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Some operators acting on weighted sequence spaces and applications

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Abstract. This paper considers the problem of constructing an evolution family for the linear nonautonomous Cauchy problem:

$$(*) \qquad \frac{\partial}{\partial t}u(t) - A(t)u(t) = 0, \qquad u(-1) = x \in \mathbb{R}^N,$$

where $A \in C([-1,1],\ \mathbb{R}^{N\times N})$. The essence of the method is that the evolution family is sought in the form of a series of Chebyshev polynomials. Then, by defining appropriate weighted sequence spaces and matrices of linear operators, we are able to obtain a sufficient condition - based only in the given data - for the representation of the solution of the initial value problem (*). The method is motivated for practical considerations in the context of Magnetic Resonance Imaging.

1. Introduction

This paper aims to present a functional analytic framework for an original method to compute the solution u of the general equation:

$$\begin{cases} u'(t) = A(t)u(t), & t \in [-1,1] \\ u(-1) = x, & x \in \mathbb{R}^N \end{cases} \tag{1}$$

where $\{A(t)\}_{t\in[-1,1]}$ is a family of continuous $N \times N$ real-valued matrices defined on a finite interval. The method is based on a series expansion of the solution in terms of Chebyshev polynomials. Through this transform, Eq. (1) is changed into an infinite system of linear differential equations with constant coefficients, the unknowns being the coefficients of the series expansion of u, to obtain a representation of the solution. Such a problem is motivated by Magnetic Resonance Imaging, where typically $N = 3$ and matrix $A(t)$ is not suited for resolution of the differential system by numerical methods, see [1].

When the matrix $A(t)$ is continuous and has periodic coefficients, according to Floquet theory, the fundamental solution, $u(t)$, for the differential system, (Eq. (1)), has the expression $u(t) = Q(t)e^{tF}$, where Q is a matrix with continuous and periodic coefficients and F is a constant matrix. However, Floquet theory gives no practical information about the way to compute matrices Q and F, and, actually, there exists no general method to compute them. In order to exploit the Floquet structure of $u(t)$, the usual procedure is to perform a Fourier expansion of the fundamental solution, leading with an infinite system of linear differential equations with constant coefficients. Then, resolution of a truncated system furnishes an approximate solution.

It is well known that classical orthogonal polynomials satisfy second order linear homogeneous differential equations of the form:

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$$a(x)y''(x) + b(x)y'(x) + \lambda_n y(x) = 0,$$

where $a(x)$ and $b(x)$ are polynomials of degrees 2 and 1, respectively, which are independent of n, and λ_n is independent of x. They have many properties in common. One is that they satisfy a differential-difference relation of the form:

$$\pi(x)p_n'(x) = (\alpha_n x + \beta_n)p_n(x) + \gamma_n p_{n-1}(x).$$

For example, Hermite polynomials satisfy:

$$H_n'(x) = 2nH_{n-1}(x),$$

and Chebyschev polynomials satisfy:

$$T_n(x) = \frac{1}{2n}\left(T_{n+1}'(x) - T_{n-1}'(x)\right), \qquad x \in [-1,1].$$

Surprisingly, the last recursive formula have recently come up in a very attractive problem in Magnetic Resonance Imaging; see [2] for this work. In that paper, the Bloch equation, which has no closed-form solution, is expanded in a Chebyshev series, which can be solved using a sparse linear algebraic system.

The goal of the present paper is to construct a functional analytic framework for the method developed in [2], so that the infinite system given by the series expansion becomes a well-posed problem in an appropriate weighted sequence space. To the knowledge of the author, this study has not been addressed in the literature in this way. One of the main difficulties that appear, is that the operators that naturally arise here are, in general, unbounded in the l^2-norm, and, hence, it was required to define them in appropriate weighted l^1-spaces. As an application of our framework, we are able to find a theoretical spectral condition - based only on the data - under which, not only is the representation of the solution for the Bloch equation, in terms of a Chebyshev expansion of u, always possible, but also for the general Eq. (1).

