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# Analytical solution for large amplitude vibrations of microbeams actuated by an electro-static force

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**Abstract.** An analytical study using Variational Iteration Method (VIM) is carried out in order to investigate the vibrations of electro-statically actuated double-clamped and simply-supported microbeams. Effects of applied voltage and residual axial load on the nonlinear natural frequency and deflection of the microbeams are studied. It shows that pre-compression in microbeams increases the amplitude of deflections for a specific applied voltage. Also, an increase in pre-tension motivates the microbeam to show more nonlinear behavior in an applied voltage. Predicted results are compared with the experimental data available in the literature and also with numerical results which shows a good agreement. It is concluded that the second order approximation of the VIM leads to highly accurate solutions which are valid for a wide range of vibration amplitudes.

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## 1. Introduction

Electrostatically actuated microbeams are often encountered in high precision applications and microelectromechanical systems (MEMS) such as signal filtering, resonant sensors and mass sensing [1]. An electrostatically actuated microbeam comprises a beam-shaped element and a fixed rigid plate electrode. When the applied voltage to the microbeam exceeds a critical value, which is called pull-in voltage, the resulting electrostatic force may lead to the instability and vibration of the system [2,3].

It should be noted that the governing equations of vibration of these systems are essentially nonlinear. Generally, in a given nonlinear problem, it is often difficult to find an analytical solution, unless a number of different simplifying assumptions is considered. Otherwise, application of different numerical techniques is unavoidable. However, it is hard to have a complete and indispensable understanding of a nonlinear problem out of these numerical results. In addition, numerical difficulties appear if a nonlinear problem has singularities or multiple solutions.

Vibrations and natural responses of MEMS have been the subject of several analytical studies in the past. Azrar et al. [4] developed a semi-analytical approach to the nonlinear dynamic response problem, based on Lagrange's principle and the harmonic balance method. An analytical method for determining the vibration modes of geometrically nonlinear beams under various edge conditions was presented by Qaisi [5]. Gou and Zhang [6] investigated nonlinear vibrations of thin beams, based on sextic cardinal spline functions and a spline-based Differential Quadrature Method (DQM). Also, Kung and Chen [7] used the DQM to calculate the natural frequencies of a fixed-fixed shaped beam. They considered the effects of the mid-plane stretching, axial residual stress and electrical field fringing.

Moreover, pull-in instability has been a subject of study in analysis of electrostatic beams. Zhang and Zhao [8] studied the pull-in instability of microstructures under electrostatic loadings. They used

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the Taylor series to expand the electrostatic loading term in the one-mode analysis method to derive the analytical solution. Zhang et al. [9] presented an analytical method for the snap-through and the pull-in instabilities of an arch-shaped beam under electrostatic loading. Because of the Taylor series expansion error, they introduced a compensated form of the Taylor series expansion on the electrostatic loading term, and modified the solution.

Nonlinear modal analysis approach based on invariant manifold method was utilized to obtain the nonlinear normal modes of a clamped-clamped beam for large amplitude displacements by Xie et al. [10]. Nayfeh and Nayfeh [11] obtained the nonlinear modes and natural frequencies of a simply-supported Euler-Bernoulli beam resting on an elastic foundation with distributed quadratic and cubic nonlinearities, using the method of multiple scales and the invariant manifold approach. Also, Nayfeh and Younis [12] and Younis and Nayfeh [13] used perturbation technique to solve the nonlinear problem of MEMS. Pirbodaghi et al. [14] used the first-order approximation of the homotopy analysis method to investigate the nonlinear free vibration analysis of Euler-Bernoulli beam. Moghimi Zand et al. [15] have used homotopy analysis method to find semi-analytic solutions to the vibrations of double-clamped microbeams subjected to suddenly applied step voltages. They have considered the effect of mid-plane stretching and residual stress in their analysis.

