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A note on higher order renormalization group method

S.A.A. Hosseini[a,*] and Z. Karimzadeh[b]

a. *Department of Mechanical Engineering, Faculty of Engineering, Kharazmi University, Mofatteh Ave., Tehran, P.O. Box 15719-14911, Iran.*
b. *Department of Chemistry, Shahid Beheshti University, Tehran, P.O. Box 19839-63113, Iran.*

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Abstract. The Renormalization Group Method (RGM) is a simple and powerful method to obtain analytical solutions for differential equations. In this paper, with some examples, we show that the application of RGM to the second order form of differential equations to determine higher order approximations may give solutions that are at variance with those solutions obtained with the Multiple Scales Method (MSM) and the Generalized Method of Averaging (GMA). However, transforming a differential equation to a complex-variable form and then applying RGM, one may obtain solutions in agreement with the MSM and GMA solutions. Furthermore, we consider a Hamiltonian 2DOF system and observe that the application of RGM to the second order form results in non-Hamiltonian RG equations, and the result is at variance with the MSM and GMA solutions. Again, this problem can be overcomed by applying RGM to a complex-variable form of the equation, obtaining solutions that are derivable from a Lagrangian that are in agreement with the MSM and GMA solutions. Therefore, using RGM, correct results may be obtained by treating the equation as a complex-variable form.

1. Introduction

Perturbation methods are amongst the most powerful in applied mathematics and engineering [1,2] and have been applied to diverse problems (e.g. [3-11]). A relatively new method is the Renormalization Group Method (RGM). This is a powerful method for determining the analytical solution of differential equations proposed by Chen et al. [12,13]. In physics, RGM extracts the features of a system, which are insensitive to details [14], and, so, this method is regarded as an asymptotic analysis [12]. The RGM may be applied to a wide range of problems that were previously treated by the Multiple Scales Method (MSM), the Generalized Method of Averaging (GMA) and the WKB method [15,16]. Kunihiro [17,18] formulated the RGM, based on the classical theory of envelopes for both scalar and vector field problems. Nozaki et al. [19] and Nozaki and Oono [20] proposed the *proto* RGM to free the classical RGM from the necessity of explicit secular terms, as much as possible. Shiwa [21] used this version of RGM to study the Swift-Hohenberg model of a cellular pattern formation. Using the same method, Tu and Cheng [22] studied the evolution of the solution of perturbed equations. Ziane [23] proved that RGM applied to an autonomous nonlinear system of equations results in solutions valid over a long time interval, and studied the relation between this method and the Poincare-Dulak normal forms and the averaging method. Mudavanhu and O'Malley [24] developed a simplified version of RGM to determine higher order approximations on larger time intervals than by GMA and MSM. Kirkinis [25] improved the RGM by rearrangement of secular terms and their grouping into the secular series that multiplies the constants of the asymptotic expansion. Furthermore, O'Malley and Kirkinis [26] introduced a method to

*. *Corresponding author. Tel./ Fax: +9826 34569555*
E-mail address: hosseinyali@gmail.com (S.A.A. Hosseini)

solve initial and boundary singularly perturbed ordinary differential equations whose solution structure can be anticipated. Their method uses renormalized expansion by separating asymptote solutions into fast and slow parts. Chiba [27] showed that RGM can be used to determine the approximate center manifold and the approximate flow on it. Furthermore, he proved [28] that a family of approximate solutions constructed by RGM determines a vector field that is approximate to the original vector field in C^1 topology. He also proved that if the RG equation has a normally hyperbolic invariant manifold, then the original equation has an invariant manifold, which is diffeomorphic to it. The same author [29] studied the higher order RG equation to refine the error in the first order solution and extended the previous results to higher order RG equations. He also obtained the simplest form of RG equations by suitably choosing the integral constants in RG equations. Deville et al. [30] detailed that the RG method may be used to determine the normal forms of autonomous and non-autonomous perturbed differential equations. In autonomous cases, they showed that the RG results are equivalent to Poincare-Birkhoff normal forms up to second order, and, in non-autonomous cases, the reduced equation is equivalent to KBM-based normal forms. Using RGM, Hosseini [31] studied the problem of spurious solutions in the higher order approximation of the forced Duffing equation in the case of primary resonance. In addition, he [32] proposed a direct method based on RGM for determining the analytical approximation of weakly nonlinear continuous systems.