This paper is organized as follows: Section 2 is devoted to recalling Chebyshev polynomials, their main properties, which we will use, and a result from the theory of (unbounded) operator matrices. In Section 3, the mathematical features of the method are presented and developed. Given $A \in C([-1,1],\mathbb{R}^{N\times N})$, our main result says the following: Defining the operator $\mathcal{B} = diag(B_0)$ with $B_0 : l_0^1(\mathbb{Z}_+) \to l_0^1(\mathbb{Z}_+)$ given by:

$$(B_0x)(n) = \begin{cases} \frac{1}{2}\sum\limits_{k=0}^{\infty} a(k)x(k), & n = 0; \\ \\ \frac{1}{2n}(x(n+1) - x(n-1)), & n \geq 1, \end{cases}$$

$a(0) := -1$, $a(1) := 1/2$ and $a(n) := \frac{2(-1)^n}{n^2-1}$, $n \geq 2$, and defining the operator:

$$\mathcal{A} = (A_{kl})_{k,l=1,\cdots,N} \subset \mathcal{L}(l^2(\mathbb{Z}_+), l_0^1(\mathbb{Z}_+)),$$

where:

$$A_{kl}x(n) = \sum_{j=0}^{\infty}\langle AT_j^k, T_n^l\rangle x(j), \qquad n \in \mathbb{Z}_+, \tag{2}$$

being a vector of N coordinates, which consists of Chebyshev polynomial $T_n(t)$ in the l-position and zeros everywhere else. We prove that if $\mathcal{I} - \mathcal{BA}$ is invertible, then the initial valued system (*) has a unique solution, $u \in C([-1,1],\mathbb{R}^N)$, such that $u' \in C([-1,1],\mathbb{R}^N)$ (Theorem 4). Moreover, the components of the solution are given by:

$$u_l(t) = \sum_{n=0}^{\infty} x_l(n)T_n^l(t), \qquad t \in [-1,1], \tag{3}$$

for each $l = 1,\cdots,N$, where:

$$x_l = \text{Projection}_l(I - \mathcal{BA})^{-1}\Psi_0. \tag{4}$$

In Section 4, as an application, we are able to solve the Bloch equation for the special case:

$$A(t) = \begin{pmatrix} 0 & \gamma g(t) & 0 \\ -\gamma g(t) & 0 & \omega(t) \\ 0 & -\omega(t) & 0 \end{pmatrix}, \tag{5}$$

of the matrix $A(t)$. In particular, we formalize and recover the results in [2] obtaining, in addition, a spectral condition to guarantee the existence and uniqueness of the solution for the corresponding problem (Theorem 5).

2. Preliminaries

The Chebyshev polynomials of the first kind can be defined by the trigonometric identity, $T_n(x) = \cos(n \arccos x)$, $x \in [-1,1]$, $n \in \mathbb{Z}_+$ (see [3,4] for the main properties). It is known that set $\{T_n; n \in \mathbb{Z}_+\}$ is orthogonal and complete in $L^2([-1,1], \mathbb{R}, d\mu)$, with respect to the measure, $d\mu(x) := (1-x^2)^{-1/2}$. More precisely, we have:

$$\int_{-1}^{1} T_n(x)T_m(x)d\mu(x) = \begin{cases} 0 & \text{if} \quad n \neq m \\ \pi & \text{if} \quad n = m = 0 \\ \frac{\pi}{2} & \text{if} \quad n = m \neq 0 \end{cases} \tag{6}$$

As an immediate consequence, a basis of the vector-valued Lebesgue space $L^2([-1,1],\mathbb{R}^N,d\mu)$ is defined in the standard way by the set of vectors:

$$\{T_k^l; k \in \mathbb{Z}_+, l = 1,\cdots,N\}, \tag{7}$$

where, for each $x \in [-1,1]$, $T_k^l(x)$ is a vector of N coordinates, which consists of the Chebyshev polynomial, $T_k(x)$, in the l-position and zeros everywhere else.

Given $f,\ g\ \in\ L^2([-1,1],\mathbb{R}^N,d\mu)$, recall that a canonical internal product on the vector-valued Lebesgue space $L^2([-1,1],\mathbb{R}^N,d\mu)$ is defined by:

$$\langle f,g\rangle_{L^2(\mathbb{R}^N)} = \int_{-1}^{1}\langle f(x),g(x)\rangle_{\mathbb{R}^N}\,d\mu(x)$$

$$= \int_{-1}^{1}\sum_{k=1}^{N} f_k(x)g_k(x)d\mu(x),$$

where f_k and g_k are the kth component of f and g, respectively.