Among several analytical methods, Variational Iteration Method (VIM) is one of the most accurate and efficient methods of studying vibrations of nonlinear systems [16]. In [17], this method was employed to analyze the large amplitude free vibration and post-buckling of unsymmetrically laminated composite beams on elastic foundation, and the accuracy of the method was investigated. Recently, Baghani [18] employed a modified version of VIM in solving a nonlinear boundary value problem. He analytically studied the size-dependent static pull-in voltage of microcantilevers, using the modified couple stress theory. He also showed that the results predicted by VIM are in excellent correspondence with the numerical results, as well as the experimental data. Baghani et al. [19] also employed this method to investigate the nonlinear response for free vibration of a conservative oscillator.

The main objective of this paper is to use the VIM to study the vibrations of electrostatically actuated double-clamped and simply supported microbeams. Analytical expressions for the nonlinear fundamental natural frequency and deflection of these microbeams are obtained using the VIM. The effects of the applied voltage and residual axial load on the nonlinear natural frequency and deflection of the microbeams are studied. Results are compared with those available from the literature and also with the results of numerical study, to show the performance of the proposed analytical solution.

The paper is organized as follows: In Section 2, the mathematical problem formulation is developed. Section 3 deals with a brief introduction to the VIM. Results of the mathematical solution for both double-clamped and simply supported microbeams are reported in Section 4. The obtained results are compared with the numerical results, as well as the experimental data available in the literature. We finally present a summary and draw conclusions in Section 5.

## 2. Problem formulation

Schematics of double-clamped and simply supported microbeams are shown in Figure 1(a) and (b), respectively. Employing von-Karman nonlinearity concept for mid-plane stretching, the deflection of both types of microbeams resulted from electrostatic force can be obtained using the following relation:

$$Y_{tt} + \frac{EI}{\rho bh} Y_{xxxx} - \frac{(N_i + N_s)}{\rho bh} Y_{xx} = \frac{b}{\rho bh} F_{es}, \qquad (1)$$

where $b$ and $h$ are width and thickness of the microbeam, respectively; $\rho$ is the density; $I$ is the moment of inertia of the cross-section about the $z$-axis; subscripts $x$ and $t$ stand for derivative with respect to the location and time, respectively; $E$ is the effective Young's modulus of the beam, and $F_{es}$ is the electrostatic force induced from the applied voltage $V$ between the microbeam and the substrate plate. $F_{es}$ is expressed as:

$$F_{es} = \frac{\varepsilon_0 V^2}{2(d_0 - Y(x,t))^2} \left(1 + \beta \frac{(d_0 - Y(x,t))}{b}\right), \qquad (2)$$

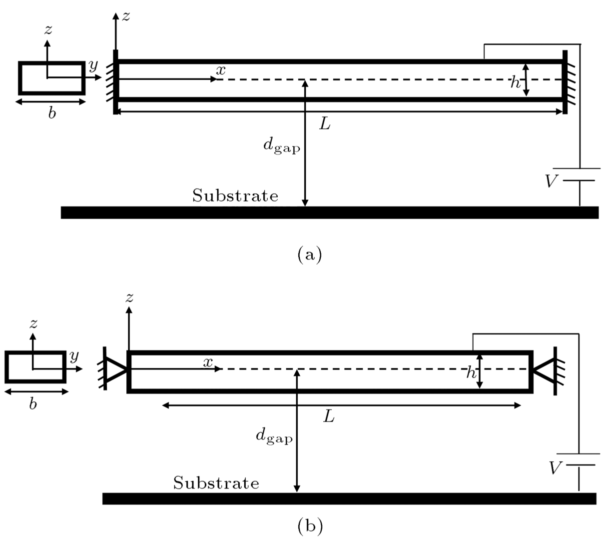


**Figure 1.** Schematic of the microbeams: (a) Double-clamped; and (b) simply supported.

where $\varepsilon_0$ is the vacuum permittivity; $d_0$ is the air initial gap. $\beta$ is a parameter describing the effect of fringing field due to the finite width of the beam, and depends on the type of the microbeam. Also, in Eq. (1), $N_i$ is the residual axial load and $N_s$ is the axial load resulted from mid-plane stretching, which is given by:

$$N_s = \frac{Ebh}{2L}\int_0^L Y_x^2 dx. \tag{3}$$

The boundary conditions for a double-clamped microbeam can be written as:

$$Y(0,t) = 0, \qquad Y_x(0,t) = 0,$$
$$Y(L,t) = 0, \qquad Y_x(L,t) = 0. \tag{4}$$

