8) is a system with p equations and $p-1$ unknowns. So, this system of equations is *overestimated.* To solve this matrix equation, the normal equation should be obtained [11]. The *normal equation* is that which minimizes the sum of the square differences between the left and right sides of the matrix equation (Eq. (8)). To compute a normal equation, both sides of Eq. (8) are multiplied by $[A]^T_{(p-1)\times p}$ as:

$$[A]^T_{(p-1)\times p}[A]_{p\times(p-1)}\{c_j\}_{(p-1)\times 1} = [A]^T_{(p-1)\times p}\{B\}_{p\times 1}. \tag{9}$$

The above equation can be written as:

$$[\bar{A}]\{c_j\} = \{\bar{B}\}, \tag{10}$$

where:

$$[\bar{A}] \equiv [A]^T_{(p-1)\times p}[A]_{p\times(p-1)}, \qquad \{\bar{B}\} \equiv [A]^T\{B\}.$$

Now, Eq. (10) is a normal equation and may be solved with any methods in linear algebra.

Consequently, finding the mode shape, i.e. solving coefficients $c_2, c_3, ..., c_p$ with respect to c_1, is reduced to solving a system of linear algebraic equations and, therefore, the procedure is fully systematic. The power of the present procedure is revealed when a large number of boundary conditions are applied (e.g. stepped structures).

In the above, the coefficient c_1 was chosen and the first column in the coefficient matrix was moved to the right hand side of the equation. Now, the question arises as to which coefficient can be chosen? The response is that every column can be chosen provided that the determinant of the matrix $[\bar{A}] \equiv [A]^T_{(p-1)\times p}[A]_{p\times(p-1)}$ does not vanish (i.e. matrix $[\bar{A}]$ must not be singular). For example, if a column exists in the coefficient matrix in which all its elements vanish, then, this column must be chosen to move to the right hand side and other columns cannot move to the right hand side. The reason is that, in this case (when a zero column does not move to the right hand side), the matrix $[\bar{A}]$ becomes singular and the normal equation does not have a solution. In the next section, a sample example is presented for this case.

3. Sample examples

For the first example, the mode shape of a simply-supported beam with length l is found. The boundary value problem corresponding to the equation of motion and boundary conditions is [2]:

$$\frac{d^4u(x)}{dx^4} - \lambda^4 u(x) = 0, \tag{11}$$

$$u(0) = u''(0) = u(l) = u''(l) = 0. \tag{12}$$

The form of mode shape is found from Eq. (11) as:

$$u = c_1\sin(\lambda x) + c_2\cos(\lambda x) + c_3\sin h(\lambda x) + c_4\cos h(\lambda x). \tag{13}$$

Applying boundary conditions (Eq. (12)) to the mode shape (Eq. (13)) gives:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin(\lambda l) & \cos(\lambda l) & \sin h(\lambda l) & \cos h(\lambda l) \\ 0 & -\lambda^2 & 0 & \lambda^2 \\ -\lambda^2\sin(\lambda l) & -\lambda^2\cos(\lambda l) & \lambda^2\sin h(\lambda l) & \lambda^2\cos h(\lambda l) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{14}$$

To obtain the eigenvalue, as usual, the determinant of the above matrix is equated to zero; the result is $\lambda = n\pi/l (n = 1, 2, ...)$.

Suppose that coefficients $c_2 ... c_4$ are solved with respect to c_1, then, Eq. (14) is written as:

$$\begin{bmatrix} 1 & 0 & 1 \\ \cos(\lambda l) & \sin h(\lambda l) & \cos h(\lambda l) \\ -\lambda^2 & 0 & \lambda^2 \\ -\lambda^2\cos(\lambda l) & \lambda^2\sin h(\lambda l) & \lambda^2\cos h(\lambda l) \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ c_4 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -\sin(\lambda l) \\ 0 \\ \lambda^2\sin(\lambda l) \end{bmatrix}. \tag{15}$$

Since $\lambda = n\pi/l (n = 1, 2, ...)$, the right hand side of Eq. (10) is zero; consequently the solution of the normal equation is $c_2 = c_3 = c_4 = 0$. Therefore, the mode shape becomes $u = c_1 \sin\frac{n\pi x}{l}$. This is the same as Eq. (11.51) in Ref. [2].

