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# Solition solutions of a few nonlinear wave equations in engineering sciences

**A.J. Mohamad Jawad[a], S. Kumar[b] and A. Biswas[c,*]**

a. *Department of Computer Engineering Techniques Al-Rafidian College, 00964, Baghdad, Iraq.*
b. *Department of Applied Sciences, Bahra Faculty of Engineering, Patiala, 147 001, Punjab, India.*
c. *Department of Mathematical Sciences, Delaware State University Dover, DE, 19901-2277, USA.*

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**Abstract.** This paper obtains the soliton and other solutions to a few nonlinear wave equations that arise on a daily basis in various engineering disciplines and other fields. The sine-cosine method is adopted to extract these solutions. The ansatz method is also implemented to obtain a singular soliton solution to the Schrödinger-Hirota equation that is studied in electrical engineering in the context of nonlinear fiber optics. In this context, both Kerr law and power law nonlinearity are going to be addressed. There are several constraint conditions that will be listed in order for the solutions to exist.



## 1. Introduction

Nonlinear wave equations play a very important role in several engineering disciplines and dictate various features in daily lives [1-27]. These equations form the essential fabric in engineering disciplines. For example, the dynamics of shallow water waves, deep water waves, pulse propagation through fiber-optic transmission cables and several other features are all governed by nonlinear wave equations. Therefore, it is imperative to take a further look into these equations from a deeper perspective.

The integrability aspect of these equations is consequently a major issue that reveals a lot of concealed features that cannot be otherwise displayed. There are several tools of integrability that become handy in order for these wave equations to be integrable. This paper will apply two such tools that will lead to the extraction of solitons and other solutions. They are the sine-cosine method and the ansatz method.

In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as the tanhsech method [1-3], the extended tanh method [4-6], the hyperbolic function method [7,8], the Jacobi elliptic function expansion method [9], the F-expansion method [10], the first integral method [11,12], and the sine-cosine method [13,14], has been used to solve different types of nonlinear systems of PDEs.

The aim of this paper is to find new exact solutions of the $(2+1)$-dimensional nonlinear Schrödinger equation, Schrödinger-Hirota equation, Gardner equation, modified KdV equation, perturbed Burgers equation, general Burgers-Fisher equation, and the $K(n+1, n+1)$ equation by the sine-cosine method. The ansatz method is also implemented to obtain a sin-

*. *Corresponding author.*
*E-mail address: biswas.anjan@gmail.com (A. Biswas)*

gular soliton solution to the Schrödinger-Hirota equation.

## 2. The sine-cosine function method

Consider the nonlinear partial differential equation in the form:

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, u_{xy}, ...) = 0, \tag{1}$$

where $u(x, y, t)$ is a traveling wave solution of nonlinear partial differential equation (Eq. (1)). We use the transformations:

$$u(x, y, t) = f(\xi), \tag{2}$$

where $\xi = x + y - \lambda t$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -\lambda \frac{d}{d\xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.),$$

$$\frac{\partial}{\partial y}(.) = \frac{d}{d\xi}(.). \tag{3}$$

We use Eq. (3) to transfer the nonlinear partial differential equation (Eq. (1)) to nonlinear Ordinary Differential Equation (ODE):

$$Q(f, f', f'', f''', ...) = 0, \tag{4}$$

where ($'$) denotes the derivative, with respect to $\xi$. The ordinary differential equation (Eq. (4)) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form [15,16]:

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi), \qquad |\xi| \leq \frac{\pi}{2\mu}, \tag{5}$$

or:

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi), \qquad |\xi| \leq \frac{\pi}{2\mu}, \tag{6}$$

where $\alpha$ and $\beta$ are parameters to be determined, and $\mu$ and $c$ are the wave number and wave speed, respectively [14]. We substitute Eq. (5) or (6) and their derivatives:

$$f'(\xi) = \alpha\beta\mu \sin^{\beta-1}(\mu\xi)\cos(\mu\xi)$$

$$f''(\xi) = \alpha\beta(\beta-1)\mu^2 \sin^{\beta-2}(\mu\xi) - \alpha\mu^2\beta^2 \sin^{\beta}(\mu\xi), \tag{7}$$

or:

$$f'(\xi) = -\alpha\beta\mu \cos^{\beta-1}(\mu\xi)\sin(\mu\xi)$$

$$f''(\xi) = \alpha\beta(\beta-1)\mu^2 \cos^{\beta-2}(\mu\xi) - \alpha\mu^2\beta^2 \cos^{\beta}(\mu\xi), \tag{8}$$

and so on into reduced Eq. (4). We balance the terms of the sine functions when Eq. (5) and its derivatives are used, or balance the terms of the cosine functions when Eq. (6) and its derivatives are used, and solve the resulting system of algebraic equations using computerized symbolic packages. We next collect all terms with the same power in $\sin^k(\mu\xi)$ or $\cos^k(\mu\xi)$, set to zero their coefficients to get a system of algebraic equations among the unknown's, $\alpha$, $\beta$ and $\mu$ and solve the subsequent system.