Also, for a simply supported beam, the boundary conditions are as follows:

$$Y(0,t) = 0, \qquad Y_{xx}(0,t) = 0,$$
$$Y(L,t) = 0, \qquad Y_{xx}(L,t) = 0. \tag{5}$$

And the initial conditions are:

$$Y(x,0) = 0, \qquad Y_t(x,0) = 0. \tag{6}$$

In order to solve the differential Eq. (1), subject to the boundary and initial conditions (4), (5) and (6), method of separation of variables is applied in which the deflection of the beam is expressed as:

$$Y(x,t) = \psi(x)w(t), \tag{7}$$

where $w(t)$ is a function of time and $\psi(x)$ is the first eigenmode, which satisfies the boundary conditions. For double-clamped and simply supported microbeams, $\psi(x)$ is given by Eqs. (8) and (9), respectively.

$$\psi(x) = \frac{-\sin\alpha(L-x) - \sin(\alpha x)\cosh(\alpha L)}{\cos(\alpha L) - \cosh(\alpha L)} + \frac{\cos(\alpha x)\sinh(\alpha L) - \sinh(\alpha x)\cos(\alpha L)}{\cos(\alpha L) - \cosh(\alpha L)} - \frac{\sinh\alpha(L-x) + \cosh(\alpha x)\sin(\alpha L)}{\cos(\alpha L) - \cosh(\alpha L)}, \tag{8}$$

$$\psi(x) = \sin(\alpha x), \qquad \alpha = \frac{\pi}{L}. \tag{9}$$

In Eq. (8), $\alpha$ is calculated based on linear vibration of the beam [8]. Substituting Eqs. (8) and (9) in Eq. (1) and using the Taylor expansion of $F_{es}$, we have [15]:

$$T(Y) = Y_{tt} + \frac{EI}{\rho bh}Y_{xxxx} - \frac{1}{\rho bh}\left(N_i + \frac{Ebh}{2L}\int_0^L Y_x^2 dx\right)Y_{xx} - \frac{\varepsilon_0 bV^2}{2\rho bh}\left(\frac{1}{d_0^2} + \frac{2Y}{d_0^3} + \frac{3Y^2}{d_0^4} + \frac{4Y^3}{d_0^5} + \frac{5Y^4}{d_0^6} + \cdots\right) - \frac{\varepsilon_0 \beta V^2}{2\rho bh}\left(\frac{1}{d_0} + \frac{Y}{d_0^2} + \frac{Y^2}{d_0^3} + \frac{Y^3}{d_0^4} + \frac{Y^4}{d_0^5} + \cdots\right) = 0. \tag{10}$$

Using the Galerkin method, the governing equation of motion is obtained as follows:

$$w_{tt} + \gamma w + Q + Rw^2 + Sw^3 + Gw^4 + Jw^5 + Pw^6 = 0. \tag{11}$$

Coefficients $\gamma, Q, R, S, G, J$ and $P$ are presented in Appendix A.

## 3. The method of solution

In this method, the studied problem is initially approximated with possible unknowns. Then, a corrected functional is constructed using a general Lagrange multiplier, which can be identified optimally via the variational theory [16]. To illustrate the basic idea of the method, we consider the following general nonlinear system:

$$\mathbb{L}[w(\tau)] + \mathbb{N}[w(\tau)] = g(\tau), \tag{12}$$

where $\mathbb{L}$ is a linear differential operator, $\mathbb{N}$ is a nonlinear analytic operator, and $g(\tau)$ is an inhomogeneous term.