Recall that $C^1([-1,1],\mathbb{R}^N)$ denotes the space of all continuously differentiable functions. Clearly:

$$C^1([-1,1],\mathbb{R}^N)\subset C([-1,1],\mathbb{R}^N)$$

$$\subset L^2([-1,1],\mathbb{R}^N,d\mu).$$

We will also use the following result due to Nagel [5].

Lemma 1. *Let E and F be Banach spaces and $A\in\mathscr{L}(E)$, $D\in\mathscr{L}(F)$, $B\in\mathscr{L}(F,E)$ and $C\in\mathscr{L}(E,F)$. Consider $\mathscr{A}\in\mathscr{L}(E\times F)$, where:*

$$\mathscr{A}=\begin{pmatrix} A & B\\ C & D\end{pmatrix}.$$

If A and D are invertible then the following assertions are equivalent:

(a) *$\mathscr{A}$ is invertible in $\mathscr{L}(E\times F)$;*

(b) *$A-BD^{-1}C$, hence, $I-BD^{-1}CA^{-1}$ is invertible in $\mathscr{L}(E)$;*

(c) *$D-CA^{-1}B$, hence, $I-CA^{-1}BD^{-1}$ is invertible in $\mathscr{L}(F)$.*

3. Some operators acting on weighted sequence spaces

Define the following space of weighted summable series:

$$l_0^1(\mathbb{Z}_+) := \left\{ x=(x(n))\subset\mathbb{R}/\sum_{n=1}^{\infty}\frac{1}{n^2}|x(n)|<\infty\right\}. \tag{8}$$

Observe that $l_0^1(\mathbb{Z}_+)$ is a Banach space endowed with the norm:

$$||x||_0 := \sum_{n=1}^{\infty}\frac{1}{n^2}|x(n)|.$$

In particular, note that the Banach space of all null sequences, $c_0(\mathbb{Z}_+)$, and the Hilbert space, $l^2(\mathbb{Z}_+)$, are contained in $l_0^1(\mathbb{Z}_+)$. The following lemma will be very important for our purposes.

Lemma 2. *Let $f\in C^1([-1,1],\mathbb{R}^N)$ be given. For all $k\in\mathbb{N}$ and $l=1,\cdots,N$ we have:*

$$2k\langle f,T_k^l\rangle_{L^2(\mathbb{R}^N)}=\langle f',T_{k-1}^l-T_{k+1}^l\rangle_{L^2(\mathbb{R}^N)}. \tag{9}$$

Proof. Using integration by parts, we obtain:

$$\langle f',T_{k-1}^l-T_{k+1}^l\rangle_{L^2(\mathbb{R}^N)}$$

$$=\int_{-1}^{1}\sum_{j=1}^{N} f_j'(x)[(T_{k-1}^l)_j(x)-(T_{k+1}^l)_j(x)]d\mu(x)$$

$$=\int_{-1}^{1} f_l'(x)[T_{k-1}(x)-T_{k+1}(x)]d\mu(x)$$

$$=\int_{0}^{\pi} f_l'(\cos(\theta))[\cos((k-1)\theta)-\cos((k+1)\theta)]d\theta$$

$$=\int_{0}^{\pi} f_l'(\cos(\theta))2\sin(k\theta)\sin(\theta)d\theta$$

$$=2\Bigg[-\sin(k\theta)f_j\ (\cos(\theta))|_0^{\pi}$$

$$+\,k\int_0^{\pi} f_l(\cos(\theta))\cos(k\theta)d\theta\Bigg]$$

$$=2k\int_0^{\pi} f_l(\cos(\theta))\cos(k\theta)d\theta$$

$$=2k\int_{-1}^{1} f_l(x)\cos(k\arccos(x))d\mu(x)$$

$$=2k\int_{-1}^{1} f_l(x)T_k(x)d\mu(x)$$

$$=2k\sum_{j=0}^{n}\int_{-1}^{1} f_j(x)(T_k^l)_j(x)d\mu(x)$$

$$=2k\langle f,T_k^l\rangle_{L^2(\mathbb{R}^N)}. \qquad\square$$

Given $y\in C^1([-1,1],\mathbb{R})$, we denote $y_1(n):=\langle y,T_n\rangle$ and $y_1'(n):=\langle y',T_n\rangle$. With the above notation, Lemma 2, in case $N=1$, reads as follows:

$$y_1(k)=\frac{1}{2k}[y_1'(k-1)-y_1'(k+1)], \tag{10}$$

for all $k\in\mathbb{N}$. We note the very remarkable fact that the value of $y_1(0)$ is unknown in the above recurrence formula. However, we are able to recover this value as data. More precisely, we obtain the following result, which is the key for the subsequent development of this paper.

