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Research Note

Research on the stochastic hybrid multi-attribute decision making method based on prospect theory

H. Yu[a,b,1], P. Liu[b,2,*] and F. Jin[b]

a. *School of Economics and Management, Shandong University of Science and Technology, Qingdao 266590, China.*
b. *School of Management Science and Engineering, Shandong University of Finance and Economics, Jinan Shandong 250014, China.*

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Abstract. With respect to stochastic hybrid multi-attribute decision making problems with interval probability and unknown attribute weight, a multi-attribute decision making method, based on the prospect theory, is proposed. To begin with, the hybrid attribute values, including real numbers, interval numbers, triangular fuzzy numbers, linguistic variables, uncertain linguistic variables and intuitionistic fuzzy values, are converted to trapezoidal fuzzy numbers, and interval probability is expressed by trapezoidal fuzzy probability. The prospect value function of the trapezoidal fuzzy numbers for every alternative, under every attribute, and every natural state, based on the decision-making reference point of each attribute and the weight function of trapezoidal fuzzy probability, is constructed, and, then, the prospect value of the attribute for every alternative is calculated through the prospect value function and the weight function. Then, a maximizing deviation method is used to determine the attribute weights and the weighted prospect value of the alternative is obtained by weighing the prospect values. All the alternatives are ranked according to the expected values of the weighted prospect values. Finally, an illustrate example is given to show the decision-making steps, and the influence on decision making of different parameter values in value and weight functions and different decision-making reference points.

1. Introduction

Multi-attribute decision-making is an important part of modern decision science, and it has a wide range of applications, socially and economically. Because of the complexity and unknowingness of objective things, and the vagueness of the human mind, most multi-attribute decision making problems are uncertain, which can be expressed by the fuzziness and randomness of the attribute values. In fuzziness, the decision attributes cannot be fully represented by quantitative data in general. Some attribute values are more suitable for using fuzzy numbers, including interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic variables, interval linguistic variables, and intuitionistic fuzzy values, etc. Typically, in a decision-making problem, there are different types of attribute value, and we call this decision problem a hybrid decision problem. In randomness, attribute values of each alternative can be expressed by random variables, which can be changed by different natural

1. *Present address: School of Mining & Safety Engineering, Shandong University of Science and Technology, Qingdao 266590, China.*
2. *Present address: School of Economics and Management, Civil Aviation University of China, Tianjin 300300, China.*
*. *Corresponding author. Tel.: +86-531-88525999*
E-mail address: peide.liu@gmail.com (P. Liu)

states. Decision makers cannot ascertain the true state in the future, but they can give a variety of possible natural states and their probability distribution.

At present, there is relatively little research into stochastic hybrid multi-attribute decision making problems in which fuzziness and randomness appear simultaneously. Liu and Liu [1] proposed a new definition of the expected value operator of a random fuzzy variable, and a random fuzzy simulation approach, which combines fuzzy simulation and random simulation, is designed to estimate the expected value of a random fuzzy variable. Then, three types of random fuzzy expected value model are presented. Liu and Zhang [2] proposed a multiple attribute decision making approach, based on a relative optimal membership degree, to deal with multiple attribute decision making problems under risk, with weight information unknown and the attribute value as a linguistic variable. Wang and Gong [3] proposed an approach based on expectation-hybrid entropy to process the multi-criteria decision-making problem in which the criteria weights are precisely known and the criteria values are interval probability fuzzy random variables. Liu et al. [4] proposed an extended TOPSIS method, based on probability theory and uncertain linguistic variables, to solve the risk multiple attribute decision making problems in which attribute weight is unknown and the attribute values take the form of uncertain linguistic variables under the interval probability. Liu and Wang [5] proposed a decision approach based on entropy weight and projection theory to solve hybrid multiple attribute decision-making problems under the risk of interval probability with weight unknown.

The above decision model and methods are mainly based on the assumptions of "full rationality". The perfect theoretical system was built through a sophisticated mathematical model and decision making can be made by selecting the maximization utility based on the expected utility theory. However, in actual decision-making, people cannot give a decision by entire rationality, and all kinds of departure can appear between actual decision-making behavior and the predictable decisions of expected utility theory. In 1947, Simon proposed the principle of "bounded rationality" in which he states that people only have a limited rationality in making a decision. In 1970s, Kahneman and Tversky [6] proposed a prospect theory, on the basis of Simon's "bounded rationality" and many individual behavior research achievements, through investigation and tests. Application of the prospect theory to multi-attribute decision-making is the developmental direction of decision theory, current research into which is less, and still in the developmental stage. The main studies are shown as follows. Lahdelma and Salminen [7] proposed the SMAA-P method that combines the piecewise linear difference functions of prospect theory with SMAA (Stochastic Multicriteria Acceptability Analysis). SMAA-P can be used in decision problems, where DM preferences (weights, reference points and coefficients of loss aversion) are difficult to assess accurately. SMAA-P can also be used to measure how robust a decision problem is, with respect to preference information. Wang et al. [8] proposed a fuzzy multi-criteria decision-making approach based on prospect theory, with regard to uncertain multi-criteria decision-making problems in which the criteria weights are incompletely certain and the criteria values of alternatives are in the form of trapezoidal fuzzy numbers. In this method, a non-linear programming model, which satisfies the maximum integrated prospect value, can be enacted, and a genetic algorithm is used to solve the model to attain the criteria weights. Hu et al. [9] proposed a multi-criteria decision making method based on linguistic evaluation and prospect theory, with respect to risk decision making problems. Firstly, the decision matrix, based on linguistic information, is transformed into a decision matrix based on intervals, and a function of difference between intervals is defined. Then, the prospect value of criteria for every alternative is calculated through the value function and the decision weight function, and the prospect value of the alternative is acquired using the weighted average method. Finally, all the alternatives are sorted and the optimal one is chosen according to the prospect values. Wang et al. [10] proposed the multi-index grey relational decision-making method based on cumulative prospect theory. Firstly, the [-1,1] linear transformation operator is used to standardize the original decision-making information, and the positive and negative ideal solution can be obtained. Then, the prospect value function is defined according to the cumulative prospect theory and gray relational analysis, and an optimization model based on the comprehensive maximization prospect values is built. The optimum weight vector can be obtained and the order of the alternatives is determined. Wang and Zhou [11] proposed a decision-making approach based on prospect theory for grey-stochastic multi-criteria decision-making problems in which probabilities and the criteria value of alternatives are both interval grey numbers, and criteria weights are not completely certain. Firstly, a prospect value function of interval grey numbers can be defined, and the prospect value of each alternative is calculated based on all other alternatives as the reference point. Then, an optimization programming model, which satisfies the algorithm of maximizing deviation, can be enacted, and the criteria weights are obtained. Finally, the order of alternatives can be obtained by comparing the prospect value of each alternative. Liu et al. [12] proposed a multi-attribute decision making method based on prospect theory, with respect to risk decision making problems with interval probability in which the attribute values take the form of uncertain linguistic variables. In this method, the uncertain linguistic variables are converted to trapezoidal fuzzy numbers, and the prospect value function of the trapezoidal fuzzy numbers, based on the decision-making reference point of each attribute and the weight function of interval probability, are obtained. The prospect value of attribute for every alternative is calculated through prospect value function and the weight function; the weighted prospect value of alternative is get by using weighted average method, and all the alternatives are ranked by the expected values of the weighted prospect values. Zhang and Fan [13] proposed a method based on the prospect theory to solve risky multiple attribute decision making problems with Decision Maker (DM's) aspirations, where attribute values and probabilities are both in the form of interval numbers. In this method, aspirations are regarded as reference points. Jiang and Cheng [14] proposed a decision-making analysis approach based on prospect theory to select the most desirable alternative from the available set of new product development alternatives. In this method, the prospect reference point is determined by considering the evaluation information of competing product alternatives. Hu et al. [15] proposed a method based on cumulative prospect theory and set pair analysis for dynamic stochastic multi-criteria decision making problems in which criteria weight is unknown and criteria values are in the form of discrete random variables. Firstly, the prospect value of alternatives is calculated on all criteria at different periods, according to the distribution function. Then, the time series weight is derived based on the binomial distribution probability density function, and the criteria weight coefficients are ascertained by the algorithm of maximizing deviation. Finally, the concepts of identity degree, contrary degree, and set pair potential are employed and, thus, the order of alternatives can be consequently determined.