The basic character of the method is to construct a correction functional for the system as follows:

$$w_{n+1}(t) = w_n(t) + \int_0^t \lambda(t)\left\{\mathbb{L}[w_n(\tau)] + \mathbb{N}\left[\tilde{w}_n(\tau)\right] - g(\tau)\right\} d\tau, \tag{13}$$

where $\lambda(t)$ is a general Lagrange multiplier which can be identified optimally via the variational theory. Also, $w_n$ is the $n$th approximate solution, and $\tilde{w}_n$ represents a restricted variation, i.e. $\delta\tilde{w}_n = 0$.

### *3.1. Implementation of the VIM in Vibrations of double-clamped and simply supported micro beams*

In view of Eq. (12), the governing Eq. (11) can be rewritten as:

$$\frac{d^2w(t)}{dt^2} + \Omega^2 w = -\mathbb{N}[w], \tag{14}$$

where:

$$\mathbb{L}[w]=\frac{d^2w(t)}{dt^2}+\Omega^2 w,$$

$$\begin{aligned}\mathbb{N}[w]=&\{\gamma w(t)+Q+Rw(t)^2+Sw(t)^3+Gw(t)^4\\&+Jw(t)^5+Pw(t)^6\}-\Omega^2 w,\qquad g(t)=0.\end{aligned}\tag{15}$$

The correction functional can be constructed in the form:

$$\begin{aligned}w_{n+1}(t)=&w_n(t)+\int_0^t\lambda(t)\bigg[\frac{d^2w_n(\tau)}{d\tau^2}\\&+\Omega^2 w_n(\tau)+\mathbb{N}[\tilde{w}_n(\tau)]\bigg]d\tau.\end{aligned}\tag{16}$$

In Eq. (16), $\tilde{w}_n$ is considered as a restricted variation, i.e. $\delta\tilde{w}_n=0$, hence $\delta\mathbb{N}[\tilde{w}_n(\tau)]=0$. Calculating the variation of Eq. (16) and noting that $\delta\mathbb{N}[\tilde{w}_n(\tau)]=0$, the following equations will be produced:

$$\begin{aligned}\delta w_{n+1}(t)=&\delta w_n(t)+\int_0^t\lambda(t)\bigg[\delta\frac{d^2w_n(\tau)}{d\tau^2}\\&+\Omega^2\delta w_n(\tau)+\delta[\tilde{w}_n(\tau)]\bigg]d\tau,\\
\delta w_{n+1}(t)=&\delta w_n(t)+\lambda\delta\frac{dw_n(\tau)}{d\tau}\bigg|_0^t-\frac{d\lambda(\tau)}{d\tau}\delta w_n(\tau)\bigg|_0^t\\&+\int_0^t\bigg[\delta w_n(\tau)\frac{d^2\lambda(\tau)}{d\tau^2}+\Omega^2\lambda(\tau)\delta w_n(\tau)\bigg]d\tau,\\
\delta w_{n+1}(t)=&\delta w_n(t)+\left(\lambda\delta\frac{dw_n(\tau)}{d\tau}\right)\bigg|_{\tau=t}\\&-\left(\frac{d\lambda(\tau)}{d\tau}\delta w_n(\tau)\right)\bigg|_{\tau=t}\\&+\int_0^t\bigg[\delta w_n(\tau)\frac{d^2\lambda(\tau)}{d\tau^2}+\Omega^2\lambda(\tau)\delta w_n(\tau)\bigg]d\tau.\end{aligned}\tag{17}$$

Separating the coefficients of $\delta w_n(\tau)|_{\tau=t}$ and $\delta\frac{dw_n(\tau)}{d\tau}\Big|_{\tau=t}$ and also $\delta w_n(\tau)|_{\tau=t}$ in the integral of Eq. (17) leads to the following stationary conditions:

$$\begin{cases}\frac{d^2\lambda(\tau)}{d\tau^2}+\Omega^2\lambda(\tau)=0\\ \lambda(\tau=t)=0\\ 1-\frac{d\lambda(\tau)}{d\tau}\Big|_{\tau=t}=0\end{cases}\tag{18}$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda(\tau)=\frac{1}{\omega}\sin\omega(\tau-t).\tag{19}$$