## 3. Applications

### 3.1. Schrödinger equation

Let us first consider the $(2 + 1)$-dimensional nonlinear Schrödinger equation [17] that reads:

$$\iota q_t + a q_{xx} - b q_{yy} + c|q|^2 q = 0, \tag{9}$$

where $a$, $b$ and $c$ are nonzero constants. Firstly, we introduce the transformations:

$$q(x, y, t) = e^{\iota\theta} u(\xi),$$

$$\theta(x, y, t) = \alpha x + \omega y + \delta t, \tag{10}$$

$$\xi(x, y, t) = k(x + ly - \lambda t),$$

where $\alpha$, $\omega$, $\delta$, $k$, $l$ and $\lambda$ are arbitrary real constants. Substituting Eq. (9) into Eq. (9), we obtain that $\lambda = 2(a\alpha - b\omega l)$ and $u(\xi)$ satisfy the ODE:

$$-(\delta + a\alpha^2 - b\omega^2)u + (a - bl^2)k^2 u'' + cu^3 = 0, \tag{11}$$

where ($'$) denotes the derivative with respect to $\xi$. We rewrite this second-order ODE as follows:

$$u'' + k_1 u^3 - k_2 u = 0, \tag{12}$$

where $k_1 = \frac{c}{(a-bl^2)k^2}$ and $k_2 = \frac{\delta + a\alpha^2 - b\omega^2}{(a-bl^2)k^2}$.

Seeking solutions of Eq. (6), we have:

$$\alpha\beta(\beta-1)\mu^2 \cos^{\beta-2}(\mu\xi) - \alpha\mu^2\beta^2 \cos^{\beta}(\mu\xi) + k_1\alpha^3 \cos^{3\beta}(\mu\xi) - k_2\alpha \cos^{\beta}(\mu\xi) = 0. \tag{13}$$

Equating the exponents and the coefficients of each pair of the cosine functions, we find the following algebraic system:

$$\beta - 2 = 3\beta, \quad \alpha\beta(\beta-1)\mu^2 + k_1\alpha^3 = 0,$$

$$-\alpha\beta^2\mu^2 - k_2\alpha = 0. \tag{14}$$

By solving the algebraic system (Eqs. (14)), we get:

$$\beta = -1, \quad \mu = \pm\iota\sqrt{k_2}, \quad \alpha = \pm\sqrt{\frac{2k_2}{k_1}}. \tag{15}$$

Then, by substituting Eq. (15) into Eq. (6), the exact solution of Eq. [12] can be written in the form:

$$u(\xi) = \pm\sqrt{\frac{2k_2}{k_1}}\sec(\pm\iota\sqrt{k_2}\xi)$$
$$= \pm\sqrt{\frac{2k_2}{k_1}}\operatorname{sech}(\sqrt{k_2}\xi). \tag{16}$$

Therefore, the solution of Eq. (9) is given as:

$$q(x,y,t) = \pm\sqrt{2\frac{(\delta + a\alpha^2 - b\omega^2)}{c}} \times \operatorname{sech}\left(\sqrt{\frac{\delta + a\alpha^2 - b\omega^2}{(a-bl^2)k^2}}k(x + ly - 2(a\alpha - b\omega l)t)\right)e^{\iota(\alpha x + \omega y + \delta t)}. \tag{17}$$

For $\alpha = \omega = \delta = l = a = c = 1$, $b = .5$, Eq. (17) reduces to:

$$q(x,y,t) = \pm\sqrt{3}\operatorname{sech}\left(\sqrt{3}(x+y-t)\right)e^{\iota(x+y+t)}. \tag{18}$$

### 3.2. Schrödinger-Hirota equation

Let us consider the nonlinear Schrödinger-Hirota equation, which governs the propagation of optical solitons in a dispersive optical fiber:

$$\iota q_t + \frac{1}{2}q_{xx} + |q|^2 q + \iota\lambda q_{xxx} = 0. \tag{19}$$

Biswas [18] studied this equation using the ansatz method for a bright and dark 1-soliton solution. The power law nonlinearity was assumed. The equation was solved also using the tanh method.