The above example is a special case. As explained in the last paragraph of Section 2, since the first column in Eq. (14) is zero (when substituting $\lambda = n\pi/l (n = 1, 2, ...)$), only this column can be moved to the right hand side. If other columns are chosen, then the normal equation does not have any solution.

If one compares our procedure with others [1,2], it is observed that it is very effective.

For the second example, the mode shape of a free-free beam with length l is obtained. The form of mode shape and the boundary conditions are:

$$u = c_1\sin(\lambda x) + c_2\cos(\lambda x) + c_3\sin h(\lambda x) + c_4\cos h(\lambda x), \tag{16}$$

$$u''(0) = u'''(0) = u''(l) = u'''(l) = 0.$$

Applying the boundary conditions to the mode shape (i.e. Eq. (6)), it is found that:

$$\begin{bmatrix} 0 & -\lambda^2 & 0 & \lambda^2 \\ -\lambda^3 & 0 & \lambda^3 & 0 \\ -\lambda^2\sin\lambda l & -\lambda^2\cos\lambda l & \lambda^2\sin h\lambda l & \lambda^2\cos h\lambda l \\ -\lambda^3\cos\lambda l & \lambda^3\sin\lambda l & \lambda^3\cos h\lambda l & \lambda^3\sin h\lambda l \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{17}$$

The eigenvalue equation becomes:

$$\cos\lambda l \cos h\lambda l = 1, \qquad (n = 1, 2, ...). \tag{18}$$

Suppose that coefficients c_1, c_3, c_4 are solved with respect to c_2. Then Eq. (17) is written as:

$$\begin{bmatrix} 0 & 0 & \lambda^2 \\ -\lambda^3 & \lambda^3 & 0 \\ -\lambda^2 \sin \lambda l & \lambda^2 \sin h\lambda l & \lambda^2 \cos h\lambda l \\ -\lambda^3 \cos \lambda l & \lambda^3 \cos \lambda l & \lambda^3 \sin h\lambda l \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} = c_2 \begin{bmatrix} \lambda^2 \\ 0 \\ \lambda^2 \cos \lambda l \\ -\lambda^3 \sin \lambda l \end{bmatrix}. \tag{19}$$

The normal equation (Eq. 20)) is shown in Box I, where Eq. (18) is used to simplify the equations. Solving Eq. (20) and using Eq. (18), it is found that:

$$\begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} = \begin{pmatrix} \cos \lambda l(\sin h\lambda l + \sin \lambda l)/(\cos^2 \lambda l - 1) \\ \cos \lambda l(\sin h\lambda l + \sin \lambda l)/(\cos^2 \lambda l - 1) \\ 1 \end{pmatrix}. \tag{21}$$

So, the mode shape becomes:

$$u(x) = c_2 \left[\frac{\cos \lambda l(\sin \lambda l + \sin h\lambda l)}{-1 + \cos^2 \lambda l} (\sin \lambda x + \sin h\lambda x) + \cos \lambda x + \cos h\lambda x \right]. \tag{22}$$

This is the same as Eq. (11.84) in Ref [2].

For the third example, flexural vibrations of a free-free stepped beam are considered (Figure 1).

Two coordinates, x_1 and x_2, are used and the corresponding displacements are denoted by u_1 and u_2, respectively.

The form of mode shapes and the boundary conditions are:

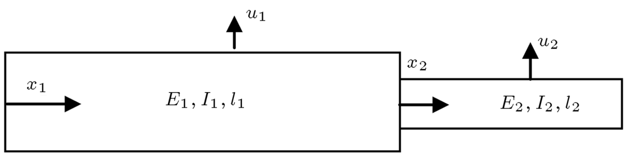

Figure 1. A stepped beam with two segments.

$$\begin{aligned} u_1(x_1) =& c_1 \sin(\lambda x_1) + c_2 \cos(\lambda x_1) \\ &+ c_3 \sin h(\lambda x_1) + c_4 \cos h(\lambda x_1), \\ u_2(x_2) =& c_5 \sin(\lambda x_2) + c_6 \cos(\lambda x_2) \\ &+ c_7 \sin h(\lambda x_2) + c_8 \cos h(\lambda x_2), \end{aligned} \tag{23}$$

$$\begin{aligned} &u_1''(0) = u_1'''(0) = 0, \\ &u_2''(l_2) = u_2'''(l_2) = 0, \\ &u_1(l_1) = u_2(0), \\ &u_1'(l_1) = u_2'(0), \\ &E_1 I_1 u_1''(l_1) = E_2 I_2 u_2''(0), \\ &E_1 I_1 u_1'''(l_1) = E_2 I_2 u_2'''(0). \end{aligned} \tag{24}$$