On the other hand, by taking into consideration the relation:

$$\begin{aligned}&\int_0^t\sin\omega(\tau-t)\left[\frac{d^2w_n(\tau)}{d\tau^2}+\omega^2 w_n(\tau)\right]d\tau\\&=-\omega w_n(t)+\omega w_n|_{t=0}\cos\omega t+\frac{dw_n}{dt}\bigg|_{t=o}\sin\omega t.\end{aligned}\tag{20}$$

Eq. (20) can be recast as:

$$\begin{aligned}w_{n+1}(t)=&w_n|_{t=0}\cos\omega t+\frac{dw_n}{dt}\bigg|_{t=o}\frac{\sin\omega t}{\omega}\\&+\frac{1}{\omega}\int_0^t\sin\omega(\tau-t)\{\mathbb{N}[w_n(\tau)]\}d\tau.\end{aligned}\tag{21}$$

Considering the initial conditions $w(0)=0$ and $\dot{w}(0)=0$, the correction functional is further reduced to:

$$w_{n+1}(t)=\frac{1}{\omega}\int_0^t\sin\omega(\tau-t)\{\mathbb{N}[w_n(\tau)]\}d\tau.\tag{22}$$

As an initial guess, $w_0(t)$ is assumed to be:

$$w_0(t)=\frac{Q}{\omega^2}(\cos(\omega t)-1).\tag{23}$$

Expanding $\mathbb{N}[w_0(t)]$, we arrive at:

$$\begin{aligned}-32\omega^{12}\mathbb{N}[w_0(\tau)]=&[-120SQ^3\omega^6-32Q\omega^{10}\gamma\\&-420JQ^5\omega^2+792PQ^6+64RQ^2\omega^8\\&+32Q\omega^{12}+224GQ^4\omega^4]\cos(\omega\tau)\\&+[240JQ^5\omega^2-16RQ^2\omega^8-495PQ^6\\&+48SQ^3\omega^6-112GQ^4\omega^4]\cos(2\omega\tau)\\&+[-90JQ^5\omega^2+32GQ^4\omega^4-8SQ^3\omega^6\\&+220PQ^6]\cos(3\omega\tau)+[-4GQ^4\omega^4\\&+20JQ^5\omega^2-66PQ^6]\cos(4\omega\tau)\\&+[12PQ^6-2JQ^5\omega^2]\cos(5\omega\tau)\\&+[-PQ^6]\cos(6\omega\tau)+[(252JQ^5\omega^2\\&-32Q\omega^{12}-462PQ^6-48RQ^2\omega^8\\&+80SQ^3\omega^6+32Q\omega^{10}\gamma-140GQ^4\omega^4].\end{aligned}\tag{24}$$

By taking into consideration the relation:

$$\frac{1}{\omega}\int_0^t \sin\omega(\tau-t)\{\cos(n\omega t)\}d\tau$$

$$=\begin{cases}\frac{\cos(n\omega t)-\cos(\omega t)}{\omega^2(n^2-1)} & n\neq 1\\ \frac{t\sin(\omega t)}{-2\omega} & n=1\end{cases} \tag{25}$$

to avoid secular terms in the next iterations, the coefficient of the $\cos(\omega\tau)$ in $\mathbb{N}[\omega_0(\tau)]$ must be vanished. The first approximation of the frequency is obtained as follows:

$$-120SQ^3\omega^6-32Q\omega^{10}\gamma-420JQ^5\omega^2+792PQ^6$$
$$+64RQ^2\omega^8+32Q\omega^{12}+224GQ^4\omega^4=0. \tag{26}$$

From Eqs. (23) and (25), for $n=1$, the first-order approximate solution is obtained as:

$$w_1(t)=c_1\cos(\omega t)+c_2\cos(2\omega t)+c_3\cos(3\omega t)$$
$$+c_4\cos(4\omega t)+c_5\cos(5\omega t)+c_6\cos(6\omega t)+c_7, \tag{27}$$

where coefficients $c_i$'s are given in Appendix B.

Substituting Eq. (27) in Eq. (22), an expression similar to Eq. (24) is obtained. Then, by setting the coefficient of $\cos(\omega\tau)$ to zero, a higher order approximation for the frequency is obtained, which is given in Appendix C. Also, the deflections of microbeams are calculated up to the second order approximation.