We introduce the transformation:

$$q = u(\xi)e^{\iota\theta}, \quad \theta = \alpha x + \omega t + \epsilon_0,$$
$$\xi = k_0(x - 2\alpha t + \chi), \tag{20}$$

where $\alpha$, $\omega$, $\epsilon_0$, $k_0$ and $\chi$ are real constants. Substituting Eq.(20) into Eq. (19), we obtain that $\alpha = -\frac{1}{3\lambda}$ and $u(\xi)$ satisfy the ODE:

$$-\left(\frac{5}{54\lambda^2} + \omega\right)u + \frac{3}{2}k_0^2 u'' + u^3 = 0, \tag{21}$$

where ($'$) denotes the derivative with respect to $\xi$. We rewrite Eq. (21) into the following form:

$$u'' + k_1 u^3 - k_2 u = 0, \tag{22}$$

where $k_1 = \frac{2}{3k_0^2}$ and $k_2 = \frac{2}{3k_0^2}\left(\frac{5}{54\lambda^2} + \omega\right)$. Seeking solutions of Eq. (5), we have:

$$\alpha\beta(\beta-1)\mu^2\sin^{\beta-1}(\mu\xi) - \alpha\beta^2\mu^2\sin^{\beta}(\mu\xi) + k_1\alpha^3\sin^{3\beta}(\mu\xi) - k_2\alpha\sin^{\beta}(\mu\xi) = 0\ . \tag{23}$$

Equating the exponents and coefficients of each pair of the sine functions, we find the following algebraic system:

$$\beta - 2 = 3\beta, \quad \alpha\beta(\beta-1)\mu^2 + k_1\alpha^3 = 0,$$
$$-\alpha\beta^2\mu^2 - k_2\alpha = 0. \tag{24}$$

By solving the algebraic system (Eqs. (24)), we get:

$$\beta = -1, \quad \mu = \pm\iota\sqrt{k_2}, \quad \alpha = \pm\sqrt{\frac{2k_2}{k_1}}. \tag{25}$$

Then, by substituting Eq. (25) into Eq. (5), the exact soliton solution of Eq. (22) can be written in the form:

$$u(\xi) = \pm\sqrt{\frac{5}{27\lambda^2} + 2\beta}\csc(\pm\iota\sqrt{k_2}\xi)$$
$$= \mp\sqrt{\frac{5}{27\lambda^2} + 2\beta}\ \operatorname{csch}(\sqrt{k_2}\xi). \tag{26}$$

The corresponding solution of Eq. (19) is:

$$q = \mp\sqrt{\frac{5}{27\lambda^2} + 2\beta} \times \operatorname{csch}\left(\sqrt{\frac{2}{3k_0^2}\left(\frac{5}{54\lambda^2} + \omega\right)} k_0(x + \frac{2}{3\lambda}t + \chi)\right)e^{\iota(-\frac{1}{3\lambda}x + \omega t + \epsilon_0)}. \tag{27}$$

For $\alpha = \omega = k_0 = 1$, $\epsilon_0 = \chi = 0$ and $\lambda = -\frac{1}{3}$, Eq. (27) becomes:

$$q = \pm\sqrt{\frac{11}{3}}\operatorname{csch}\left(\frac{\sqrt{11}}{3}(x-2t)\right)e^{\iota(x+t)}. \tag{28}$$

### 3.3. Gardner equation

Let us consider the Gardner equations [19,20]:

$$u_t - 6(u + \epsilon^2 u^2)u_x + u_{xxx} = 0. \tag{29}$$

This equation, known as the mixed KdV-mKdV equation, is very widely studied in various areas of physics that include plasma physics, fluid dynamics, quantum field theory, solid state physics and others [20].

We introduce the transformation $\xi = k(x - \lambda t)$, where $k$ and $\lambda$ are real constants. Eq. (29) transforms to the ODE:

$$-k\lambda u' - 3k(u^2)' - 2\epsilon^2(u^2)' + k^3 u''' = 0. \tag{30}$$

We integrate Eq. (30) and neglect the integration constant to get the following ordinary differential equation:

$$\lambda u + 3u^2 + 2\epsilon^2 u^3 - k^2 u'' = 0. \tag{31}$$

Seeking the solution in Eq. (6), we have:

$$\lambda\alpha\cos^{\beta}(\mu\xi)+3\alpha^2\cos^{2\beta}(\mu\xi)+2\epsilon^2\alpha^3\cos^{3\beta}(\mu\xi)$$
$$-\beta(\beta-1)k^2\mu^2\alpha\cos^{\beta-2}(\mu\xi)$$
$$+\beta^2\mu^2k^2\alpha\cos^{\beta}(\mu\xi)=0. \tag{32}$$

Equating the exponents and coefficients of each pair of cosine functions, we find the following relations for $\beta$:

$$\beta(\beta-1)(\beta-2)\neq 0 \qquad \beta=-1. \tag{33}$$

Substituting Eq. (33) into Eq. (32), we get:

$$\lambda\alpha\cos^{-1}(\mu\xi)+3\alpha^2\cos^{-2}(\mu\xi)+2\epsilon^2\alpha^3\cos^{-3}(\mu\xi)$$
$$-2k^2\mu^2\alpha\cos^{-3}(\mu\xi)$$
$$+\mu^2k^2\alpha\cos^{-1}(\mu\xi)=0. \tag{34}$$