In the above stochastic fuzzy multi-attribute decision-making research based on prospect theory, the hybrid multi-attribute decision making is less. Fan et al. [16] proposed a decision analysis method based on cumulative prospect theory for hybrid multi-attribute decision making problems with decision maker aspirations. However, the randomness of the decision problems is not considered. At the same time, in fuzziness, these decision making problems considered only three types of attribute information: real numbers, interval numbers and linguistic variables. Obviously, the decision-making problems and the method have certain limitations. This paper will propose a multi-attribute decision-making method, based on prospect theory, with respect to stochastic hybrid multi-attribute decision making problems with interval probability and unknown attribute weight, where attribute values take the form of real numbers, interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic variables, uncertain linguistic variables and intuitionistic fuzzy values, respectively.

In order to do so, the remainder of this paper is as follows. In Section 2, we briefly review some basic concepts of interval probability, trapezoidal fuzzy numbers, linguistic variables, uncertain linguistic variables, intuitionistic fuzzy sets and prospect theory. In Section 3, we establish a multi-attribute decision-making method based on prospect theory, with respect to stochastic hybrid multi-attribute decision making problems with interval probability and unknown attribute weight. In Section 4, we give an example to illustrate the application of the proposed method, and the influence on decision making for different parameters in value function and weight function and different decision-making reference points. In Section 5, we conclude the paper.

2. Preliminaries

2.1. Interval probability

Definition 1 [17]. There are n real interval numbers $[L_i, U_i](i = 1, 2, ..., n)$, if they meet $0 \leq L_i \leq U_i \leq 1(i = 1, 2, ..., n)$, then they can be used to describe the probability of basic events in event set Ω; they are also called n-dimensional interval probability, abbreviated to n-PRI. For convenience, vectors $L = (L_1, L_2, ..., L_n)^T$ and $U = (U_1, U_2, ..., U_n)^T$ are introduced, then, n-PRI can be denoted as n-PRI (L, U).

Definition 2 [17]. For an n-PRI (L, U), if there are n positive real numbers, $p_1, p_2, ..., p_n$, and they meet $\sum_{i=1}^{n} p_i = 1$, $L_i \leq p_i \leq U_i(i = 1, 2, ..., n)$, then, n-PRI(L, U) is called reasonable, otherwise, it is unreasonable.

Theorem 1 [18]. A n-PRI is reasonable, iff:

$$\sum_{i=1}^{n} L_i \leq 1 \leq \sum_{i=1}^{n} U_i.$$

Yager and Kreinovich [18] proposed that if n-PRI(L, U) is reasonable, then, the probability intervals, $[L_i, U_i](i = 1, 2, ..., n)$, can be transformed into more precise probability intervals, $[\bar{L}_i, \bar{U}_i](i = 1, 2, ..., n)$, where:

$$\bar{L}_i = \max\Big(L_i, 1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} U_j\Big), \ \bar{U}_i = \min\Big(U_i, 1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} L_j\Big). \quad (1)$$

2.2. Trapezoidal fuzzy numbers

1. The definition of trapezoidal fuzzy numbers.

Definition 3 [19]. $\tilde{a} = (a^L, a^{ML}, a^{MU}, a^U)$ is called a trapezoidal fuzzy number if its membership function, $a(x) : R \to [0, 1]$, meets:

$$a(x) = \begin{cases} \frac{x - a^L}{a^{ML} - a^L}, & x \in (a^L, a^{ML}) \\ 1, & x \in (a^{ML}, a^{MU}) \\ \frac{x - a^U}{a^{MU} - a^U}, & x \in (a^{MU}.a^U) \\ 0, & x \in (-\infty, a^L) \cup (a^U, \infty) \end{cases} \tag{2}$$

The element of a trapezoidal fuzzy number is $x \in R$ (R represents the set of real numbers). Its membership function shows that the extent of element x belongs to fuzzy set, $\tilde{a}$, and it is a regular, continuous convex function. In four sets of data, a^L, a^{ML}, a^{MU} and a^U, if any two of them are equal, the trapezoidal fuzzy number will be degraded into a triangular fuzzy number. For example, if $a^{ML} = a^{MU}$, then $\tilde{a} = (a^L, a^{ML}, a^U)$. If any three of them are equal, or $a^L = a^{ML}, a^{MU} = a^U$, the trapezoidal fuzzy number will be degraded into an interval number, for example, if $a^L = a^{ML} = a^{MU}$, then, $\tilde{a} = [a^L, a^U]$. If four of them are equal, the trapezoidal fuzzy number will be degraded into a real number, i.e. $\tilde{a} = a^L$.

2. The operation rules of the trapezoidal fuzzy numbers.

Let $\tilde{a} = [a^L, a^{ML}, a^{MU}, a^U]$ and $\tilde{b} = [b^L, b^{ML}, b^{MU}, b^U]$ be the two trapezoidal fuzzy numbers, and $\lambda \geq 0$, then, the operation rules are shown as follows [19]:

$$\tilde{a} + \tilde{b} = [a^L + b^L, a^{ML} + b^{ML}, a^{MU} + b^{MU}, a^U + b^U], \tag{3}$$

$$\tilde{a} - \tilde{b} = [a^L - b^U, a^{ML} - b^{MU}, a^{MU} - b^{ML}, a^U - b^U], \tag{4}$$

$$\tilde{a}\tilde{b} = [a^L b^L, a^{ML} b^{ML}, a^{MU} b^{MU}, a^U b^U], \tag{5}$$

$$\lambda\tilde{a} = [\lambda a^L, \lambda a^{ML}, \lambda a^{MU}, \lambda a^U], \tag{6}$$

$$\tilde{a}^\lambda = [(a^L)^\lambda, (a^{ML})^\lambda, (a^{MU})^\lambda, (a^U)^\lambda]. \tag{7}$$

3. The distance of the trapezoidal fuzzy numbers.

Definition 4 [19]. Let $\tilde{a} = [a^L, a^{ML}, a^{MU}, a^U]$ and $\tilde{b} = [b^L, b^{ML}, b^{MU}, b^U]$ be two trapezoidal fuzzy numbers, then, the distance of $\tilde{a}$ and $\tilde{b}$ can be defined in Eq. (8) shown in Box I.

4. The comparison method of the two trapezoidal fuzzy numbers.

Definition 5 [8]. Let $\tilde{a} = [a^L, a^{ML}, a^{MU}, a^U]$ and $\tilde{b} = [b^L, b^{ML}, b^{MU}, b^U]$ be two trapezoidal fuzzy numbers, then:

(i) If $a^L \geq b^L$, $a^{ML} \geq b^{ML}$, $a^{MU} \geq b^{MU}$, $a^U \geq b^U$, then $\tilde{a} \geq \tilde{b}$.

(ii) If conditions $a^L \geq b^L$, $a^{ML} \geq b^{ML}$, $a^{MU} \geq b^{MU}$, $a^U \geq b^U$, are not met, but meet

$$\frac{a^L + a^{ML} + a^{MU} + a^U}{4} \geq \frac{b^L + b^{ML} + b^{MU} + b^U}{4},$$

then, $\tilde{a} \geq \tilde{b}$.

2.3. Linguistic variables and uncertain linguistic variables

Let $S = (s_0, s_1, ..., s_{l-1})$ be a linguistic term set, where s_α is called the linguistic variable, and l is the odd number. In practice, l can obtain 3, 5, 7, 9 etc., respectively, and S can be defined as:

$$S = (s_0, s_1, s_2) = (\text{poor, fair, good}),$$

$$S = (s_0, s_1, s_2, s_3, s_4) = (\text{very poor, poor, fair, good, very good}),$$

$$S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (\text{very poor, poor, slightly poor, fair, slightly good,good, very good}),$$

$$d(\tilde{a}, \tilde{b}) = \sqrt{\frac{(a^L - b^L)^2 + (a^{ML} - b^{ML})^2 + (a^{MU} - b^{MU})^2 + (a^U - b^U)^2}{4}}. \tag{8}$$

Box I

$$S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8)$$
$$= (\text{extremely poor, very poor, poor, slightly poor,}$$
$$\text{fair, slightly good, good, very good,}$$
$$\text{extremely good}).$$

For any linguistic term set, S, the elements in S can satisfy the following additional characteristics:

1. The set is ordered: $s_i < s_j$, if, and only if, $i < j$;
2. There is the negation operator: $\text{neg}(s_i) = s_j$, and $j = l - i - 1$;
3. There is maximum operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$;
4. There is minimum operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$.