## 4. Results and discussion

Variational Iteration Method (VIM) is applied to study the nonlinear frequency and deflection of electrostatically actuated double-clamped and simply supported microbeams. Expressions for the frequency and deflection of the micro beams up to the second order approximation are obtained. Also, the deflection of the micro beams is calculated numerically, using 4th order Runge-Kutta method.

### 4.1. Double-clamped microbeam

The procedure explained in previous section is applied to study the vibrations of a double-clamped microbeam. In order to demonstrate the accuracy of the VIM, values of nonlinear frequency $\omega_{NL}$ of four different isotropic double-clamped microbeams with lengths of 210, 310, 410 and 510 $\mu$m have been obtained, and compared with previous works, which are summarized in Table 1.

The parameters of the microbeam are as follows:

Thickness = 1.5 $\mu$m,

Width = 100 $\mu$m,

Initial gap = 1.18 $\mu$m,

Residual axial load = 0.0009 N,

$\rho = 2332$ kg/m$^3$,

$\varepsilon_0 = 8.854\times 10^{-12}Fm^{-1}$

Effective Young's modulus = 166 GPa.

In Table 1, $\omega_0$ is the linear frequency calculated from the initial guess, and $\omega_1$ and $\omega_2$ are the values obtained from the first and the second iterations, respectively.

It can be observed that the results obtained from the VIM and those reported by [2,7,15] are in a good agreement. Table 1 also shows the advantage of using this method from the convergence view point. As shown, only a few iterations are required to calculate the results with appropriate accuracy.

Variations of the linear and nonlinear frequencies obtained from the first and second iterations with the applied voltage for different values of residual axial load are illustrated in Figure 2. These values are also compared with the results of [15] where a good agreement is observed. It is shown that for values of voltage lower than 10 V, the difference between the linear and nonlinear frequencies is almost negligible. For higher values of voltage, effect of nonlinearity is dominant and has caused the linear frequency to diverge from the nonlinear solution. Also, for voltages greater than a critical value, no real solution can be found for the problem. It should be noted that the

**Table 1.** Nonlinear frequencies at different beam lengths and comparison with the previous works results and experimental data.

| Length ($\mu$m) | Ref. [2] | | Ref. [7] | Ref. [15] | Present study | | |
|---|---|---|---|---|---|---|---|
| | Measured | Calculated | | | $\omega_0$ | $\omega_1$ | $\omega_2$ |
| 210 | 322.05 | 324.7 | 324.7 | 324.78 | 322.4036 | 322.3534 | 322.3534 |
| 310 | 163.22 | 164.35 | 163.46 | 163.16 | 161.9797 | 161.9549 | 161.9549 |
| 410 | 102.17 | 103.8 | 103.7 | 103.42 | 101.6387 | 101.5353 | 101.5354 |
| 510 | 73.79 | 74.8 | 73.46 | 74.38 | 73.2741 | 73.2200 | 73.2200 |

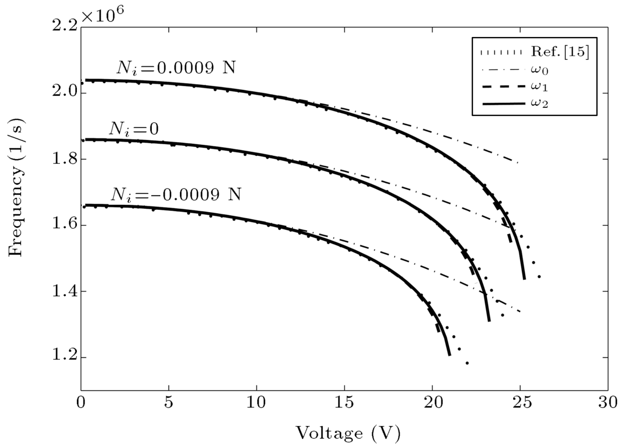


**Figure 2.** Variation of natural frequencies with voltage for a double-clamped microbeam with $L = 210$ $\mu$m and different values of residual axial load.