Equating the exponents and coefficients of each pair of cosine functions, we obtain a system of algebraic equations:

$$\cos^{-3}(\mu\xi): 2\epsilon^2\alpha^3-2\alpha k^2\mu^2=0,$$
$$\cos^{-2}(\mu\xi): 3\alpha^2=0,$$
$$\cos^{-1}(\mu\xi): \lambda\alpha+\alpha\mu^2k^2=0. \tag{35}$$

By solving algebraic system (Eq. (34)), we get:

$$\beta=-1, \quad \lambda=-\mu^2k^2, \quad \alpha=\mp\frac{k\mu}{\epsilon}. \tag{36}$$

Then, by substituting Eq. (35) into Eq. (6), the exact soliton solution of Eq. (29) is obtained as follows:

$$u(x,t)=\mp\frac{k\mu}{\epsilon}\sec(\mu k(x+\mu^2k^2t)),$$
$$0<\mu k(x+\mu^2k^2t)<\pi. \tag{37}$$

For $\mu=k=\epsilon=1$, Eq. (37) reduces to:

$$u(x,t)=\pm\sec(x+t). \tag{38}$$

### 3.4. The (1 + 1)-dimensionalnonlinear dispersive equation

Consider the (1 + 1)-dimensional nonlinear dispersive equation:

$$u_t-\delta u^2u_x+u_{xxx}=0, \tag{39}$$

where $\delta$ is a nonzero positive constant. This equation is called the modified KdV equation [21], which arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops, and it exhibits compaction solitons with compact support. To find the traveling wave solutions of Eq. (39), He et al. [22] used the exp-function method, and Elsayed and Shorog [21] used the $(\frac{G'}{G})$-expansion method.

Let us now solve Eq. (39) by the proposed method. We introduce the transformation $\xi=k(x-\lambda t)$, where $k$ and $\lambda$ are real constants. Eq. (39) transforms to the ODE:

$$-k\lambda u'-\frac{\delta}{3}k(u^3)'+k^3u'''=0. \tag{40}$$

We integrate Eq. (40) once with zero constant to get the following ODE:

$$\lambda u+\frac{\delta}{3}(u^3)-k^2u''=0. \tag{41}$$

Seeking the solution in Eq. (6), we have:

$$\lambda\alpha\cos^{\beta}(\mu\xi)+\frac{\delta}{3}\alpha^3\cos^{3\beta}(\mu\xi)$$
$$-\beta(\beta-1)k^2\mu^2\alpha\cos^{\beta-2}(\mu\xi)$$
$$+\beta^2\mu^2k^2\alpha\cos^{\beta}(\mu\xi)=0. \tag{42}$$

Equating the exponents and coefficients of each pair of cosine functions, we find the following algebraic system:

$$\beta=-1,$$
$$\frac{\delta}{3}\alpha^3-2\alpha k^2\mu^2=0,$$
$$\lambda\alpha+\alpha\mu^2k^2=0. \tag{43}$$

By solving algebraic system Eq. (43), we get:

$$\beta=-1, \quad \lambda=-\mu^2k^2, \quad \alpha=\mp\sqrt{\frac{6}{\delta}}k\mu. \tag{44}$$

Then, by substituting Eq. (44) into Eq. (6), the exact soliton solution of Eq. (39) can be written in the form:

$$u(x,t)=\mp\sqrt{\frac{6}{\delta}}k\mu\sec(\mu k(x+\mu^2k^2t)),$$
$$0<\mu k(x+\mu^2k^2t)<\pi. \tag{45}$$

### 3.5. Perturbed Burgers equation

In this section, the study is going to be focused on the perturbed Burgers equation. The solitary wave ansatz method will be adopted to obtain the exact 1-soliton solution of the Burgers equation in $(1+1)$ dimensions. the search is going to be for a topological 1-soliton solution. The perturbed Burgers equation is given by the following form [23]:

$$u_t + auu_x + bu_{xx} = cu^2u_x + duu_{xx}$$
$$+ \gamma(u_x)^2 + \delta u_{xxx}. \tag{46}$$

Eq. (46) appears in the study of gas dynamics and also in the free surface motion of waves in heated fluids. The perturbation terms are obtained from long-wave perturbation theory. Eq. (46) shows up in the long-wave small-amplitude limit of extended systems dominated by dissipation, where dispersion is also present at a higher order [23].