In the above papers, and other studies, RGM is applied to second order forms, as well as to the complex-variable form of differential equations. Transformation of the nonlinear second-order ordinary differential equations to the complex-variable form results in first-order differential equations, and the number of equations becomes double. Since both forms are equivalent, it seems that the corresponding RG equations should also be identical. In this paper, we show that this is not the case. With three examples, we show that the application of RGM to the second order form and to the complex-variable form may give different solutions. The results of RGM, when treating the equations in second order form, is at variance with MSM and GMA, while application of RGM to complex-variable forms leads to solutions that are in agreement with the solutions obtained by these methods. Moreover, we consider a Hamiltonian 2DOF system of ordinary differential equations and apply RGM to the second order form. We observe that the RG equations are non-Hamiltonian and are at variance with those results obtained by MSM and GMA. To remedy this problem, we apply RGM to a complex-variable form of equations and it is shown that the RG equations are derivable from a Lagrangian (and, therefore, has a Hamiltonian structure) and are in agreement with the results of MSM and GMA.

Therefore, one may conclude that in application of the RGM, it is necessary to treat the equations in complex-variable form. It is interesting that this situation also may occur in MSM. Rega et al. [33] applied the MSM to the second order form of differential equations governing the displacement of a suspended cable near the first crossover and found that in the absence of damping and external forces, the modulation equations are not derivable from a Lagrangian, despite the fact that the system is conservative. Nayfeh [34] remedied this problem by application of MSM to the space state form of the governing equations.

2. Application of the RGM to Duffing and Rayleigh equations in the second order form

First, we apply the RGM [30] to the Duffing equation:

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0. \tag{1}$$

Substituting the naive expansion:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2, \tag{2}$$

into Eq. (1), we find:

$$O(\varepsilon^0): \qquad \frac{d^2u_0}{dt^2} + u_0 = 0, \tag{3}$$

$$O(\varepsilon^1): \qquad \frac{d^2u_1}{d^2t} + u_1 = -u_0^3, \tag{4}$$

$$O(\varepsilon^2): \qquad \frac{d^2u_2}{dt^2} + u_2 = -3u_0^2 u_1. \tag{5}$$

The solution of Eq. (3) is:

$$u_0(t) = Ae^{i(t-\tau)} + cc, \tag{6}$$

where $i = \sqrt{-1}$, A is a complex constant, τ is an initial time and cc stands for "complex conjugate".

Substituting Eq. (6) into Eq. (4) and solving the result, it is found:

$$u_1(t) = -\frac{1}{8}A^3 e^{i(t-\tau)} + \frac{3}{2} i A^2 \bar{A}(t-\tau) e^{i(t-\tau)} + \frac{1}{8}A^3 e^{3i(t-\tau)} + cc, \tag{7}$$

where an overbar denotes a complex conjugate. In the above, the homogenous parts of the solutions are chosen, so that $u_1(\tau) = 0$. To renormalize the integration constant, A, we absorb the homogenous parts of the solutions into it and generate a new integration constant, $A = A(\tau)$ [30]. Removing the non-secular resonance terms at first order, we obtain:

$$\tilde{u}_1(t) = \frac{3}{2} iA^2\bar{A}(t-\tau)e^{i(t-\tau)} + \frac{1}{8}A^3 e^{3i(t-\tau)} + cc. \quad (8)$$