Lemma 3. *Let $y \in C^1([-1,1],\mathbb{R})$ be given and denote $y(-1)=y_0$. Then:*

$$y_1(0) = y_0 + \frac{1}{2}\sum_{n=0}^{\infty} a(n) y_1'(n), \tag{11}$$

where $a(0) := -1$, $a(1) := 1/2$ and $a(n) := \frac{2(-1)^n}{n^2-1}$, $n \geq 2$.

Proof. Since $y \in C([-1,1],\mathbb{R})$, we have the representation:

$$y(x) = \sum_{n=0}^{\infty} \langle y, T_n\rangle T_n(x) = \sum_{n=0}^{\infty} y_1(n) T_n(x),$$

for all $x \in [-1,1]$. Hence, for all $x \in [-1,1]$, we have:

$$y(x) = y_1(0)T_0(x) + \sum_{n=1}^{\infty} y_1(n)T_n(x).$$

In particular:

$$y_0 = y(-1) = y_1(0) + \sum_{n=1}^{\infty} y_1(n) T_n(-1), \tag{12}$$

since $T_0(-1) = 1$. On the other hand, multiplying Eq. (10) by $T_n(-1)$ we obtain:

$$\begin{aligned}\sum_{n=1}^{\infty} T_n(-1)y_1(n) &= \sum_{n=1}^{\infty}\frac{1}{2n}T_n(-1)y_1'(n-1)\\ &- \sum_{n=1}^{\infty}\frac{1}{2n}T_n(-1)y_1'(n+1)\\ &= \sum_{n=0}^{\infty}\frac{1}{2(n+1)}T_{n+1}(-1)y_1'(n)\\ &- \sum_{n=2}^{\infty}\frac{1}{2(n-1)}T_{n-1}(-1)y_1'(n)\\ &= \sum_{n=2}^{\infty}\left[\frac{1}{2(n+1)}T_{n+1}(-1)\right.\\ &\left. - \frac{1}{2(n-1)}T_{n-1}(-1)\right] y_1'(n)\\ &+ \frac{1}{2}T_1(-1)y_1'(0) + \frac{1}{4}T_2(-1)y_1'(1).\end{aligned}$$

Finally, since $T_n(-1) = (-1)^n$, we get from Eq. (12):

$$\begin{aligned}y_0 = &y(-1) = y_1(0) - \frac{1}{2}y_1'(0) + \frac{1}{4}y_1'(1)\\ &+ \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}y_1'(n),\end{aligned}$$

and, hence, the conclusion follows. □

We next proceed to construct a bounded operator on the weighted space, $l_0^1(\mathbb{Z}_+)$, which has the property of transforming symbols y_l' into symbols y_l plus a rest. Note that it resembles a similar property of Laplace transforms, with respect to the first derivative of a function. More precisely, we define:

$$(B_0x)(n) = \begin{cases} \frac{1}{2}\sum_{k=0}^{\infty} a(k)x(k), & n=0;\\ \frac{1}{2n}(x(n-1)-x(n+1)), & n \geq 1\end{cases}$$

where $a(k)$ is defined in Lemma 3, and denote:

$$\delta_0(n) := \begin{cases} 1, & n=0;\\ 0, & n\neq 0.\end{cases}$$

Moreover, for each $y \in L^2([-1,1],\mathbb{R}^N, d\mu)$, we write:

$$y_l(n) := \langle y, T_n^l\rangle, \qquad n \in \mathbb{Z}_+, \qquad l \in \{2,3,\cdots N\}.$$

Theorem 1. *Let $y \in C^1([-1,1],\mathbb{R}^N)$. Then, $B_0 \in \mathcal{B}(l_0^1(\mathbb{Z}_+))$ and for all $l \in \{1,\cdots,N\}$, we have:*

$$B_0 y_l' = y_l + y_{0l}\delta_0, \tag{13}$$

where $y_{0l} = y_l(-1)$.