To solve Eq. (46) by the proposed method, we introduce the transformation $\xi = k(x - \lambda t)$, where $k$ and $\lambda$ are real constants. Eq. (46) transforms to the ODE:

$$-\lambda ku' + akuu' + bk^2u'' = kcu^2u'$$
$$+ dk^2uu'' + \gamma k^2(u')^2 + \delta k^3u'''. \tag{47}$$

Seeking the solution in Eq. (6), we have:

$$\lambda\alpha\beta\cos^{\beta-1}(\mu\xi)\sin(\mu\xi)$$
$$- a\alpha^2\beta\mu\cos^{2\beta-1}(\mu\xi)\sin(\mu\xi)$$
$$+ bk\alpha\beta(\beta-1)\mu^2\cos^{\beta-2}(\mu\xi)$$
$$- bk\alpha\beta^2\mu^2\cos^{\beta}(\mu\xi)$$
$$+ c\alpha^3\beta\mu\cos^{3\beta-1}(\mu\xi)\sin(\mu\xi)$$
$$+ dk\alpha^2\beta(\beta-1)\mu^2\cos^{2\beta-2}(\mu\xi)$$
$$+ dk\alpha^2\beta^2\mu^2\cos^{2\beta}(\mu\xi)$$
$$- \gamma k\alpha^2\beta^2\mu^2\cos^{2\beta-2}(\mu\xi)$$
$$+ \gamma k\alpha^2\beta^2\mu^2\cos^{2\beta}(\mu\xi)$$
$$+ \alpha\beta(\beta-1)(\beta-2)\mu^3\delta k^2\cos^{\beta-3}(\mu\xi)\sin(\mu\xi)$$
$$- \alpha\beta^3\mu^3\delta k^2\cos^{\beta-1}(\mu\xi)\sin(\mu\xi) = 0. \tag{48}$$

From Eq. (46), we have:

$$2\beta - 2 = 3\beta - 1, \tag{49}$$

so that:

$$\beta = -1. \tag{50}$$

Setting the coefficients of the different powers of the cosine function to zero yields:

$$-dk\alpha^2\beta(\beta-1)\mu^2 - \gamma k\alpha^2\beta^2\mu^2$$
$$+ \alpha\beta(\beta-1)(\beta-2)\mu^3\delta k^2 = 0,$$
$$bk\alpha\beta(\beta-1)\mu^2 - a\alpha^2\beta\mu = 0,$$
$$(dk + \gamma k)\alpha\beta\mu + \lambda - \beta^2\mu^2k^2\delta = 0. \tag{51}$$

By solving system (Eq. (51)), we get:

$$\delta = \frac{(2d+\gamma)b}{3a}, \quad \alpha = -\frac{2bk}{a}\mu,$$
$$\lambda = (4d - 5\gamma)\frac{bk^2\mu^2}{3a}. \tag{52}$$

Then, by substituting Eq. (52) into Eq. (6), the exact soliton solution of Eq. (46) can be written in the form:

$$u(x,t) = -\frac{2bk}{a}\mu\sec(\mu k(x - (4d-5\gamma)\frac{bk^2\mu^2}{3a}t)). \tag{53}$$

### *3.6. The general Burgers-Fisher equation*

Let us consider the following general Burgers-Fisher equation [24]:

$$u_t - au^nu_x + bu_{xx} + cu(1-u^n) = 0, \tag{54}$$

where $a$, $b$ and $c$ are nonzero constants. We introduce the transformation $\xi = k(x - \lambda t)$, where $k$ and $\lambda$ are real constants. The traveling wave variable $\xi$ permits us to convert Eq. (54) into the following ODE:

$$-\lambda ku' - aku^nu' + bk^2u'' + cu(1-u^n) = 0. \tag{55}$$

Seeking the solution in Eq. (6), we have:

$$\lambda k\alpha\beta\mu\cos^{\beta-1}(\mu\xi)\sin(\mu\xi)$$
$$- ak\alpha^{n+1}\beta\mu\cos^{(n+1)\beta-1}(\mu\xi)\sin(\mu\xi)$$
$$+ bk^2\alpha\beta(\beta-1)\mu^2\cos^{eta-2}(\mu\xi)$$
$$- (bk^2\alpha\beta^2\mu^2 - c\alpha)\cos^{\beta}(\mu\xi)$$
$$- c\alpha^{n+1}\cos^{n+1}(\mu\xi) = 0. \tag{56}$$

From Eq. (56), we have:

$$(n+1)\beta = \beta - 1, \tag{57}$$

so that:

$$\beta = -\frac{1}{n}. \tag{58}$$

When the exponent pair $(n+1)\beta - 1 = \beta - 2$ is equated, we have the same value of $\beta$. Thus, setting their coefficients to zero yields:

$$c\alpha^{n+1} + \lambda k\alpha\beta\mu = 0,$$
$$bk^2\alpha\beta(\beta-1)\mu^2 - ak\alpha^{n+1}\beta\mu = 0. \tag{59}$$

By solving the system (59), we get:

$$\lambda=-\frac{bc(n+1)}{a},\quad \alpha=\left(\frac{b(n+1)}{an}k\mu\right)^{\frac{1}{n}}. \tag{60}$$