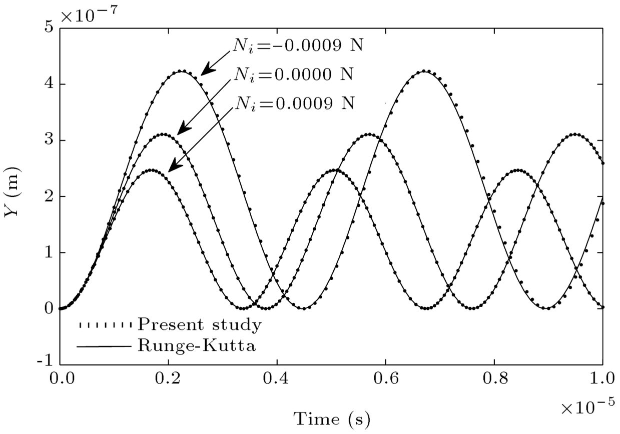


**Figure 3.** Effect of residual axial load on deflection of the double-clamped microbeam at $x = L/2$ with $L = 210$ $\mu$m at 19 V.

values of frequency obtained from the first and second iterations are very close in whole studied interval of voltage, which implies the high rate of convergence in the VIM. It is also observed that pretension in microbeams results in increase of nonlinear frequency.

Figure 3 shows the effect of residual axial load on deflection of the microbeam at $x = L/2$. Results of the VIM obtained at 19 V for a microbeam with $L = 210$ $\mu$m are compared with the numerical results of 4th order Runge-Kutta method. A good agreement is observed even at this relatively high voltage. Also, it can be seen that precompression increases the amplitude of deflections for a specific applied voltage.

### *4.2. Simply-supported microbeam*

The VIM is applied to analyze the vibration of a simply supported microbeam actuated by an electrostatic force. The difference between the solutions obtained for double-clamped and simply supported beams is due to the different boundary conditions, which results in different expressions for $\psi(x)$.

Figure 4 depicts the variation of linear and nonlinear frequencies of the simply supported microbeam with respect to the applied voltage. As observed in this figure, the values of frequencies fall to zero at higher voltages, called pull-in voltage. In such voltages, the instability occurs which should be considered as an important issue in design procedures [18].

Figure 5 shows the deflection of the microbeam at $x = L/2$ with length $L = 210$ $\mu$m for different axial loads at voltage 10. The slight difference between the results of current study and that of the numerical study is due to the applied relatively high voltage, which is close to the value of the critical voltage. We also investigated the effect of beam length ($L$) and voltage ($V$) on the nonlinear frequency in Figure 6 where the residual axial load is zero. As shown, at smaller beam lengths, the value of frequency increases. This increase is much more significant at beam lengths smaller than $L = 100$ $\mu$m.

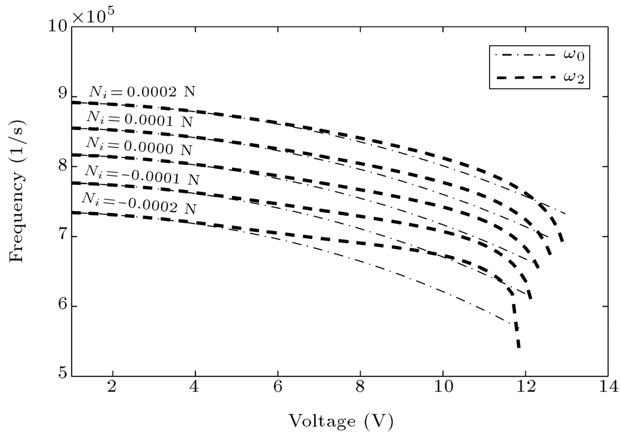


**Figure 4.** Variation of natural frequency with voltage for a simply supported microbeam with $L = 210$ $\mu$m and different values of residual axial load.