Definition 6 [20]. Let $\tilde{s} = [s_a, s_b]$, $s_a, s_b \in S$ and $a \leq b$. s_a and s_b are the lower and upper limits of $\tilde{s}$, respectively, then, $\tilde{s}$ is called an uncertain linguistic variable.

2.4. Intuitionistic fuzzy sets

Definition 7 [21]. Let $X = \{x_1, x_2, ..., x_n\}$ be a universe of discourse. An Intuitionistic Fuzzy Set (IFS), A in X, is given by $A = \{< x, u_A(x), v_A(x) > x \in X\}$, where $u_A : X \to [0,1]$ and $v_A : X \to [0,1]$, with the condition $0 \leq u_A(x) + v_A(x) \leq 1$, $\forall x \in X$. The numbers, $u_A(x)$ and $v_A(x)$, are the membership degree and non-membership degree of the element, x, to the set, A, respectively.

To given element x the pair $(u_A(x), v_A(x))$ is called an Intuitionistic Fuzzy Value (IFV). For convenience, pair $(u_A(x), v_A(x))$ can be denoted as $\tilde{a} = (u_{\tilde{a}}, v_{\tilde{a}})$, such that $u_{\tilde{a}} \in [0,1]$, $v_{\tilde{a}} \in [0,1]$ and $0 \leq u_{\tilde{a}} + v_{\tilde{a}} \leq 1$.

2.5. Prospect theory

The prospect theory thinks that the decision maker will select the course of action based on the prospect value. The prospect value can be jointly determined by the value function and the probability weight function, and it is shown as follows [6]:

$$V = \sum_{i=1}^{k} \left(w(p_i) v(\Delta x_i) \right), \tag{9}$$

where V is the prospect value, and $v(\Delta x)$ is the value function, which is reflected by the decision makers' subjective feelings. $\Delta x_i = x_i - x_0$ is used to express the value of the deviations from the existing wealth to a certain reference point, where x_0 is the reference point. When the wealth is larger than the reference point, we can define the outcome as the gains; otherwise, we define the outcome as the losses. $w(p)$ is the probability weight function, which is the monotone increasing function of probability.

1. The value function.

   Kahneman and Tversky [6] think the value function is a power function, and define it as follows:

$$v(x) = \begin{cases} x^{\alpha}, & x \geq 0 \\ -\theta(-x)^{\beta}, & x < 0 \end{cases} \tag{10}$$

   where x represents the gains or the losses; the gains are the positive values and the losses are the negative values. α and β show the concave-convex degree of the value function in the gain and loss regions, respectively, and the conditions of $0 < \alpha < 1$ and $0 < \beta < 1$ show the diminishing sensibility. The values of α and β are larger, and the decision maker tends to risk. θ shows that the loss region is steeper than the gain region, and $\theta > 1$ shows the losses aversion. Obviously, we have $v(0) = 0$. Regarding parameters α, β and θ in Eq. (10), Kahneman and Tversky [6] obtained $\alpha = \beta = 0.88$ and $\theta = 2.25$, and Abdellaoui [22] suggested that the values of α and β are equal to 0.89 and 0.92, respectively [22].

2. The probability weight function.

   Tversky and Kahneman [23] believe that the probability weight is the subjective judgment of the decision maker based on the probability, p, of the event outcome, and it is neither the probability nor the linear function of the probability. It is the corresponding weight on the probability. The probability weight function is defined as follows [23]:

$$w^{+}(p) = \frac{p^{\gamma}}{\left(p^{\gamma} + (1-p)^{\gamma}\right)^{1/\gamma}}, \tag{11}$$

$$w^{-}(p) = \frac{p^{\delta}}{\left(p^{\delta} + (1-p)^{\delta}\right)^{1/\delta}}, \tag{12}$$

   where $w^{+}(p)$ and $w^{-}(p)$ represent the nonlinear weight function of the gains and the losses, respectively, γ is the risk gain attitude coefficient, and δ is the risk loss attitude coefficient. Tversky and Kahneman [23] obtained that $\gamma = 0.61$ and $\delta = 0.72$, and Richard and Wu [24] believe that $\gamma = 0.74$, $\delta = 0.74$.

3. The decision making method

3.1. The description of the decision making problems

Suppose that there is a stochastic hybrid multi-attribute decision making problem, which has the set of alternatives, $A = (a_1, a_2, ..., a_m)$, and the set of attributes, $C = (c_1, c_2, ..., c_n)$. Let $\omega = (\omega_1, \omega_2, ..., \omega_n)$ be the attribute weight vector, and $\omega_j (j = 1, 2, ..., n)$ be completely unknown. For each attribute, c_j, there are l_j possible statuses, $\Theta_j = (\theta_1, \theta_2, ..., \theta_{l_j})$, and $p_j^t = [p_j^{Lt}, p_j^{Ut}]$ is the interval probability of status $\theta_t (1 \leq t \leq l_j)$, which belongs to attribute c_j, where $0 \leq p_j^{Lt} \leq p_j^{Ut} \leq 1$, $\sum_{t=1}^{l_j} p_j^{Lt}$ and $\sum_{t=1}^{l_j} p_j^{Ut} \geq 1$. The status value under status θ_t for attribute c_j, with respect to alternative a_i, is x_{ij}^t, which can be represented by real numbers, interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic variables, uncertain linguistic variables or intuitionistic fuzzy values. Based on these, we can evaluate the alternatives.

3.2. The conversion of different data types to trapezoidal fuzzy numbers

1. linguistic variables conversion into trapezoidal fuzzy numbers.

   If status value x_{ij}^t is expressed by the linguistic label, s_k, from linguistic set, $S = (s_0, s_1, ..., s_{l-1})$, and l is an odd number, x_{ij}^i can be converted to the trapezoidal fuzzy number, $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}]$.

   The transformation method from the linguistic variable $s_k (k = 0, 1, ..., l-1)$ to the trapezoidal fuzzy number $[a_k^L, a_k^{ML}, a_k^{MU}, a_k^U]$ is shown as follows:

$$\begin{cases} a_0^L = a_0^{ML} = a_0^{MU} = 0 \\ a_k^{MU} = a_k^{ML} + \frac{1}{2\times l-3} \quad (1 \leq k \leq l-2) \\ a_k^U = a_k^{MU} + \frac{1}{2\times l-3} \quad (0 \leq k \leq l-2) \\ a_{k+1}^L = a_k^{MU} \quad (0 \leq k \leq l-2) \\ a_{k+1}^{ML} = a_k^U \quad (0 \leq k \leq l-2) \\ a_{l-1}^{MU} = a_{l-1}^U = 1 \end{cases} \tag{13}$$

   So, we have:

$$[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}] = [a_k^L, a_k^{ML}, a_k^{MU}, a_k^U].$$

   According to Eqs. (13), when, $l = 7$, we can get the relationship from the linguistic set to their corresponding trapezoidal fuzzy numbers, which is shown as follows:

$$s_0 = [0, 0, 0, 0.091],$$

$$s_1 = [0, 0.091, 0.182, 0.273],$$

$$s_2 = [0.182, 0.273, 0.364, 0.455],$$

$$s_3 = [0.364, 0.455, 0.545, 0.636],$$

$$s_4 = [0.545, 0.636, 0.727, 0.818],$$

$$s_5 = [0.727, 0.818, 0.909, 1],$$

$$s_6 = [0.909, 1, 1, 1].$$

2. Uncertain linguistic variables conversion into trapezoidal fuzzy numbers.

   Suppose status value x_{ij}^t is expressed by the uncertain linguistic variable, $[s_a, s_b]$, and the converted trapezoidal fuzzy number can be expressed by $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}]$. If s_a and s_b are represented by trapezoidal fuzzy numbers, $[a^L, a^{ML}, a^{MU}, a^U]$ and $[b^L, b^{ML}, b^{MU}, b^U]$, respectively, then, we have $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}] = [a^L, a^{ML}, b^{MU}, b^U]$.