Proof. First note that:

$$\begin{aligned}||T_n^l||_{L^2}^2 &= \int_{-1}^{1} ||T_n^l(x)||_{\mathbb{R}^N}^2 d\mu(x)\\ &= \int_{-1}^{1} |T_n(x)|^2 d\mu(x) = \pi/2,\end{aligned}$$

for all $n \in \mathbb{N}$ and $||T_0^l||_{L^2}^2 = \pi$. Hence, by definition and the Cauchy Schwarz inequality, we obtain for each $y \in L^2([-1,1],\mathbb{R}^N, d\mu)$ that:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2}|y_l(n)| &= \sum_{n=1}^{\infty}\frac{1}{n^2}|\langle y, T_n^l\rangle|\\ &\leq \sum_{n=1}^{\infty}\frac{1}{n^2}||y||_{L^2}||T_n^l||_{L^2}\\ &= ||y||_{L^2}\sqrt{\pi/2}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.\end{aligned} \tag{14}$$

It proves that $y_l \in l_0^1(\mathbb{Z}_+)$ and, hence, that operator B_0 is well defined on the weighted space, $l_0^1(\mathbb{Z}_+)$. Let $x \in l_0^1(\mathbb{Z}_+)$ be given. Since the sequence $(n^2 a(n))$ is bounded, we obtain:

$$\frac{1}{2}\sum_{j=1}^{\infty}|a(j)x(j)| = \frac{1}{2}\sum_{j=1}^{\infty}|j^2 a(j)||\frac{1}{j^2}x(j)| \leq 2||x||_0,$$

and:

$$||B_0x||_0 = \sum_{n=1}^{\infty} \frac{1}{2n^3}|x(n-1) - x(n+1)|$$

$$\leq \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}|x(k)| = \frac{1}{2}||x||_0,$$

proving that B_0 is bounded. It remains to prove the identity (Eq. (13)). Let $n \geq 1$. Then, by Lemma 2:

$$B_0y_l'(n) = \frac{1}{2n}(y_l'(n-1) - y_l'(n+1))$$

$$= \frac{1}{2n}(\langle y', T_{n-1}^l\rangle - \langle y', T_{n+1}^l\rangle)$$

$$= \frac{1}{2n}\langle y', T_{n-1}^l - T_{n+1}^l\rangle$$

$$= \langle y, T_n^l\rangle = y_l(n).$$

On the other hand, in case $n = 0$, the following identity is obtained by Lemma 3:

$$B_0y_l'(0) = \sum_{k=0}^{\infty} a(k)y_l'(k) = y_l(0) + y_{0l},$$

proving the Theorem. □

Let $A \in C([-1,1], \mathbb{R}^{N\times N})$ and k, $l \in \{1, \cdots, N\}$ be given. Define A_{kl} by:

$$A_{kl}x(n) = \sum_{j=0}^{\infty}\langle AT_j^k, T_n^l\rangle x(j), \qquad n \in \mathbb{Z}_+. \tag{15}$$

Note that, in general, the operators A_{kl} are not bounded on $l_0^1(\mathbb{Z}_+)$. However, we have the following result.

Theorem 2. Let $A \in C([-1,1], \mathbb{R}^{N\times N})$. Then, $A_{kl} \in \mathcal{B}(l^2(\mathbb{Z}_+); l_0^1(\mathbb{Z}_+))$ for each $k, l \in \{1, \cdots, N\}$.

Proof. Since $A \in C([-1,1], \mathbb{R}^{N\times N})$, observe that for each $k \in \{1, \cdots, N\}$ fixed, we have:

$$\sup_{x\in[-1,1]}\sum_{m=1}^{N}|A^{mk}(x)|^2 := C_k < \infty.$$

Then:

$$||AT_n^k||_{L^2}^2 = \int_{-1}^{1}||(AT_n^k)(x)||_{\mathbb{R}^N}^2 d\mu(x)$$

$$= \int_{-1}^{1}||A(x)T_n^k(x)||_{\mathbb{R}^N}^2 d\mu(x)$$

$$= \int_{-1}^{1}\sum_{m=1}^{N}|A^{mk}(x)T_n(x)|^2 d\mu(x) \leq 2C_k. \tag{16}$$

Hence, for $x \in l^2(\mathbb{Z}_+)$ we have:

$$\sum_{j=0}^{\infty}|\langle AT_j^k, T_n^l\rangle||x(j)|$$

$$\leq \left(\sum_{j=0}^{\infty}|\langle AT_j^k, T_n^l\rangle|^2\right)^{1/2}\left(\sum_{j=0}^{\infty}|x(j)|^2\right)^{1/2}$$

$$= ||AT_n^k||_{L^2}||x||_{l^2} \leq M||x||_{l^2},$$

where we made use of Cauchy-Schwarz inequality and Parseval's identity. Therefore, there exists a constant, $C > 0$, such that:

$$||A_{kl}x||_0 = \sum_{n=1}^{\infty}\frac{1}{n^2}|A_{kl}x(n)|$$

$$\leq \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n^2}|\langle AT_j^k, T_n^l\rangle||x(j)|$$

$$\leq C||x||_{l^2}. \qquad \square \tag{17}$$

4. Application to nonautonomous systems

Let $A \in C([-1,1], \mathbb{R}^{N\times N})$ be given. Suppose that the system $y'(t) = A(t)y(t)$ has a solution, $y \in C^1([-1,1], \mathbb{R}^N)$, which satisfies the initial condition, $y(-1) = y_0$. Then, the system can be written as:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_N'(t) \end{pmatrix} = \begin{pmatrix} A^{11}(t) & A^{12}(t) & \cdots & A^{1N}(t) \\ A^{21}(t) & A^{22}(t) & \cdots & A^{2N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ A^{N1}(t) & A^{N2}(t) & \cdots & A^{NN}(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{pmatrix}, \tag{18}$$

where $A^{lm} \in C([-1,1], \mathbb{R})$ are the entries of the matrix $A(t)$.

Expanding in series y and y' in terms of the basis given by the Chebyshev polynomials T_j^l, we obtain that the system $y'(t) = A(t)y(t)$ can be rewritten as the following identity in the Hilbert space $L^2([-1,1], \mathbb{R}^N, d\mu)$:

$$\sum_{l=1}^{N}\sum_{j=0}^{\infty}\langle y', T_j^l\rangle T_j^l = \sum_{l=1}^{N}\sum_{j=0}^{\infty}\langle y, T_j^l\rangle AT_j^l$$

$$= \sum_{l=1}^{N}\sum_{j=0}^{\infty}\langle y, T_j^l\rangle\sum_{m=1}^{N}\sum_{i=0}^{\infty}\langle AT_j^l, T_i^m\rangle T_i^m$$

$$= \sum_{l=1}^{N}\sum_{j=0}^{\infty}\sum_{m=1}^{N}\sum_{i=0}^{\infty}\langle y, T_j^l\rangle\langle AT_j^l, T_i^m\rangle T_i^m$$

$$= \sum_{m=1}^{N}\sum_{i=0}^{\infty}\left(\sum_{l=1}^{N}\sum_{j=0}^{\infty}\langle AT_j^l, T_i^m\rangle\langle y, T_j^l\rangle\right) T_i^m,$$

where $(AT_j^l)(t) := A(t)T_j^l(t)$ for each $j \in \mathbb{Z}_+$ and $l = 1, \cdots, N$. From the orthogonality of T_j^l, we obtain that the above identity is equivalent to the system:

$$\langle y', T_k^m\rangle = \sum_{l=1}^{N}\sum_{j=0}^{\infty}\langle AT_j^l, T_k^m\rangle\langle y, T_j^l\rangle, \tag{19}$$

where $k \in \mathbb{Z}_+$ and $m = 1, \ldots, N$.

Theorem 3. *Let $A \in C([-1,1], \mathbb{R}^{N\times N})$ and $y \in C^1([-1,1], \mathbb{R}^N)$ be given. The system (Eq. (19)) is equivalent to the system:*

$$y_m' = \sum_{l=1}^{N} A_{lm} y_l, \tag{20}$$

where $m = 1, \ldots, N$.