Substituting Eqs. (6) and (8) into Eq. (5) and solving the result, we find:

$$\begin{aligned} u_2(t) = &\frac{1}{64}A^4\left(21\bar{A} - A\right)e^{i(t-\tau)} - \frac{15}{16}iA^3\bar{A}^2(t-\tau)e^{i(t-\tau)} \\ &- \frac{9}{8}A^3\bar{A}^2(t-\tau)^2 e^{i(t-\tau)} - \frac{21}{64}A^4\bar{A}e^{3i(t-\tau)} \\ &+ \frac{9}{16}iA^4\bar{A}(t-\tau)e^{3i(t-\tau)} + \frac{1}{64}A^5 e^{5i(t-\tau)} + cc. \quad (9) \end{aligned}$$

Again, in Eq. (9), the homogenous parts of the solutions are chosen, so that $u_2(\tau) = 0$. Removing the non-secular resonance terms at Eq. (9), it is found that:

$$\begin{aligned} \tilde{u}_2(t) = &-\frac{15}{16}iA^3\bar{A}^2(t-\tau)e^{i(t-\tau)} \\ &- \frac{9}{8}A^3\bar{A}^2(t-\tau)^2 e^{i(t-\tau)} - \frac{21}{64}A^4\bar{A}e^{3i(t-\tau)} \\ &+ \frac{9}{16}i(t-\tau)e^{3i(t-\tau)} + \frac{1}{64}A^5 e^{5i(t-\tau)} + cc. \end{aligned} \quad (10)$$

With the application of the RG condition [29]:

$$\left.\frac{d\left(u_0 + \varepsilon\tilde{u}_1 + \varepsilon^2\tilde{u}_2\right)}{d\tau}\right|_{\tau=t} = 0, \quad (11)$$

it is found that:

$$\frac{dA}{dt} = \frac{3}{2}iA^2\bar{A}\varepsilon - \frac{15}{16}iA^3\bar{A}^2\varepsilon^2 + O(\varepsilon^3). \quad (12)$$

As Kunihiro [17,18] stated, with the application of the RG condition (Eq. (11)), an envelope for the family (Eq. (2)) parameterized by τ is constructed. In other words [17]. "Actually, what the RG method does may be said to construct an approximate but global solution from the ones with a local nature which were obtained in the perturbation theory". The other interpretation [23] is that the naïve perturbation approximation (Eq. (2)) has general secular terms. To remove these secular terms, the free parameter, τ, is introduced. Now, since the naïve solution (Eq. (2)) is originally independent of the parameter, τ, the approximate solution should not depend on τ. Consequently, the RG condition (Eq. (11)) is applied. It is observed that RGM requires neither assumptions about the structure of the perturbation series (e.g. time scales in MSM) nor the use of asymptotic matching.

Neglecting the higher order terms, $O(\varepsilon^3)$, the RG equation [29] is found as:

$$\frac{dA}{dt} = \frac{3}{2}iA^2\bar{A}\varepsilon - \frac{15}{16}iA^3\bar{A}^2\varepsilon^2. \quad (13)$$

In a similar fashion, we apply RGM to the Rayleigh equation:

$$\frac{d^2u}{dt} + u - \varepsilon\frac{du}{dt} + \varepsilon\frac{1}{3}\left(\frac{du}{dt}\right)^3 = 0, \quad (14)$$

that results in the RG equation as:

$$\frac{dA}{dt} = \frac{1}{2}A\left(1 - A\bar{A}\right)\varepsilon + \frac{1}{16}iA\left(A^2\bar{A}^2 - 2\right)\varepsilon^2. \quad (15)$$

Eqs. (13) and (15) were previously obtained in [26,30], respectively.

3. Application of the RGM to Duffing and Rayleigh equations in the complex-variable form

Now, the RGM is applied to Duffing and Rayleigh equations in the complex-variable form. Transformation of the nonlinear second-order ordinary differential equations to the complex-variable form results in the first-order differential equations, and the number of equations becomes double.