It is assumed that:

$$\begin{aligned} &l_1 = 1, \quad l_2 = 2, \quad E_1 = 10^9, \\ &E_2 = 2 \times 10^9, \quad I_1 = 10^{-5}, \quad I_2 = 4 \times 10^{-5}. \end{aligned} \tag{25}$$

Substituting Eqs. (23) and (24) into Eq. (6), one can compute the matrix $[A]$. This matrix is a function of λ. To obtain the eigenvalues, the determinant of this matrix is equated to zero. The first eigevalue is $\lambda = 1.728$.

Now, the coefficients, $c_2, c_3, ..., c_8$, are solved with respect to c_1. The solution of Eq. (10) for this case is:

$$\begin{bmatrix} c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix} = \begin{bmatrix} -2.240 \\ 1.032 \\ -1.804 \\ -0.2076 \\ 1.185 \\ -0.704 \\ 0.714 \end{bmatrix}. \tag{26}$$

$$\begin{bmatrix} \lambda^2 + 1 + (\lambda^2 - 1)\cos^2 \lambda l & -2\lambda^2 - \sin h\lambda l \sin \lambda l & -\sin \lambda l \cos h\lambda l - \lambda^2 \cos \lambda l \sin h\lambda l \\ -2\lambda^2 - \sin h\lambda l \sin \lambda l & \lambda^2 + (1 + \lambda^2)\cos h^2 \lambda l - 1 & (1 + \lambda^2) \sin h\lambda l \cos h\lambda l \\ -\sin \lambda l \cos h\lambda l - \lambda^2 \cos \lambda l \sin h\lambda l & (1 + \lambda^2)\sin h\lambda l \cos h\lambda l & 1 + (1 + \lambda^2)\cos h^2 \lambda l - \lambda^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix}$$
$$= c_2 \begin{bmatrix} (-1 + \lambda^2) \sin \lambda l \cos \lambda l \\ \sin h\lambda l \cos \lambda l - \lambda^2 \cos h\lambda l \sin \lambda l \\ 2 - \lambda^2 \sin h\lambda l \sin \lambda l \end{bmatrix}. \tag{20}$$

Box I

Therefore, the first mode shape is found as:

$$u_1(x_1) = \sin(1.728x_1) - 2.24\cos(1.728x_1) + 1.032\sin h(1.728x_1) - 1.804\cos h(1.728x_1),$$

$$u_2(x_2) = -0.207\sin(1.728x_2) + 1.185\cos(1.728x_2) - 0.704\sin h(1.728x_2) + 0.714\cos h(1.728x_2). \quad (27)$$

Since the solution of the above problem was not found in the literature, one can directly validate the result. That is, we substitute solution (27) into Eq. (11) and boundary conditions (24), and it is observed that this solution satisfies these equations. So, Eq. (27) is our desired solution. It is seen that the proposed method is especially suitable for implementation by computer.

4. Conclusions

A procedure to determine mode shapes, based on the concept of a normal equation, was presented. In this approach, first, an arbitrary column of the coefficient matrix is moved to the right hand side of the equation, and then, with the use of the normal equation concept, the resulted algebraic equation is solved. The method is systematic and easy. This procedure is very effective, especially when implemented by computer. Some examples were solved using the proposed method. The usefulness of the method is more obvious when one deals with multi-step structures with many segments. The proposed method can also be applied to discrete systems.

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Biography

Seyyed Ali Asghar Hosseini was born in Qom, Iran, in 1978. He received his BS degree from Iran University of Science and Technology, and MS and PhD degrees from Tarbiat Modares University, Tehran, Iran. He is currently Assistant Professor in the Mechanical Engineering Department of Kharazmi University, Tehran, Iran. His research interests include nonlinear vibrations, rotor dynamics, and random vibrations.