Proof. For $k \geq 1$ and $m \in \{1, \cdots, N\}$, we have:

$$y_m'(k) = \langle y', T_k^m\rangle,$$

and:

$$\sum_{l=1}^{N} A_{lm} y_l(k) = \sum_{l=1}^{N}\sum_{j=0}^{\infty}\langle AT_j^l, T_k^m\rangle y_l(j)$$

$$= \sum_{l=1}^{N}\sum_{j=0}^{\infty}\langle AT_j^l, T_k^m\rangle\langle y, T_j^l\rangle.$$

This way, the problem of solving Eq. (18) is equivalent to solving the following.

Problem. From Eq. (20), determine y_l for all $l = 1, \cdots, N$.

Applying the operator B_0 to both sides of the Eq. (20), we obtain the identity:

$$y_m = \sum_{l=1}^{N} B_0 A_{lm} y_l - y_{0m}\delta_0. \tag{21}$$

Define the operator matrix, $\mathcal{A} : l^2(\mathbb{Z}_+)^N \to l_0^1(\mathbb{Z}_+)^N$, as follows:

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{N1} \\ A_{12} & A_{22} & \cdots & A_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1N} & A_{2N} & \ldots & A_{NN} \end{pmatrix}. \tag{22}$$

Define now the operator matrix $\mathcal{B} : l_0^1(\mathbb{Z}_+)^N \to l_0^1(\mathbb{Z}_+)^N$, as follows:

$$\mathcal{B} = \begin{pmatrix} B_0 & 0 & \ldots & \ldots & 0 \\ 0 & B_0 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & B_0 & 0 \\ 0 & 0 & \ldots & 0 & B_0 \end{pmatrix}. \tag{23}$$

Consequently, Eq. (21) can be rewritten as a problem of the form:

$$\Psi = \mathcal{B}\mathcal{A}\Psi + \Psi_0, \tag{24}$$

where $\Psi := (y_1, \cdots y_N) \in l_p^1(\mathbb{Z}_+)^N$ are vectors with entries, y_l, defined by $y_l(j) := \langle y, T_j^l\rangle$ for all $j \in \mathbb{Z}_+$ and $\Psi_0 := (y_{01}, y_{02}, \cdots, y_{0N})$.

In what follows, we denote by $\rho(S)$ the resolvent set of the operator, S. We arrive at the following theorem, which is the main result of this section.

Theorem 4. *Suppose that $A \in C([-1,1], \mathbb{R}^{N\times N})$ and $1 \in \rho(\mathcal{B}\mathcal{A})$. Then, the system*

$$y'(t) = A(t)y(t),$$

with initial condition $y(-1) = y_0$ has a unique solution, $y \in C([-1,1], \mathbb{R}^N)$, such that $y' \in C([-1,1], \mathbb{R}^N)$.

Proof. By hypothesis and Eq. (24), there exists $x_l \in l_0^1(\mathbb{Z}_+)$, $(l = 1, \cdots, N)$ such that:

$$\Psi = (I - \mathcal{B}\mathcal{A})^{-1}\Psi_0, \tag{25}$$

where $\Psi := (x_1, \cdots x_N)$. Define:

$$y_l(t) = \sum_{n=0}^{\infty} x_l(n) T_n^l(t), \qquad t \in [-1, 1].$$

$l = 1, \cdots, N$. Now, let $y := (y_1, \cdots, y_N)$, and the conclusion follows by construction. □

5. Application to the Bloch equation

In matrix form, the Bloch equation reads [1]:

$$M'(t) = A(t)M(t) + b, \tag{26}$$

where:

$$A(t) = \begin{pmatrix} -1/T_1 & \gamma B & -\gamma B_y(t) \\ -\gamma B & -1/T_2 & \gamma B_x(t) \\ \gamma B_y(t) & -\gamma B_x(t) & -1/T_1 \end{pmatrix},$$

and:

$$b = \begin{pmatrix} 0 \\ 0 \\ \eta_0/T_1 \end{pmatrix}.$$

We observe that matrix $A(t)$ is not suited for resolution of the differential system by a numerical method. Indeed, the coefficients γB are of magnitude 10^6, whereas the others are of magnitude 1.

Under perturbation, the matrix $A(t)$ is not constant. However, the matrix $A(t)$ is continuous and has periodic coefficients (see [2]).

Define $A \in C([-1,1], \mathbb{R}^{3\times 3}, d\mu)$, as follows:

$$A(t) = \begin{pmatrix} 0 & \gamma g(t) & 0 \\ -\gamma g(t) & 0 & \omega(t) \\ 0 & -\omega(t) & 0 \end{pmatrix}, \tag{27}$$

where we assume that $g(t)$ and $\omega(t)$ are continuous functions in $[-1,1]$.