Then, by substituting Eq. (60) into Eq. (6), the exact soliton solution of Eq. (54) can be written in the form:

$$u(x,t)=\left[\frac{b(n+1)}{an}k\mu\sec(k\mu(x+\frac{bc(n+1)}{a}t)\right]^{\frac{1}{n}}. \tag{61}$$

### *3.7. The $K(n+1,\ n+1)$ equation*

Let us consider the following $K(n+1,\ n+1)$ equation [25]:

$$u_t+a(u^{n+1})_x+b(u(u^n)_{xx})_x=0, \tag{62}$$

where $a$ and $b$ are nonzero constants. We introduce the transformation $\xi=k(x-\lambda t)$, where $k$ and $\lambda$ are real constants. The traveling wave variable, $\xi$, permits us to convert Eq. (62) into the following ODE:

$$\begin{aligned}-\lambda u'u^{2-n}&+a(n+1)u^2u'+bnk^2u^2u'''\\&+bk^2n(3n-2)uu'u''\\&+bn(n-1)^2k^2(u')^2=0.\end{aligned} \tag{63}$$

Seeking the solution in Eq. (6), we have:

$$\begin{aligned}&\lambda\beta\mu\alpha^{3-n}\cos^{(3-n)\beta-1}(\mu\xi)\sin(\mu\xi)\\&-a(n+1)\alpha^3\beta\mu\cos^{(3\beta-1)}(\mu\xi)\sin(\mu\xi)\\&-bnk^2\alpha^3\beta(\beta-1)(\beta-2)\\&\times\mu^3\cos^{3\beta-3}(\mu\xi)\sin(\mu\xi)\\&+bnk^2\alpha^3\beta^3\mu^3\cos^{3\beta-1}(\mu\xi)\sin(\mu\xi)\\&-bk^2n(3n-2)\alpha^3\beta^2(\beta-1)\\&\times\mu^3\cos^{3\beta-3}(\mu\xi)\sin(\mu\xi)\\&+bk^2n(3n-2)\alpha^3\beta^3\mu^3\cos^{3\beta-1}(\mu\xi)\\&\times\sin(\mu\xi)-bn(n-1)^2k^2\beta^3\mu^3\alpha^3\\&\times\cos^{3\beta-3}(\mu\xi)\sin(\mu\xi)=0.\end{aligned} \tag{64}$$

From Eq. (64), equating exponents yield:

$$(3-n)\beta-1=3\beta-3, \tag{65}$$

so that:

$$\beta=\frac{2}{n}. \tag{66}$$

Thus, setting coefficients of Eq. (64) to zero yields:

$$\begin{aligned}&\lambda\beta\mu\alpha^{3-n}-bnk^2\alpha^3\beta(\beta-1)(\beta-2)\mu^3\\&\qquad-bk^2n(3n-2)\alpha^3\beta^2(\beta-1)\mu^3\\&\qquad-bn(n-1)^2k^2\beta^3\mu^3\alpha^3=0,\\&-\alpha(n+1)\alpha^3\beta\mu+bnk^2\alpha^3\beta^3\mu^3\\&\qquad+bk^2n(3n-2)\alpha^3\beta^3\mu^3=0.\end{aligned} \tag{67}$$

By solving the algebraic system (Eq. (67)), we get:

$$\alpha=\left[\frac{2(3n-1)}{a(n+1)(2n+1)(n-4)}\lambda\right]^{\frac{1}{n}},$$

$$\mu=\frac{1}{2k}\sqrt{\frac{an(n+1)}{b(3n-1)}}. \tag{68}$$

Then by substituting Eq. (68) into Eq. (7), the exact soliton solution of Eq. (62) can be written in the form:

$$\begin{aligned}u(x,t)=&\left[\frac{2(3n-1)}{a(n+1)(2n+1)(n-4)}\lambda\cos^2\right.\\&\left.\times\left(\sqrt{\frac{an(n+1)}{4b(3n-1)}}(x-\lambda t)\right)\right]^{\frac{1}{n}},\ n\neq 4\end{aligned} \tag{69}$$

For $\lambda=-1,\ n=2,\ a=\frac{1}{3},\ b=\frac{1}{10}$, we have:

$$u(x,t)=\cos(x+t). \tag{70}$$

## 4. Ansatz method

This section will describe the ansatz method that will be applied to SHE in order to obtain the singular soliton solution. The parameter domains will be identified during the course of the derivation of the solutions. The study will be split into two sections, namely, Kerr law nonlinearity and power law nonlinear media. The results of the power law nonlinearity will collapse to the ones from the Kerr law, on setting the power law nonlinearity parameter value to unity.