The Duffing Eq. (1) in state space form is:

$$\frac{du}{dt} - w = 0, \quad (16)$$

$$\frac{dw}{dt} + u + \varepsilon u^3 = 0. \quad (17)$$

Using transformation, $u = \zeta + \bar{\zeta}$, $w = i(\zeta - \bar{\zeta})$, the above equations become:

$$\frac{d\zeta}{dt} + \frac{d\bar{\zeta}}{dt} - i\zeta + i\bar{\zeta} = 0, \quad (18)$$

$$\frac{d\zeta}{dt} - \frac{d\bar{\zeta}}{dt} - i\left(\zeta + \bar{\zeta}\right) - \varepsilon i\left(\zeta + \bar{\zeta}\right)^3 = 0. \quad (19)$$

Solving Eqs. (18) and (19) for $\frac{d\zeta}{dt}$ and $\frac{d\bar{\zeta}}{dt}$, it is obtained that:

$$\frac{d\zeta}{dt} = i\zeta + \frac{1}{2}i\varepsilon\left(\zeta + \bar{\zeta}\right)^3, \quad (20)$$

$$\frac{d\bar{\zeta}}{dt} = -i\bar{\zeta} - \frac{1}{2}i\varepsilon\left(\zeta + \bar{\zeta}\right)^3. \quad (21)$$

Eq. (21) is just a complex conjugate of Eq. (20). Therefore, we apply the RGM to Eq. (20). Substituting the naive expansion;

$$\zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2, \quad (22)$$

into Eq. (20), we obtain:

$$O(\varepsilon^0): \quad \frac{d\zeta_0}{dt} - i\zeta_0 = 0, \tag{23}$$

$$O(\varepsilon^1): \quad \frac{d\zeta_1}{dt} - i\zeta_1 = \frac{1}{2}i\left(\zeta_0 + \bar{\zeta}_0\right)^3, \tag{24}$$

$$O(\varepsilon^2): \quad \frac{d\zeta_2}{dt} - i\zeta_2 = \frac{3}{2}i\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_0 + \bar{\zeta}_0\right)^2. \tag{25}$$

The solution of Eq. (23) with initial time τ is:

$$\zeta_0 = Ae^{i(t-\tau)}. \tag{26}$$

Substituting Eq. (26) into Eq. (24) and solving the result, it is obtained:

$$\begin{aligned}\zeta_1 =& \frac{1}{8}\left(\bar{A}^3 + 6A\bar{A}^2 - 2A^3\right)e^{i(t-\tau)} \\ &+ \frac{3}{2}i\bar{A}A^2(t-\tau)e^{i(t-\tau)} - \frac{3}{4}A\bar{A}^2e^{-i(t-\tau)} \\ &- \frac{1}{8}\bar{A}^3e^{-3i(t-\tau)} + \frac{1}{4}A^3e^{3i(t-\tau)}.\end{aligned} \tag{27}$$

As before, by removing the non-secular resonance terms in Eq. (27), it is found that:

$$\begin{aligned}\tilde{\zeta}_1 =& \frac{3}{2}i\bar{A}A^2(t-\tau)e^{i(t-\tau)} - \frac{3}{4}A\bar{A}^2e^{-i(t-\tau)} \\ &- \frac{1}{8}\bar{A}^3e^{-3i(t-\tau)} + \frac{1}{4}A^3e^{3i(t-\tau)}.\end{aligned} \tag{28}$$

By substituting Eqs. (26) and (28) into Eq. (25), solving the result with initial time, τ, and removing the non-secular resonance terms, we obtain:

$$\begin{aligned}\tilde{\zeta}_2 =& -\frac{51}{16}i\bar{A}^2A^3(t-\tau)e^{i(t-\tau)} \\ &- \frac{9}{8}A^3\bar{A}^2(t-\tau)^2e^{i(t-\tau)} + \frac{69}{32}A^2\bar{A}^3e^{-i(t-\tau)} \\ &+ \frac{9}{8}iA^2\bar{A}^{\,3}(t-\tau)e^{-i(t-\tau)} - \frac{15}{16}A^4\bar{A}e^{3i(t-\tau)} \\ &+ \frac{9}{8}iA^4\bar{A}(t-\tau)e^{3i(t-\tau)} + \frac{21}{64}\bar{A}^4Ae^{-3i(t-\tau)} \\ &+ \frac{9}{16}i\bar{A}^4A(t-\tau)e^{-3i(t-\tau)} + \frac{3}{64}A^5e^{5i(t-\tau)} \\ &- \frac{1}{32}\bar{A}^5e^{-5i(t-\tau)}.\end{aligned} \tag{29}$$