3. Intuitionistic fuzzy values conversion into trapezoidal fuzzy numbers.

   Suppose status value x_{ij}^t is expressed by the intuitionistic fuzzy value, (u, v), and the converted trapezoidal fuzzy number can be expressed by $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}]$. Because the intuitionistic fuzzy value, (u, v), can be converted to interval number, $[u, 1-v]$, we have $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}] = [a^L, a^{ML}, b^{MU}, b^U]$.

4. Expression of real numbers, interval numbers and triangular fuzzy numbers by trapezoidal fuzzy numbers.

   Suppose status value x_{ij}^t is expressed by the real number, a, and the converted trapezoidal fuzzy number can be expressed by $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}]$, then, we have $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}] = [a, a, a, a]$.

   Suppose status value x_{ij}^t is expressed by the interval number, $[a, b]$, and the converted trapezoidal fuzzy number can be expressed by $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}]$, then, we have $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}] = [a, a, b, b]$.

   Suppose status value x_{ij}^t is expressed by the triangular fuzzy number, $[a, b, c]$, and the converted trapezoidal fuzzy number can be expressed by $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}]$, then, we have $[a_{ij}^{Lt}, a_{ij}^{MLt}, a_{ij}^{MUt}, a_{ij}^{Ut}] = [a, b, b, c]$.

   After the above conversion, the different types of attribute value can be transformed into trapezoidal fuzzy numbers.

3.3. The decision making steps

1. Make the interval probability more precisely.

   According to Eq. (1), the interval probability, under different natural statuses for different attributes, can be made more precisely, and we can get more precise probabilities, $[\bar{p}_j^{L1}, \bar{p}_j^{U1}]$

$[\bar{p}_j^{L2}, \bar{p}_j^{U2}]...[\bar{p}_j^{Ll_j}, \bar{p}_j^{Ul_j}]$ $(j = 1, 2, ..., n)$. Further, they can be expressed by trapezoidal fuzzy probabilities, $[\bar{p}_j^{L1}, \bar{p}_j^{L1}, \bar{p}_j^{U1}, \bar{p}_j^{U1}][\bar{p}_j^{L2}, \bar{p}_j^{L2}, \bar{p}_j^{U2}, \bar{p}_j^{U2}]$ $...[\bar{p}_j^{Ll_j}, \bar{p}_j^{Ll_j}, \bar{p}_j^{Ul_j}, \bar{p}_j^{Ul_j}]$ $(j = 1, 2, ..., n)$.

2. Convert the different data types to trapezoidal fuzzy numbers.

According to the method proposed in Section 3.2, we can transform the different data types to trapezoidal fuzzy numbers. Suppose the converted trapezoidal fuzzy number of the tth natural status for the jth attribute, with respect to the ith alternative, is $x_{ij}^t = (x_{ij}^{Lt}, x_{ij}^{MLt}, x_{ij}^{MUt}, x_{ij}^{Ut})$.

3. Standardize the decision matrix.

There is an incommensurability measure between different attributes, i.e. different attributes have different metrics. Generally speaking, we cannot directly use the initial attribute values to do the comprehensive evaluation and ranking of alternatives; we must standardize the decision matrix in order to eliminate the influences from different dimensions, units, levels and types of all attributes

For benefit attribute:

$x_{ij}^t = (x_{ij}^{Lt}, x_{ij}^{MLt}, x_{ij}^{MUt}, x_{ij}^{Ut})$,

it can be standardized as follows [19]:

$$r_{ij}^t = \left(\frac{x_{ij}^{Lt}}{x_j^{+U}}, \frac{x_{ij}^{MLt}}{x_j^{+U}}, \frac{x_{ij}^{MUt}}{x_j^{+U}}, \frac{x_{ij}^{Ut}}{x_j^{+U}} \right), \tag{14}$$

where $r_j^{+U} = \max_{i,t}(r_{ij}^{Ut})$.

For cost attribute $x_{ij}^t=(x_{ij}^{Lt}, x_{ij}^{MLt}, x_{ij}^{MUt}, x_{ij}^{Ut})$, it can be standardized as follows [19]:

$$r_{ij}^t = \left(\frac{x_j^{-L}}{x_{ij}^{Ut}}, \frac{x_j^{-L}}{x_{ij}^{MUt}}, \frac{x_j^{-L}}{x_{ij}^{MLt}}, \frac{x_j^{-L}}{x_{ij}^{Lt}} \right), \tag{15}$$

where $r_j^{-L} = \min_{i,t}(r_{ij}^{Lt})$.

The standardization steps may be omitted if the decision data does not have the difference in the dimensions, units, levels and types of all attributes.

4. Select the decision making reference points.

The decision making reference points can be determined by the decision makers' risk preference and their mentation. In traditional stochastic fuzzy multi-attribute decision-making, because there are no specified reference points, we can use the following methods to specify them: (1) zero point; (2) mean value; (3) the middle value of sorting from the biggest to the smallest for one attribute under different alternatives; (4) the worst point; (5) the best point, and (6), the expected value of each attribute. We select the expected value of each attribute as the decision-making reference point of this attribute.

(i) Probability normalization can be calculated by:

$$\bar{p}_j^t = \frac{\bar{p}_j^{Lt} + \bar{p}_j^{Ut}}{\sum_{t=1}^{l_j}(\bar{p}_j^{Lt} + \bar{p}_j^{Ut})}. \tag{16}$$

(ii) The expected value of the trapezoidal fuzzy number is:

$$\bar{r}_{ij}^t = \frac{1}{4}(r_{ij}^{Lt} + r_{ij}^{MLt} + r_{ij}^{MUt} + r_{ij}^{Mt}). \tag{17}$$

(iii) The expected value of each attribute is:

$$r_j^0 = \frac{1}{m}\sum_{i=1}^{m}\sum_{t=1}^{l_j}(\bar{r}_{ij}^t \bar{p}_j^t). \tag{18}$$

5. Calculate the value of the value function.

The value function of the trapezoidal fuzzy number can be calculated as follows:

$$z_{ij}^t = \Big(v(r_{ij}^{Lt} - r_j^{U0}), v(r_{ij}^{MLt} - r_j^{MU0}), v(r_{ij}^{MUt} - r_j^{ML0}), v(r_{ij}^{Ut} - r_j^{L0}) \Big), \tag{19}$$

where:

$$v(\Delta r) = \begin{cases} \Delta r^{\alpha}, & \Delta r \geq 0 \\ -\theta(-\Delta r)^{\beta}, & \Delta r < 0 \end{cases}$$

6. Convert the trapezoidal fuzzy probability into trapezoidal probability weight.

According to Eqs. (11) and (12), we can convert the t trapezoidal fuzzy probabilities, $(p_j^1, p_j^2, ..., p_j^{l_j})$, into trapezoidal probability weights, $(w_{ij}^1, w_{ij}^2, ..., w_{ij}^{l_j})$, i.e.,

$$[w(\bar{p}_j^{L1}), w(\bar{p}_j^{L1}), w(\bar{p}_j^{U1}), w(\bar{p}_j^{U1})]$$

$$[w(\bar{p}_j^{L2}), w(\bar{p}_j^{L2}), w(\bar{p}_j^{U2}), w(\bar{p}_j^{U2})]...$$

$$[w(\bar{p}_j^{Ll_j}), w(\bar{p}_j^{Ll_j}), w(\bar{p}_j^{Ul_j}), w(\bar{p}_j^{Ul_j})]$$

$$(j = 1, 2, ..., n),$$

where $w(\bar{p}_j^{Lt})$, $w(\bar{p}_j^{MLt})$, $w(\bar{p}_j^{MUt})$, $w(\bar{p}_j^{Ut})$ can get w^+ or w^- according to the positive or negative numbers of their corresponding $(r_{ij}^{Lt} - r_j^0)$,$(r_{ij}^{MLt} - r_j^0)$,$(r_{ij}^{MUt} - r_j^0)$ and $(r_{ij}^{Ut} - r_j^0)$. If they are positive, then $w(\bar{p}_j^{Lt})$, $w(\bar{p}_j^{MLt})$, $w(\bar{p}_j^{MUt})$ and $w(\bar{p}_j^{Ut})$ can get w^+, otherwise, w^- can be adopted.