### *4.1. Kerr law nonlinearity*

The Schrödinger-Hirota Equation (SHE) with Kerr law nonlinearity, to be studied in this subsection, is given by Biswas [18,26,27]:

$$\iota q_t+aq_{xx}+b|q|^2q=\iota c(q_{xxx}+k|q|^2q_x)=0. \tag{71}$$

In this case, the first term is the evolution term, while the second term is the group velocity dispersion term, The coefficient of $b$ is the Kerr law nonlinearity term that is also due to the self-phase modulation. The coefficient of $c$ is the third order dispersion term and, finally, the last term is due to nonlinear dispersion. Eq. (71) represents the model for the propagation of optical solitons through dispersive Kerr law optical fibers. The details of the derivation of this equation have been discussed earlier on several occasions [18]. This paper will focus on derivation of the singular soliton solution to Eq. (71) by the ansatz method. In order to proceed, the starting hypothesis is taken to be:

$$q(x,t) = A\text{csch}^p(B(x-vt))e^{\iota(-\alpha x+\omega t+\theta)}, \tag{72}$$

where $A$ and $B$ are free parameters and $v$ is the velocity of the singular soliton. The unknown index or parameter is $p$, whose value will be derived. From the phase component, $\alpha$ represents the soliton frequency, $\omega$ is the soliton wave number and $\theta$ is the phase constant.

Now by substituting Eq. (72) into Eq. (71) and separating into real and imaginary components, we are led to the following pair of equations:

$$\begin{aligned}&(\omega + a\alpha^2 + c\alpha^3)\text{csch}^p(\tau)\\&\quad-(a+3c\alpha)B^2p\{p\ textcsch^p(\tau)\\&\quad+(p+1)\text{csch}^{p+2}(\tau)\}\\&\quad-(b+c\alpha k)A^2csch^{3p}(\tau)=0,\end{aligned} \tag{73}$$

$$\begin{aligned}&(v+2a\alpha+3c\alpha^2)\text{csch}^p(\tau)-ckA^2\text{csch}^{3p}(\tau)\\&\quad-c(p+1)(p+2)B^2\text{csch}^{p+2}(\tau)=0,\end{aligned} \tag{74}$$

where $\tau = B(x-vt)$. From Eq. (73), by the balancing principle, equating the exponents gives:

$$p = 1. \tag{75}$$

Then, from Eq. (73) setting the coefficients of the linearly independent functions for $\text{csch}^{p+j}(\tau)$, $j = 0, 2$ to zero yields the values of the soliton free parameters as:

$$A = \sqrt{-\frac{2(\omega + a\alpha^2 + c\alpha^3)}{b + c\alpha k}}, \tag{76}$$

and:

$$B = \sqrt{\frac{\omega + a\alpha^2 + c\alpha^3}{a + 3c\alpha}}, \tag{77}$$

which introduces the respective constraints as:

$$(\omega + a\alpha^2 + c\alpha^3)(b + c\alpha k) < 0, \tag{78}$$

and:

$$(\omega + a\alpha^2 + c\alpha^3)(a + 3c\alpha) < 0. \tag{79}$$

Again, from the imaginary part, the balancing principle yields the same value of the parameter, $p$. Similarly, the linearly independent functions from Eq. (74), give:

$$B = A\sqrt{-\frac{k}{6}}, \tag{80}$$

and:

$$v = \frac{c(\omega + a\alpha^2 + c\alpha^3) - \alpha(a+3c\alpha)(2a+2c\alpha)}{a+3c\alpha}. \tag{81}$$

Now, Eq. (80) reveals an immediate constraint condition, given by:

$$k < 0. \tag{82}$$

Finally, by substituting the values of the free parameters, $A$ and $B$, from Eqs. (76) and (77) into Eq. (80), we are given the additional constraint relation as:

$$3(b + c\alpha k) - k(a + 3c\alpha) = 0. \tag{83}$$

Hence, finally, the singular 1-soliton solution to the SHE, with Kerr law nonlinearity, is given by:

$$q(x,t) = A\text{csch}[B(x-vt)]e^{\iota(-\alpha x+\omega t+\theta)}, \tag{84}$$

where the free parameters, $A$ and $B$, are, respectively, given by Eqs. (76) and (77), while the velocity, $v$, of the soliton is given by Eq. (81). The singular solitons will exist provided the constraint conditions given in Eqs. (78), (79), (82), and (83) hold.