With application of the RG condition:

$$\left.\frac{d\left(\zeta_0 + \varepsilon\tilde{\zeta}_1 + \varepsilon^2\tilde{\zeta}_2\right)}{d\tau}\right|_{\tau=t} = 0, \tag{30}$$

the RG equation is found as:

$$\frac{dA}{dt} = \frac{3}{2}iA^2\bar{A}\varepsilon - \frac{51}{16}iA^3\bar{A}^2\varepsilon^2. \tag{31}$$

This solution was previously presented in [29]. It is noted that if RGM were applied to the state-space form of the Duffing equation, i.e. Eqs. (16) and (17), the RG equation would become the same as Eq. (13), not Eq. (31).

In the same method, by writing the Rayleigh Eq. (14) in complex-variable form, using transformation, $u = \zeta + \bar{\zeta}$, $\frac{du}{dt} = i(\zeta - \bar{\zeta})$, we find:

$$\frac{d\zeta}{dt} = i\zeta + \frac{1}{2}\varepsilon\left(1 - \zeta\bar{\zeta}\right)\left(\zeta - \bar{\zeta}\right) + \frac{1}{6}\varepsilon\left(\zeta^3 - \bar{\zeta}^3\right), \tag{32}$$

and the RG equation becomes:

$$\frac{dA}{dt} = \frac{1}{2}A\left(1 - A\bar{A}\right)\varepsilon + \frac{1}{16}iA\left(4A\bar{A} - 3A^2\bar{A}^2 - 2\right)\varepsilon^2. \tag{33}$$

Obviously, the higher order parts of Eqs. (13) and (31) are not identical. Also, the higher order parts of Eqs. (15) and (33) are not the same. The solutions obtained by treating the equations in complex-variable form, i.e. Eqs. (31) and (33), are identical to Eqs. (20) and (107) of paper [35], using MSM and GMA. In summary, application of the RGM to the complex-variable form of Duffing and Rayleigh equations are in agreement with those obtained with MSM and GMA, and at variance with the results obtained by application of the RGM to the second order form and state-space form of these equations.

In a theorem, Chiba [28] showed that for a differential equation associated with a given vector field, a family of approximate solutions obtained by the RG method defines a vector field which is close to the original vector field in the C^1 topology. All his proof was carried out on vector fields. So, the differential equations were represented in first-order form. To state and prove the above theorem, he assumed some norm conditions. Sufficient conditions for the system to satisfy these norm conditions are [28]:

(i) The unperturbed equation has a diagonalizable constant matrix, all of whose eigenvalues lie on the imaginary axis;

(ii) The nonlinear part is polynomial in x and periodic in t;

(iii) The nonlinear part is polynomial in x and almost periodic in t, and whose set of Fourier exponents has no accumulation points.

The nonlinear part in our system satisfies conditions (ii) and (iii). The complex variable form satisfies, exactly, assumption (i). Consequently, the most appropriate form in which to apply RGM is the complex-variable form.