7. Calculate the value of the prospect function.

According to Eq. (9), we can calculate the prospect function value z_{ij} of the jth attribute, with respect to the ith alternative:

$$z_{ij}=\sum_{t=1}^{l_j}(w_{ij}^t z_{ij}^t)=(z_{ij}^L,z_{ij}^{ML},z_{ij}^{MU},z_{ij}^U)$$

$$=\left(\sum_{t=1}^{l_j}\left(w(\bar{p}_j^{Lt})v(r_{ij}^{Lt}-r_j^0)\right),\right.$$

$$\sum_{t=1}^{l_j}\left(w(\bar{p}_j^{Lt})v(r_{ij}^{MLt}-r_j^0)\right),$$

$$\sum_{t=1}^{l_j}\left(w(\bar{p}_j^{Ut})v(r_{ij}^{MUt}-r_j^0)\right),$$

$$\left.\sum_{t=1}^{l_j}\left(w(\bar{p}_j^{Ut})v(r_{ij}^{Ut}-r_j^0)\right)\right). \tag{20}$$

8. Calculate the attribute weights.

   The stochastic fuzzy multi-attribute decision-making problems have been converted into fuzzy multi-attribute decision-making problems by the prospect value function calculated by the above steps. In order to determine the attribute weights, the maximizing deviation method has been adopted. Suppose $D_{ij}(\omega)$ is used to represent the deviation between the prospect function value, z_{ij}, of the jth attribute, with respect to the ith alternative, and the sum of prospect function value of the jth attribute, with respect to all alternatives, then, we have $D_{ij}(\omega)=\sum_{k=1}^m d(z_{ij},z_{kj})\omega_j$, $k,\ i=1,2,...,m$, $j=1,2,...,n$ where $d(z_{ij},z_{kj})$ represents the distance between the prospect function value z_{ij} of the jth attribute, with respect to the ith alternative, and the prospect function value z_{kj} of the jth attribute, with respect to the kth alternative, with conditions $d(z_{ij},z_{kj})\geq 0$ and $\omega_j\geq 0$. Total deviations, $D_j(\omega)$, under the jth attribute for all alternatives to other alternatives can be expressed as $D_j(\omega)=\sum_{i=1}^m D_{ij}(\omega)=\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj})\omega_j$, $j=1,2,...,n$, and the total deviations, $D(\omega)$, under all attributes are expressed as $D(\omega)=\sum_{j=1}^n D_j(\omega)=\sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj})\omega_j$. According to the principle of maximizing deviation, the attribute weight vector, $\omega=(\omega_1,\omega_2,...,\omega_n)$, can be selected by maximizing the total deviations of all alternatives on all alternatives. To do this, we can construct the following optimization model [25]:

$$\max D(\omega)=\sum_{j=1}^n D_j(\omega)=\sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj})\omega_j$$

$$\text{s.t.}\quad \sum_{j=1}^n \omega_j^2=1. \tag{21}$$

We can construct the Lagrange multiplier function as follows:

$$L(\omega_j,\lambda)=\sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj})\omega_j$$

$$+\lambda(\sum_{j=1}^n\omega_j^2-1). \tag{22}$$

By solving the above model, we can get:

$$\begin{cases}\frac{\partial L(\omega_j,\lambda)}{\partial\omega_j}=\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj})+2\lambda\omega_j=0\\ \frac{\partial L(\omega_j,\lambda)}{\partial\lambda}=\sum_{j=1}^n\omega_j^2-1=0\end{cases} \tag{23}$$

$$\begin{cases}2\lambda=-\sqrt{\sum_{j=1}^n(\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj}))^2}\\ \omega_j=\frac{\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj})}{\sqrt{\sum_{j=1}^n(\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj}))^2}}\end{cases} \tag{24}$$

Renormalizing attribute weights, we can get:

$$\omega_j=\frac{\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj})}{\sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^m d(z_{ij},z_{kj})},\quad j=1,2,...,n \tag{25}$$

9. Calculate the weighted prospect function value of the ith alternative:

$$z_i=\sum_{j=1}^n(\omega_j\times z_{ij})=(z_i^L,z_i^{ML},z_i^{MU},z_i^U)$$

$$=\left(\sum_{j=1}^n\left(\omega_j\times\sum_{t=1}^{l_j}\left(w(\bar{p}_j^{Lt})v(r_{ij}^{Lt}-r_j^0)\right)\right),\right.$$

$$\sum_{j=1}^n\left(\omega_j\times\sum_{t=1}^{l_j}\left(w(\bar{p}_j^{Lt})v(r_{ij}^{MLt}-r_j^0)\right)\right),$$

$$\sum_{j=1}^n\left(\omega_j\times\sum_{t=1}^{l_j}\left(w(\bar{p}_j^{Ut})v(r_{ij}^{MUt}-r_j^0)\right)\right),$$

$$\left.\sum_{j=1}^n\left(\omega_j\times\sum_{t=1}^{l_j}\left(w(\bar{p}_j^{Ut})v(r_{ij}^{Ut}-r_j^0)\right)\right)\right). \tag{26}$$

10. Rank all alternatives.

    Since the weighted prospect values of all alternatives are trapezoidal fuzzy numbers, all alternatives can be sorted by their expectation values, which can be calculated as follows:

$$E_i=\frac{z_i^L+z_i^{ML}+z_i^{MU}+z_i^U}{4}. \tag{27}$$

The greater the expectation value, E_i, of the ith alternative is, the better the alternative is.

Table 1. The decision making data table.

	c_1				c_2			
	θ_1	θ_2	θ_3	θ_4	θ_1	θ_2	θ_3	θ_4
	[0.1,0.2]	[0.2,0.7]	[0.3,0.4]	[0.1,0.5]	[0.0,0.2]	[0.2,0.6]	[0.3,0.6]	[0.2,0.4]
a_1	[0.7,0.9]	[0.7,0.7]	[0.4,0.7]	[0.9,0.9]	(0.2,0.4,0.5)	(0.5,0.7 ,0.8)	(0.5,0.7,0.9)	(0.8,0.9,1.0)
a_2	[0.9,0.9]	[0.6,0.7]	[0.9,0.9]	[0.7,0.8]	(0.8,0.9,1.0)	(0.5,0.6 ,0.7)	(0.7,0.7,0.7)	(0.5,0.7,0.8)
a_3	[0.7,0.8]	[0.9,1.0]	[0.4,0.5]	[0.5,0.5]	(0.1,0.2,0.3)	(0.8,1.0 ,1.0)	(0.8,0.9,0.9)	(0.2,0.3,0.4)

	c_3			c_4		
	θ_1	θ_2	θ_3	θ_1	θ_2	θ_3
	[0.2,0.5]	[0.3,0.4]	[0.3,0.5]	[0.3,0.5]	[0.2,0.4]	[0.4,0.6]
a_1	$[s_5, s_5]$	$[s_5, s_6]$	$[s_1, s_2]$	(0.6,0.2)	(0.3,0.4)	(0.5,0.2)
a_2	$[s_6, s_6]$	$[s_2, s_3]$	$[s_1, s_2]$	(0.3,0.4)	(0.8,0.0)	(0.3,0.4)
a_3	$[s_3, s_4]$	$[s_4, s_5]$	$[s_2, s_3]$	(0.4,0.4)	(0.7,0.2)	(0.9,0.1)

4. Illustrative example

The company is planning to set up a new factory. There are three alternatives (a_1, a_2, a_3), and four attributes are considered by decision makers which are direct benefits c_1, indirect benefits, c_2, social benefits, c_3, and pollution loss, c_4. The attribute weight is completely unknown. According to the market forecast, there are four natural statuses in attribute c_1 and c_2, including very good, θ_1, good, θ_2, fair, θ_3, and poor, θ_4, and there are three natural statuses in attribute c_3 and c_4, including very good, θ_1, good, θ_2, and fair, θ_3. The probability of each natural status is expressed as the interval probability. The attribute value of each natural status for each attribute, with respect to each alternative, is expressed as hybrid types, including interval numbers, triangular fuzzy numbers, uncertain linguistic variables and intuitionistic fuzzy values. Linguistic set. $S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6) =$ (very poor, poor, slightly poor, fair, slightly good, good, very good) is adopted by the decision makers. The decision making data is shown in Table 1, and we obtain the best alternative according to the above information and the method proposed in this paper.