The Bloch Eq. (26), with $A(t)$, as in Eq. (27), was studied in [2] in the context of Magnetic Resonance Imaging. We find, after a calculation using the given definitions in the previous section, that $A_{11} = A_{13} = A_{22} = A_{31} = A_{33} = 0$ and:

$$A_{12}x(n) = A_{21}x(n)$$

$$= -\gamma \sum_{j=0}^{\infty} \left[\int_{-1}^{1} g(s)T_j(s)T_n(s)d\mu(s) \right] x(j),$$

as well as:

$$A_{23}x(n) = A_{32}x(n)$$

$$= \sum_{j=0}^{\infty} \left[\int_{-1}^{1} \omega(s)T_j(s)T_n(s)d\mu(s) \right] x(j).$$

Define G, $\Omega : l^2(\mathbb{Z}_+) \to l_0^1(\mathbb{Z}_+)$ by:

$$Gx(n) = \sum_{j=0}^{\infty} \left[\int_{-1}^{1} g(s)T_j(s)T_n(s)d\mu(s) \right] x(j),$$

and:

$$\Omega x(n) = \sum_{j=0}^{\infty} \left[\int_{-1}^{1} \omega(s)T_j(s)T_n(s)d\mu(s) \right] x(j),$$

and, hence:

$$\mathcal{A} = \begin{pmatrix} 0 & -\gamma G & 0 \\ -\gamma G & 0 & \Omega \\ 0 & \Omega & 0 \end{pmatrix}.$$

Therefore, we obtain the following matrix of bounded operators:

$$I - \mathcal{B}\mathcal{A} = \begin{pmatrix} I & \gamma B_0 G & 0 \\ \gamma B_0 G & I & -B_0\Omega \\ 0 & -B_0\Omega & I \end{pmatrix}. \tag{28}$$

In order to study the invertibility of the operator-valued matrix (Eq. (29)), we use Lemma 1, obtaining the following result.

Theorem 5. *Suppose that G and Ω are bounded operators in $l_0^1(\mathbb{Z}_+)$. If $1 \in \rho((B_0\Omega)^2)$ and $1/\gamma^2 \in \rho((B_0G)^2)$, then the Bloch Eq. (26), with $A(t)$, given by Eq. (27) has a unique solution.*

Proof. We use Lemma 1, Statement (b), with $E := l_0^1(\mathbb{Z}_+) \times l_0^1(\mathbb{Z}_+)$ and $F := l_0^1(\mathbb{Z}_+)$. Taking:

$$A := \begin{pmatrix} I & \gamma B_0 G \\ \gamma B_0 G & I \end{pmatrix}, \qquad B := \begin{pmatrix} 0 \\ B_0\Omega \end{pmatrix},$$

$$C := \begin{pmatrix} 0 & -B_0\Omega \end{pmatrix}, \qquad D := \begin{pmatrix} I \end{pmatrix},$$

we obtain that the operator-valued matrix (Eq. (29)) is invertible if, and only if, the operator-valued matrix:

$$\begin{pmatrix} I & 0 \\ \gamma(B_0\Omega)^2(B_0G) & I - (B_0\Omega)^2 \end{pmatrix} (I - \gamma^2(B_0G)^2)^{-1},$$

is invertible, and hence the result follows by hypothesis and Theorem 4.

We observe that in view of the definitions, we have an explicit expression for operators B_0G and $B_0\Omega$. For example:

$$B_0(Gx)(n) = \frac{1}{2n} \sum_{j=0}^{\infty} \Big[\int_{-1}^{1} g(s)T_j(s)[T_{n+1}(s)$$

$$- T_{n-1}(s)]d\mu(s) \Big] x(j), \quad n \geq 1.$$

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Biographies

Carlos Lizama is currently Titular Professor at the University of Santiago of Chile and Vice-Dean of the Faculty of Sciences. He has more than 21 years in academia during which time, he has done research, extension and teaching at all levels. At present, he has more than 90 research publications in international journals with ISI ranking. From 1993 to date he has worked on various competitive projects with external financing such as Fondecyt and Rings projects of Science and Technology (PIA-CONICYT).