4. Application of the RGM to a 2DOF system of equations

We consider the following 2DOF system of differential equations with quadratic nonlinearity [35]:

$$\frac{d^2u}{dt^2}+\omega_1^2u-2\varepsilon\delta uv=0,$$

$$\frac{d^2v}{dt^2}+\omega_2^2v-\varepsilon\delta u^2=0. \tag{34}$$

To compare the results, the two-to-one internal resonance, $\omega_2 \approx 2\omega_1$, is considered. For the sake of simplicity, it is assumed that $\omega_1 = 1$ and $\omega_2 = 2$. First, we apply the RGM to the above equation. Substituting the naïve expansion:

$$u=u_0+\varepsilon u_1+\varepsilon^2u_2, \qquad v=v_0+\varepsilon v_1+\varepsilon^2 v_2, \tag{35}$$

into Eq. (34), we find:

$$O(\varepsilon^0):\quad \frac{d^2u_0}{dt^2}+u_0=0,$$

$$\frac{d^2v_0}{dt^2}+4v_0=0, \tag{36}$$

$$O(\varepsilon^1):\quad \frac{d^2u_1}{dt^2}+u_1=2\delta u_0v_0,$$

$$\frac{d^2v_1}{dt^2}+4v_1=\delta u_0^2, \tag{37}$$

$$O(\varepsilon^2):\quad \frac{d^2u_2}{dt^2}+u_2=2\delta(v_0u_1+v_1u_0),$$

$$\frac{d^2v_2}{dt^2}+4v_2=2\delta u_0u_1. \tag{38}$$

The solution of Eqs. (36) is:

$$u_0=Ae^{i(t-\tau)}+cc, \qquad v_0=Be^{2i(t-\tau)}+cc. \tag{39}$$

By substituting Eqs. (39) into Eq. (37), solving the result with initial time τ and removing the non-secular resonance terms, we obtain:

$$\tilde{u}_1=-i\delta\bar{A}B(t-\tau)e^{i(t-\tau)}-\frac{1}{4}\delta ABe^{3i(t-\tau)}+cc,$$

$$\tilde{v}_1=-\frac{1}{4}i\delta A^2(t-\tau)e^{2i(t-\tau)}+\frac{1}{4}\delta A\bar{A}+cc. \tag{40}$$

Similarly, the solution in $O(\varepsilon^2)$ becomes:

$$\tilde{u}_2=\frac{1}{8}i\delta^2A\left(6B\bar{B}-5A\bar{A}\right)(t-\tau)e^{i(t-\tau)}$$

$$+\frac{1}{8}\delta^2A\left(4B\bar{B}-A\bar{A}\right)(t-\tau)^2e^{i(t-\tau)}$$

$$+\mathrm{NRT}+cc,$$

$$\tilde{v}_2=-\frac{1}{4}\delta^2A\bar{A}B(t-\tau)^2e^{2i(t-\tau)}+\mathrm{NRT}+cc, \tag{41}$$

where NRT stands for "non-resonance term". Finally, RG equations are found as:

$$\frac{dA}{dt}=-i\delta\bar{A}B\varepsilon+i\delta^2\left(-\frac{5}{8}\bar{A}A^2+\frac{3}{4}AB\bar{B}\right)\varepsilon^2, \tag{42}$$

$$\frac{dB}{dt}=-\frac{1}{4}i\delta A^2\varepsilon. \tag{43}$$

It is obvious that Eqs. (42) and (43) are not derivable from a Lagrangian, because the coefficient of $AB\bar{B}$ in Eq. (42) is $\frac{3}{4}i\delta^2$, while the coefficient of $A\bar{A}B$ in Eq. (43) is 0. The RG Eqs. (42) and (43) are not Hamiltonian, in spite of the fact that Eq. (34) is Hamiltonian. In other words, in this case, the RGM reduces a Hamiltonian system to a non-Hamiltonian one.

Now, we treat Eq. (34) in complex-variable form. Using transformation:

$$u=\zeta+\bar{\zeta}, \qquad \frac{du}{dt}=i\left(\zeta-\bar{\zeta}\right),$$

$$v=\varsigma+\varsigma, \qquad \frac{dv}{dt}=2i\left(\varsigma-\bar{\varsigma}\right). \tag{44}$$

Eq. (34) becomes:

$$\frac{d\zeta}{dt}-i\zeta=-i\delta\varepsilon\left(\zeta+\bar{\zeta}\right)\left(\varsigma+\bar{\varsigma}\right),$$

$$\frac{d\varsigma}{dt}-2i\varsigma=-\frac{1}{4}i\delta\varepsilon\left(\zeta+\bar{\zeta}\right)^2. \tag{45}$$