4.1. The steps of this example

1. Making the interval probability more precisely, we can get:

$$[\bar{p}_1^{L1}, \bar{p}_1^{U1}] = [0.1, 0.2], \quad [\bar{p}_1^{L2}, \bar{p}_1^{U2}] = [0.2, 0.5],$$

$$[\bar{p}_1^{L3}, \bar{p}_1^{U3}] = [0.3, 0.4], \quad [\bar{p}_1^{L4}, \bar{p}_1^{U4}] = [0.1, 0.4],$$

$$[\bar{p}_2^{L1}, \bar{p}_2^{U1}] = [0.0, 0.2], \quad [\bar{p}_2^{L2}, \bar{p}_2^{U2}] = [0.2.0.5],$$

$$[\bar{p}_2^{L3}, \bar{p}_2^{U3}] = [0.3, 0.6], \quad [\bar{p}_2^{L4}, \bar{p}_2^{U4}] = [0.2, 0.4],$$

$$[\bar{p}_3^{L1}, \bar{p}_3^{U1}] = [0.2, 0.4], \quad [\bar{p}_3^{L2}, \bar{p}_3^{U2}] = [0.3, 0.4],$$

$$[\bar{p}_3^{L3}, \bar{p}_3^{U3}] = [0.3, 0.5], \quad [\bar{p}_4^{L1}, \bar{p}_4^{U1}] = [0.3, 0.4],$$

$$[\bar{p}_4^{L2}, \bar{p}_4^{U2}] = [0.2, 0.3], \quad [\bar{p}_4^{L3}, \bar{p}_4^{U3}] = [0.4, 0.5].$$

2. Converting the different data types to trapezoidal fuzzy numbers, we can get the converted decision making matrix, as shown in Table 2.

3. Standardize the decision matrix.

   The unified dimension and type in the decision matrix does not need to be standardized.

4. Select the decision making reference points. According to Eqs. (16)-(18), we can get:

   (i) Probability normalization:

$$\bar{p}_1^1 = 0.136, \quad \bar{p}_1^2 = 0.318, \quad \bar{p}_1^3 = 0.318,$$

$$\bar{p}_1^4 = 0.227, \quad \bar{p}_2^1 = 0.083, \quad \bar{p}_2^2 = 0.292,$$

$$\bar{p}_2^3 = 0.375, \quad \bar{p}_2^4 = 0.250, \quad \bar{p}_3^1 = 0.286,$$

$$\bar{p}_3^2 = 0.333, \quad \bar{p}_3^3 = 0.381, \quad \bar{p}_4^1 = 0.333,$$

$$\bar{p}_4^2 = 0.238, \quad \bar{p}_4^3 = 0.429.$$

   (ii) The expected value of trapezoidal fuzzy numbers:

$$\bar{r}_{11}^1 = 0.800, \quad \bar{r}_{11}^2 = 0.700, \quad \bar{r}_{11}^3 = 0.550,$$

$$\bar{r}_{11}^4 = 0.900, \quad \bar{r}_{21}^1 = 0.900, \quad \bar{r}_{21}^2 = 0.650,$$

$$\bar{r}_{21}^3 = 0.900, \quad \bar{r}_{21}^4 = 0.750, \quad \bar{r}_{31}^1 = 0.750,$$

$$\bar{r}_{31}^2 = 0.950, \quad \bar{r}_{31}^3 = 0.450, \quad \bar{r}_{31}^4 = 0.500,$$

$$\bar{r}_{12}^1 = 0.375, \quad \bar{r}_{12}^2 = 0.675, \quad \bar{r}_{12}^3 = 0.700,$$

Table 2. The converted decision making matrix.

c_1	θ_1	θ_2	θ_3	θ_4
p	[0.10,0.10,0.20,0.20]	[0.20,0.20,0.50,0.50]	[0.30,0.30,0.40, 0.40]	[0.10,0.10,0.40,0.40]
a_1	[0.70,0.70,0.90,0.90]	[0.70,0.70,0.70,0.70]	[0.40,0.40,0.70,0 .70]	[0.90,0.90,0.90,0.90]
a_2	[0.90,0.90,0.90,0.90]	[0.60,0.60,0.70,0.70]	[0.90,0.90,0.90, 0.90]	[0.70,0.70,0.80,0.80]
a_3	[0.70,0.70,0.80,0.80]	[0.90,0.90,1.00,1.00]	[0.40,0.40,0.50, 0.50]	[0.50,0.50,0.50,0.50]

c_2	θ_1	θ_2	θ_3	θ_4
p	[0.00,0.00,0.20,0.20]	[0.20,0.20,0.50,0.50]	[0.30,0.30,0.60, 0.60]	[0.20,0.20,0.40,0.40]
a_1	[0.20,0.40,0.40,0.50]	[0.50,0.70,0.70,0.80]	[0.50,0.70,0.70,0 .90]	[0.80,0.90,0.90,1.00]
a_2	[0.80,0.90,0.90,1.00]	[0.50,0.60,0.60,0.70]	[0.70,0.70,0.70, 0.70]	[0.50,0.70,0.70,0.80]
a_3	[0.10,0.20,0.20,0.30]	[0.80,1.00,1.00,1.00]	[0.80,0.90,0.90, 0.90]	[0.20,0.30,0.30,0.40]

c_3	θ_1	θ_2	θ_3
p	[0.20,0.20,0.40,0.40]	[0.30,0.30,0.40,0.40]	[0.30,0.30,0.50,0.50]
a_1	[0.667,0.833,0.833,1.000]	[0.667,0.833,1.000,1.000]	[0.000,0.167,0.333,0.500]
a_2	[0.833,1.000,1.000,1.000]	[0.167,0.333,0.500,0.667]	[0.000,0.167,0.333,0.500]
a_3	[0.333,0.500,0.667,0.833]	[0.500,0.667,0.833,1.000]	[0.167,0.333,0.500,0.667]

c_4	θ_1	θ_2	θ_3
p	[0.30,0.30,0.40,0.40]	[0.20,0.20,0.30,0.30]	[0.40,0.40,0.50,0.50]
a_1	[0.60,0.60,0.80,0.80]	[0.30,0.30,0.60,0.60]	[0.50,0.50,0.80,0.80]
a_2	[0.30,0.30,0.60,0.60]	[0.80,0.80,1.00,1.00]	[0.30,0.30,0.60,0.60]
a_3	[0.40,0.40,0.60,0.60]	[0.70,0.70,0.80,0.80]	[0.90,0.90,0.90,0.90]

$$\bar{r}_{12}^4 = 0.900, \quad \bar{r}_{22}^1 = 0.900, \quad \bar{r}_{22}^2 = 0.600,$$

$$\bar{r}_{22}^3 = 0.700, \quad \bar{r}_{22}^4 = 0.675, \quad \bar{r}_{32}^1 = 0.200,$$

$$\bar{r}_{32}^2 = 0.950, \quad \bar{r}_{32}^3 = 0.875, \quad \bar{r}_{32}^4 = 0.300,$$

$$\bar{r}_{13}^1 = 0.833, \quad \bar{r}_{13}^2 = 0.875, \quad \bar{r}_{13}^3 = 0.250,$$

$$\bar{r}_{23}^1 = 0.958, \quad \bar{r}_{23}^2 = 0.417, \quad \bar{r}_{23}^3 = 0.250,$$

$$\bar{r}_{23}^1 = 0.583, \quad \bar{r}_{33}^2 = 0.750, \quad \bar{r}_{33}^3 = 0.417,$$

$$\bar{r}_{14}^1 = 0.700, \quad \bar{r}_{14}^2 = 0.450, \quad \bar{r}_{14}^3 = 0.650,$$

$$\bar{r}_{24}^1 = 0.450, \quad \bar{r}_{24}^2 = 0.900, \quad \bar{r}_{24}^3 = 0.450,$$

$$\bar{r}_{34}^1 = 0.500, \quad \bar{r}_{34}^2 = 0.750, \quad \bar{r}_{34}^3 = 0.900.$$