### *4.2. Power law nonlinearity*

In this subsection, the SHE with power law nonlinearity will be addressed. The SHE is given by:

$$\iota q_t + aq_{xx} + b|q|^{2n}q + \iota c(q_{xxx} + k|q|^{2n}q_x) = 0. \tag{85}$$

Here, in addition to the regular parameters in the Kerr law case, the additional parameter is the power law index, $n$, that dictates the strength of nonlinearity. In order to solve Eq. (85), the starting hypothesis is the same as Eq. (72). Substituting this hypothesis into Eq. (85) and then splitting into real and imaginary parts yields:

$$\begin{aligned}&(\omega + a\alpha^2 + c\alpha^3)\text{csch}^p(\tau)\\&\quad-(a+3c\alpha)B^2p\{p\text{csch}^p(\tau)\\&\quad+(p+1)\text{csch}^{p+2}(\tau)\}\\&\quad-(b+c\alpha k)A^{2n}\text{csch}^{(2n+1)p}(\tau)=0,\end{aligned} \tag{86}$$

and:

$$\begin{aligned}&(v+2a\alpha+3c\alpha^2)\text{csch}^p(\tau)-ckA^{2n}\text{csch}^{(2n+1)p}(\tau)\\&\quad-c(p+1)(p+2)B^2\text{csch}^{p+2}(\tau)=0,\end{aligned} \tag{87}$$

respectively. As in the Kerr law case, the balancing principle gives:

$$p = \frac{1}{n}, \tag{88}$$

on equating the exponents $(2n + 1)p$ and $p + 2$. Again, from Eq. (86), the coefficients of the linearly independent functions, $\mathrm{csch}^{p+j}(\tau)$ for $j = 0, 2$, when set to zero, reveal:

$$A = \left[ -\frac{(n+1)(\omega + a\alpha^2 + c\alpha^3)}{a + 3c\alpha} \right]^{\frac{1}{2n}}, \tag{89}$$

and:

$$B = n\sqrt{\frac{\omega + a\alpha^2 + c\alpha^3}{a + 3c\alpha}}, \tag{90}$$

which means that the same constraint conditions as given by Eqs. (78) and (79) must hold in order for the singular soliton to exist.

Similarly, from the imaginary part equation, the same value of the unknown parameter, $p$, is obtained, by the aid of the balancing principle. Now, from the coefficients of the linearly independent functions in Eq. (87), the value of the free parameter, $B$, is:

$$B = nA^n\sqrt{-\frac{k}{(n+1)(2n+1)}}, \tag{91}$$

while the velocity, $v$, is still given by Eq. (81). Hence, from Eqs. (89)-(91), the additional constraint condition that falls out is:

$$(2n + 1)(b + c\alpha k) - k(a + 3c\alpha) = 0. \tag{92}$$

Therefore, the singular 1-soliton solution to the SHE, with power law nonlinearity, is given by:

$$q(x, t) = A\mathrm{csch}^{\frac{1}{n}}[B(x - vt)]e^{\iota(-\alpha x+\omega t+\theta)}, \tag{93}$$

where the free parameters, $A$ and $B$, are, respectively given by Eqs. (89) and (90), while the velocity, $v$, of the soliton is given by Eq. (81). The singular solitons will exist provided the constraint conditions given in Eqs. (78), (79), (82), and (92) hold. Thus, under these conditions, the SHE with power law nonlinearity supports the singular 1-soliton solution to Eq. (85). On a final note, all results of power law nonlinearity collapse to those with Kerr law nonlinearity when the power law parameter, $n$, is set to unity.

## 5. Conclusion

In this paper, the sine-cosine function method has been successfully applied to find the solution to nonlinear partial differential equations. The method is used to find a new exact solution. The ansatz method is also implemented to obtain a singular soliton solution to the Schrödinger-Hirota equation. Thus, we can say that the sine-cosine function method can be extended to solve the problems of nonlinear partial differential equations, which arise in the theory of solitons and other areas.

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## Biographies

**Anwar Ja'afar Mohamed Jawad** was born in Baghdad, Iraq, in 1961. He obtained a BS degree in Mechanical Engineering from Baghdad University in 1983, and MS and PhD degrees in Operations Research and Applied Mathematics from the University of Technology, Baghdad, in 1989 and 2000, respectively. He is currently Assistant professor in Applied Mathematics at Al-Rafidain University College, Baghdad, Iraq. His research interests include differential equations and numerical analysis. He has published more than 30 manuscripts in international journals on solving nonlinear partial differential equations. He has taught mathematics and numerical analysis to graduate and postgraduate students in Iraqi and Syrian universities, and has supervised many MS and PhD theses.

**Sachin Kumar** received his PhD degree from Thapar University Patiala, India, and is currently a post-doctoral fellow in the Department of Mathematics at Punjab University, Chandigarh, India. His research interests include symmetries and exact solutions of nonlinear partial differential equations. He has published a number of research papers in peer reviewed journals.

**Anjan Biswas** earned his BS degree in Mathematics from St. Xavier's College in Calcutta, India, and MS and MPhil degrees in Applied Mathematics from the University of Calcutta, India. Subsequently, he obtained MA and PhD degrees in Applied Mathematics from the University of New Mexico, Albuquerque, USA. Currently, he is Associate Professor of Mathematics in the Department of Mathematical Sciences at Delaware State University, Dover, USA. His research interests include the theory of solitons.