Substituting the naive expansion:

$$\zeta=\zeta_0+\varepsilon\zeta_1+\varepsilon^2\zeta_2,$$

$$\varsigma=\varsigma_0+\varepsilon\varsigma_1+\varepsilon^2\varsigma_2, \tag{46}$$

into Eq. (45), we obtain:

$$O(\varepsilon^0):\quad \frac{d\zeta_0}{dt}-i\zeta_0=0,$$

$$\frac{d\varsigma_0}{dt}-2i\varsigma_0=0, \tag{47}$$

$$O(\varepsilon^1):\quad \frac{d\zeta_1}{dt}-i\zeta_1=i\delta\left(\zeta_0+\bar{\zeta}_0\right)\left(\varsigma_0+\bar{\varsigma}_0\right),$$

$$\frac{d\varsigma_1}{dt}-2i\varsigma=-\frac{1}{4}i\delta\left(\zeta_0+\bar{\zeta}_0\right)^2, \tag{48}$$

$$O(\varepsilon^2):\quad \frac{d\zeta_2}{dt}-i\zeta_2=-i\delta\left(\varsigma_0+\bar{\varsigma}_0\right)\left(\zeta_1+\bar{\zeta}_1\right)$$

$$-i\delta\left(\zeta_0+\bar{\zeta}_0\right)\left(\varsigma_1+\bar{\varsigma}_1\right),$$

$$\frac{d\varsigma_2}{dt}-2i\varsigma_2=-\frac{1}{2}i\delta\left(\zeta_0+\bar{\zeta}_0\right)\left(\zeta_1+\bar{\zeta}_1\right), \tag{49}$$

The solution of Eq. (47) with initial time τ is:

$$\zeta_0 = Ae^{i(t-\tau)}, \quad \varsigma_0 = Be^{2i(t-\tau)}. \tag{50}$$

By substituting Eq. (50) into Eq. (48), solving the result with initial time τ and removing the non-secular resonance terms we obtain:

$$\begin{aligned}\tilde{\zeta}_1 =& -\delta\bar{A}Bi(t-\tau)e^{i(t-\tau)} + \frac{1}{2}\delta A\bar{B}e^{-i(t-\tau)} \\ &- \frac{1}{2}\delta ABe^{3i(t-\tau)} + \frac{1}{4}\delta\bar{A}\bar{B}e^{-3i(t-\tau)}, \\ \tilde{\varsigma}_1 =& -\frac{1}{4}i\delta A^2(t-\tau)e^{2i(t-\tau)} + \frac{1}{16}\delta\bar{A}^2e^{-2i(t-\tau)} \\ &+ \frac{1}{4}\delta A\bar{A}.\end{aligned} \tag{51}$$

Similarly, for solutions in $O(\varepsilon^2)$, we have:

$$\begin{aligned}\tilde{\zeta}_2 =& -\frac{1}{16}i\delta^2A\left(9A\bar{A}+4B\bar{B}\right)(t-\tau)e^{i(t-\tau)} \\ &- \frac{1}{8}\delta^2A\left(A\bar{A}-4B\bar{B}\right)(t-\tau)^2e^{i(t-\tau)} \\ &+ \frac{1}{8}i\delta^2\bar{A}\left(A\bar{A}-4B\bar{B}\right)(t-\tau)e^{-i(t-\tau)} \\ &+ \frac{1}{32}\delta^2\bar{A}\left(11A\bar{A}-4B\bar{B}\right)e^{-i(t-\tau)}, \\ \tilde{\varsigma}_2 =& -\frac{1}{8}i\delta^2A\bar{A}B(t-\tau)e^{2i(t-\tau)} \\ &- \frac{1}{4}\delta^2A\bar{A}B(t-\tau)^2e^{2i(t-\tau)} \\ &+ \frac{1}{8}i\delta^2A\bar{A}\bar{B}(t-\tau)e^{-2i(t-\tau)} \\ &+ \frac{1}{16}\delta^2A\bar{A}\bar{B}e^{-2i(t-\tau)} \\ &+ \frac{1}{4}\delta^2A^2\bar{B} + \frac{1}{4}i\delta^2A^2B(t-\tau) \\ &- \frac{1}{4}i\delta^2\bar{A}^2B(t-\tau).\end{aligned} \tag{52}$$