(iii) The expected value of each attribute, i.e., the decision making reference points of each attribute:

$$r_1^0 = 0.720, \quad r_2^0 = 0.698,$$

$$r_3^0 = 0.569, \quad r_4^0 = 0.636.$$

5. Calculate the value of the value function.

According to Eq. (19), the values of the value function of trapezoidal fuzzy numbers can be obtained as follows (suppose parameters $\alpha = 0.89$, $\beta = 0.92$ and $\theta = 2.25$):

$$z_{11}^1 = [-0.061, -0.061, 0.218, 0.218],$$

$$z_{11}^2 = [-0.061, -0.061, -0.061, -0.061],$$

$$z_{11}^3 = [-0.788, -0.788, -0.061, -0.061],$$

$$z_{11}^4 = [0.218, 0.218, 0.218, 0.218],$$

$$z_{21}^1 = [0.218, 0.218, 0.218, 0.218],$$

$$z_{21}^2 = [-0.319, -0.319, -0.061, -0.061],$$

$$z_{21}^3 = [0.218, 0.218, 0.218, 0.218],$$

$$z_{21}^4 = [-0.061, -0.061, 0.106, 0.106],$$

$$z_{31}^1 = [-0.061, -0.061, 0.106, 0.106],$$

$z_{31}^2 = [0.218, 0.218, 0.322, 0.322]$,

$z_{31}^3 = [-0.788, -0.788, -0.558, -0.558]$,

$z_{31}^4 = [-0.558, -0.558, -0.558, -0.558]$,

$z_{12}^1 = [-1.185, -0.738, -0.738, -0.507]$,

$z_{12}^2 = [-0.507, 0.004, 0.004, 0.131]$,

$z_{12}^3 = [-0.507, 0.004, 0.004, 0.241]$,

$z_{12}^4 = [0.131, 0.241, 0.241, 0.345]$,

$z_{22}^1 = [0.131, 0.241, 0.241, 0.345]$,

$z_{22}^2 = [-0.507, -0.265, -0.265, 0.004]$,

$z_{22}^3 = [0.004, 0.004, 0.004, 0.004]$,

$z_{22}^4 = [-0.507, 0.004, 0.004, 0.131]$,

$z_{32}^1 = [-1.402, -1.185, -1.185, -0.964]$,

$z_{32}^2 = [0.131, 0.345, 0.345, 0.345]$,

$z_{32}^3 = [0.131, 0.241, 0.241, 0.241]$,

$z_{32}^4 = [-1.185, -0.964, -0.964, -0.738]$,

$z_{13}^1 = [0.126, 0.306, 0.306, 0.472]$,

$z_{13}^2 = [0.126, 0.306, 0.472, 0.472]$,

$z_{13}^3 = [-1.340, -0.975, -0.596, -0.193]$,

$z_{23}^1 = [0.306, 0.472, 0.472, 0.472]$,

$z_{23}^2 = [-0.975, -0.596, -0.193, 0.126]$,

$z_{23}^3 = [-1.340, -0.975, -0.596, -0.193]$,

$z_{33}^1 = [-0.596, -0.193, 0.126, 0.306]$,

$z_{33}^2 = [-0.193, 0.126, 0.306, 0.472]$,

$z_{33}^3 = [-0.975, -0.596, -0.193, 0.126]$,

$z_{14}^1 = [-0.105, -0.105, 0.200, 0.200]$,

$z_{14}^2 = [-0.824, -0.824, -0.105, -0.105]$,

$z_{14}^3 = [-0.358, -0.358, 0.200, 0.200]$,

$z_{24}^1 = [-0.824, -0.824, -0.105, -0.105]$,

$z_{24}^2 = [0.200, 0.200, 0.407, 0.407]$,

$z_{24}^3 = [-0.824, -0.824, -0.105, -0.105]$,

$z_{34}^1 = [-0.595, -0.595, -0.105, -0.105]$,

$z_{34}^2 = [0.087, 0.087, 0.200, 0.200]$,

$z_{34}^3 = [0.306, 0.306, 0.306, 0.306]$.

6. Convert the trapezoidal fuzzy probability into trapezoidal probability weight.

   When $\gamma = 0.61$ and $\delta = 0.72$, we can get the trapezoidal probability weight function values as follows:

$w_{11}^1 = [0.163, 0.163, 0.261, 0.261]$,

$w_{11}^2 = [0.254, 0.254, 0.464, 0.464]$,

$w_{11}^3 = [0.329, 0.329, 0.397, 0.397]$,

$w_{11}^4 = [0.186, 0.186, 0.370, 0.370]$,

$w_{21}^1 = [0.186, 0.186, 0.261, 0.261]$,

$w_{21}^2 = [0.254, 0.254, 0.464, 0.464]$,

$w_{21}^3 = [0.318, 0.318, 0.370, 0.370]$,

$w_{21}^4 = [0.163, 0.163, 0.370, 0.370]$,

$w_{31}^1 = [0.163, 0.163, 0.261, 0.261]$,

$w_{31}^2 = [0.261, 0.261, 0.421, 0.421]$,

$w_{31}^3 = [0.329, 0.329, 0.397, 0.397]$,

$w_{31}^4 = [0.163, 0.163, 0.397, 0.397]$,

$w_{12}^1 = [0.000, 0.000, 0.254, 0.254]$,

$w_{12}^2 = [0.254, 0.261, 0.421, 0.421]$,

$w_{12}^3 = [0.329, 0.318, 0.474, 0.474]$,

$w_{12}^4 = [0.261, 0.261, 0.370, 0.370]$,

$w_{22}^1 = [0.000, 0.000, 0.261, 0.261]$,

$w_{22}^2 = [0.254, 0.254, 0.464, 0.421]$,

$w_{22}^3 = [0.318, 0.318, 0.474, 0.474]$,

$w_{22}^4 = [0.254, 0.261, 0.370, 0.370]$,

$w_{32}^1 = [0.000, 0.000, 0.254, 0.254]$,

$w_{32}^2 = [0.261, 0.261, 0.421, 0.421]$,

$w_{32}^3 = [0.318, 0.318, 0.474, 0.474]$,

$$w_{32}^4 = [0.254, 0.254, 0.397, 0.397],$$

$$w_{13}^1 = [0.261, 0.261, 0.370, 0.370],$$

$$w_{13}^2 = [0.318, 0.318, 0.370, 0.370],$$

$$w_{13}^3 = [0.329, 0.329, 0.464, 0.464],$$

$$w_{23}^1 = [0.261, 0.261, 0.370, 0.370],$$

$$w_{23}^2 = [0.329, 0.329, 0.397, 0.370],$$

$$w_{23}^3 = [0.329, 0.329, 0.464, 0.464],$$

$$w_{33}^1 = [0.254, 0.254, 0.370, 0.370],$$

$$w_{33}^2 = [0.329, 0.318, 0.370, 0.370],$$

$$w_{33}^3 = [0.329, 0.329, 0.464, 0.421],$$

$$w_{14}^1 = [0.329, 0.329, 0.370, 0.370],$$

$$w_{14}^2 = [0.254, 0.254, 0.329, 0.329],$$

$$w_{14}^3 = [0.397, 0.397, 0.421, 0.421],$$

$$w_{24}^1 = [0.329, 0.329, 0.397, 0.397],$$

$$w_{24}^2 = [0.261, 0.261, 0.318, 0.318],$$

$$w_{24}^3 = [0.397, 0.397, 0.464, 0.464],$$

$$w_{34}^1 = [0.329, 0.329, 0.397, 0.397],$$

$$w_{34}^2 = [0.261, 0.261, 0.318, 0.318],$$

$$w_{34}^3 = [0.370, 0.370, 0.421, 0.421].$$

7. Calculate the value of the prospect function.
   According to Eq. (20), the prospect function value z_{ij} of the jth attribute, with respect to the ith alternative, can be obtained as follows:

$$z_{11} = [-0.244, -0.244, 0.085, 0.085],$$

$$z_{21} = [0.019, 0.019, 0.148, 0.148],$$

$$z_{31} = [-0.303, -0.303, -0.280, -0.280],$$

$$z_{12} = [-0.261, 0.065, -0.095, 0.168],$$

$$z_{22} = [-0.256, -0.065, -0.057, 0.142],$$

$$z_{32} = [-0.225, -0.078, -0.424, -0.279],$$

$$z_{13} = [-0.368, -0.143, 0.011, 0.260],$$

$$z_{23} = [-0.681, -0.393, -0.178, 0.132],$$

$$z_{33} = [-0.535, -0.205, 0.070, 0.341],$$

$$z_{14} = [-0.386, -0.386, 0.124, 0.124],$$

$$z_{24} = [-0.546, -0.546, 0.039, 0.039],$$

$$z_{34} = [-0.060, -0.060, 0.151, 0.151].$$

8. Calculate the attribute weights.
   According to Eq. (25), the attribute weight can be obtained as follows:

$$\omega_1 = 0.306, \quad \omega_2 = 0.234,$$

$$\omega_3 = 0.197, \quad \omega_4 = 0.262.$$

9. Calculate the weighted prospect function value of the ith alternative.
   According to Eq. (26), the weighted prospect function value of the ith alternative can be obtained as follows:

$$z_1 = [-0.309, -0.189, 0.039, 0.149],$$

$$z_2 = [-0.331, -0.230, 0.007, 0.115],$$

$$z_3 = [-0.267, -0.167, -0.132, -0.044].$$

10. Rank all alternatives.
    According to Eq. (27), the expected value of the weighted prospect function for the ith alternative can be obtained as:

$$E_1 = -0.0776, \quad E_2 = -0.1098, \quad E_3 = -0.1525.$$

    Based on the expected values, we can sort the alternatives:

$$a_1 \succ a_2 \succ a_3.$$

4.2. Discussion

In order to further explore what the influence is of prospect value function parameters and decision-making reference points on the results of decision-making, we re-rank all alternatives by combining the different parameters in the value function and probability weight function, the different decision-making reference points, and the expected utility theory. The ranking results are shown in Table 3 (suppose the attribute weight is fixed by $\omega = (0.306, 0.234, 0.197, 0.262)$).

These results show that the decision making results, based on the prospect theory and the expected utility theory, are not completely consistent, and the different parameters and different decision-making reference points also have a certain influence on the order.

Table 3. The decision-making results by combining the different parameters, the different decision-making reference points, and the expected utility theory.

The expected utility theory	
Expected values $E(z_i)$	**Ranking**
$E_1 = 0.765$ $E_2 = 0.735$ $E_3 = 0.758$	$a_1 \succ a_2 \succ a_3$

The prospect theory

	The minimum of reference points		**The maximum of reference points**	
	$r_1^0 = 0, r_2^0 = 0$ $r_3^0 = 0, r_4^0 = 0$		$r_1^0 = 1, r_2^0 = 1$ $r_1^0 = 1, r_2^0 = 1$	
	Expected values $E(z_i)$	**Ranking**	**Expected values $E(z_i)$**	**Ranking**
$\alpha = 0.88$ $\beta = 0.88$ $\gamma = 0.61$ $\delta = 0.72$	$E_1 = 0.798$ $E_2 = 0.782$ $E_3 = 0.780$	$a_1 \succ a_2 \succ a_3$	$E_1 = -0.904$ $E_2 = -0.947$ $E_3 = -0.927$	$a_1 \succ a_3 \succ a_2$
$\alpha = 0.88$ $\beta = 0.88$ $\gamma = 0.74$ $\delta = 0.74$	$E_1 = 0.835$ $E_2 = 0.814$ $E_3 = 0.820$	$a_1 \succ a_3 \succ a_2$	$E_1 = -0.905$ $E_2 = -0.950$ $E_3 = -0.927$	$a_1 \succ a_3 \succ a_2$
$\alpha = 0.89$ $\beta = 0.92$ $\gamma = 0.61$ $\delta = 0.72$	$E_1 = 0.795$ $E_2 = 0.780$ $E_3 = 0.778$	$a_1 \succ a_2 \succ a_3$	$E_1 = -0.869$ $E_2 = -0.914$ $E_3 = -0.897$	$a_1 \succ a_3 \succ a_2$
$\alpha = 0.89$ $\beta = 0.92$ $\gamma = 0.74$ $\delta = 0.74$	$E_1 = 0.832$ $E_2 = 0.811$ $E_3 = 0.818$	$a_1 \succ a_3 \succ a_2$	$E_1 = -0.871$ $E_2 = -0.917$ $E_3 = -0.896$	$a_1 \succ a_3 \succ a_2$

	The middle of reference points		**Expected values**	
	$r_1^0 = 0.5, r_2^0 = 0.5$ $r_3^0 = 0.5, r_4^0 = 0.5$		$r_1^0 = 0.720, r_2^0 = 0.698$ $r_1^0 = 569, r_2^0 = 636$	
	Expected values $E(z_i)$	**Ranking**	**Expected values $E(z_i)$**	**Ranking**
$\alpha = 0.88$ $\beta = 0.88$ $\gamma = 0.61$ $\delta = 0.72$	$E_1 = 0.196$ $E_2 = 0.157$ $E_3 = 0.150$	$a_1 \succ a_2 \succ a_3$	$E_1 = -0.088$ $E_2 = -0.121$ $E_3 = -0.166$	$a_1 \succ a_2 \succ a_3$
$\alpha = 0.88$ $\beta = 0.88$ $\gamma = 0.74$ $\delta = 0.74$	$E_1 = 0.210$ $E_2 = 0.165$ $E_3 = 0.167$	$a_1 \succ a_3 \succ a_2$	$E_1 = -0.082$ $E_2 = -0.120$ $E_3 = -0.155$	$a_1 \succ a_2 \succ a_3$
$\alpha = 0.89$ $\beta = 0.92$ $\gamma = 0.61$ $\delta = 0.72$	$E_1 = 0.197$ $E_2 = 0.160$ $E_3 = 0.155$	$a_1 \succ a_2 \succ a_3$	$E_1 = -0.078$ $E_2 = -0.110$ $E_3 = -0.153$	$a_1 \succ a_2 \succ a_3$
$\alpha = 0.89$ $\beta = 0.92$ $\gamma = 0.74$ $\delta = 0.74$	$E_1 = 0.211$ $E_2 = 0.168$ $E_3 = 0.172$	$a_1 \succ a_3 \succ a_2$	$E_1 = -0.071$ $E_2 = -0.108$ $E_3 = -0.141$	$a_1 \succ a_2 \succ a_3$

5. Conclusion

Stochastic fuzzy multi-attribute decision-making problems are widely used in real decision making, and decision-making based on the prospect theory is more in line with actual decision-making behavior. This paper proposes a hybrid multi-attribute decision making method, based on the prospect theory, for stochastic fuzzy multi-attribute decision-making problems with interval probability, where the attribute values take the form of hybrid data types, including real numbers, interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic variables, uncertain linguistic variables and intuitionistic fuzzy values, and the decision making steps are proposed. This paper analyzes what the influence is of different parameter values in the value function and probability weight function, the different decision-making reference points, and the expected utility theory on the results of decision-making. The method proposed in this paper is easy to use and understand. It has enriched and developed the theory and method of stochastic fuzzy multi-attribute decision-making problems, and has provided a new idea for solving these problems. In the future, we will continue to study applications of the proposed method in real decision making, such as risk assessment of projects, etc.

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Biographies

Hongliang Yu was born in China, in 1970. He obtained a BS degree from the Chinese Department of Liaocheng University, and an MS degree in Business Administration from Shandong Economic University, China. He is currently studying for a PhD degree in Resource Economics and Management. His main research fields are resource economics and management, decision support and business administration.

Peide Liu was born in China in 1966. He obtained BS and MS degrees in Electronic Technology from Southeast University, China, and a PhD degree in Information Management from Beijing Jiaotong University, China. He is currently working as full Professor in the Shandong University of Finance and Economics, China. His main research fields are technology and information management, decision support and electronic-commerce.

Fan Jin was born in China, in 1967. She obtained a PhD degree in Economics from Fudan University, China, and is currently Professor in Shandong University of Finance and Economics, China. Her main research fields are industrial economics, decision support, knowledge management, and electronic-commerce.