With the application of the RG condition;

$$\left.\frac{d\left(\zeta_0+\varepsilon\tilde{\zeta}_1+\varepsilon^2\tilde{\zeta}_2\right)}{d\tau}\right|_{\tau=t} = 0,$$

$$\left.\frac{d\left(\varsigma_0+\varepsilon\tilde{\varsigma}_1+\varepsilon^2\tilde{\varsigma}_2\right)}{d\tau}\right|_{\tau=t} = 0, \tag{53}$$

the RG equations are found as:

$$\begin{aligned}\frac{dA}{dt} &= -\delta i\bar{A}B\varepsilon - i\delta^2A\left(\frac{9}{16}A\bar{A}+\frac{1}{4}B\bar{B}\right)\varepsilon^2, \\ \frac{dB}{dt} &= -\frac{1}{4}i\delta A^2\varepsilon - \frac{1}{8}i\delta^2A\bar{A}B\varepsilon^2.\end{aligned} \tag{54}$$

Eqs. (54) are derivable from Lagrangian:

$$\begin{aligned}L =& 2\left(\bar{A}\dot{A}-A\dot{\bar{A}}\right) + 4\left(\bar{B}\dot{B}-B\dot{\bar{B}}\right) + i\delta B\bar{A}^2\varepsilon \\ &+ i\delta\bar{B}A^2\varepsilon + \frac{1}{2}i\delta^2A\bar{A}B\bar{B}\varepsilon^2 + \frac{9}{16}i\delta^2A^2\bar{A}^2\varepsilon^2.\end{aligned} \tag{55}$$

Consequently, the Hamiltonian structure of Eqs. (45), which is in complex-variable form, is preserved in the RG reduction process. Moreover, Eqs. (54) are in agreement with Eqs. (172) and (173) in [35], obtained by MSM and GMA, and are at variance with Eqs. (42) and (43) obtained by treating the equations in second order form. In summary, application of RGM to complex-variable forms results in reduced equations that are in agreement with those obtained by MSM and GMA and preserves the Hamiltonian structure of the equations. But, this is not the case when treating the equations in second order form.

5. Conclusion

We applied the RGM to second order and complex-variable forms of some nonlinear differential equations. Transformation of the nonlinear second-order ordinary differential equations to the complex-variable form results in first-order differential equations and the number of equations becomes double. With three examples, we have shown that the application of RGM to the complex-variable form of equations results in solutions that preserve the original structure of the equations and are in full agreement with those results obtained by MSM and GMA. However, treating the equations in second order form may lead to erroneous results. For example, if the nonlinear differential equations have a Hamiltonian structure, the approximation solution from the RGM possesses the original Hamiltonian structure.

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Biographies

Seyed Ali Asghar Hosseini was born in Qom, Iran, in 1978. He received his BS degree from Iran University of Science and Technology, and MS and PhD degrees from Tarbiat Modares University, Tehran, Iran. He is currently Assistant Professor in the Mechanical Engineering Department of Kharazmi University, Tehran, Iran. His research interests include nonlinear vibrations, rotor dynamics, and random vibrations.

Zohreh Karimzadeh was born in Tehran, Iran, in 1980. She received her BS degree from Mazandaran University, Iran, and MS and PhD degrees from Shahid Beheshti University, Tehran, Iran. She is currently invited lecturer in the Chemistry Department of Alzahra University, Tehran, Iran. Her research interests include solutions thermodynamics and computational statistical mechanics.