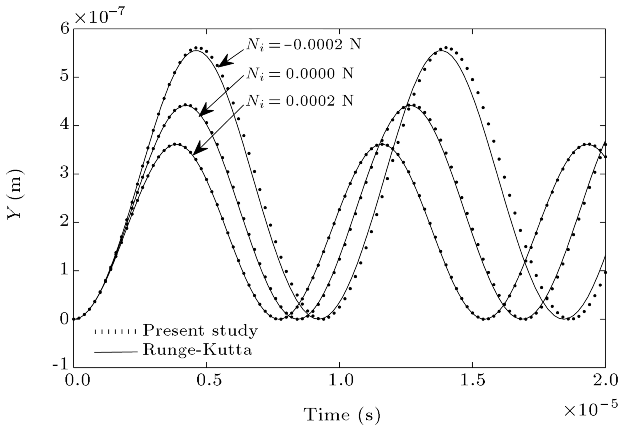


**Figure 5.** Effect of residual axial load on deflection of the simply supported microbeam at $x = L/2$ with $L = 210$ $\mu$m at 10 V.

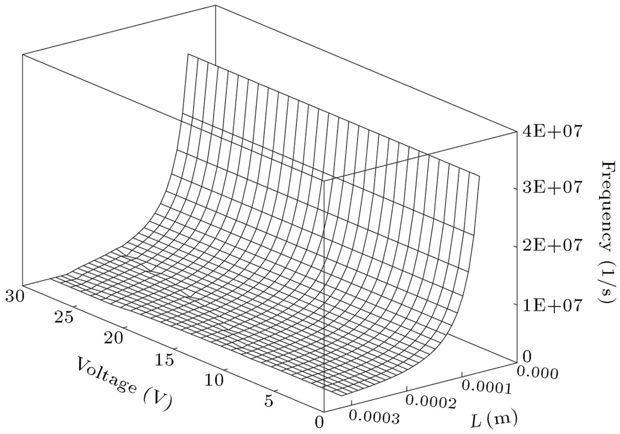


**Figure 6.** Effect of beam length ($L$) and voltage ($V$) on the nonlinear frequency where the residual axial load is zero.

## 5. Summary and conclusion

Variational Iteration Method (VIM) has been applied to study the vibrations of electrostatically actuated double-clamped and simply supported microbeams. The geometrical nonlinearity has been modeled using von-Karman's assumptions. Galerkin's decomposition method has been used to obtain the nonlinear ordinary differential equation of motion. Analytical Expressions for the nonlinear natural frequency and deflection of these microbeams have been obtained using the VIM. The effects of the applied voltage and residual axial load on the nonlinear natural frequency and deflection of the microbeams have been studied. It is shown that by increasing the applied voltage, higher iterations are required to find the results accurately. It is also concluded that pretension in microbeams results in higher values of nonlinear frequency. Also, precompression in microbeams increases the amplitude of deflections for a specific applied voltage. Having a higher rate of convergence and high accuracy, VIM is shown to be an efficient method for studying the behavior of nonlinear systems.

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## Appendix A

Coefficients required for determining Eq. (11) are presented as follows:

$$Q = -\frac{\varepsilon_0 V^2}{2Md_0}\left(\frac{b}{d_0}+\beta\right)\int_0^L \psi dx,$$

$$M = \rho bh \int_0^L \psi^2 dx,$$

$$R = -\frac{\varepsilon_0 V^2}{2Md_0^3}\left(\frac{3b}{d_0}+\beta\right)\int_0^L \psi^3 dx,$$

$$\gamma = \frac{1}{M}\int_0^L \psi\Bigg(EI\psi_{xxxx} - N_i\psi_{xx} - \frac{b\varepsilon_0 V^2\psi}{d_0^3} - \frac{\beta\varepsilon_0 V^2\psi}{2d_0^2}\Bigg)dx,$$

$$G = -\frac{\varepsilon_0 V^2}{2Md_0^5}\left(\frac{5b}{d_0}+\beta\right)\int_0^L \psi^5 dx,$$

$$S = \frac{-1}{2ML}\int_0^L \psi\Bigg(Ebh\psi_{xx}\int_0^L \psi_x^2 dx + \frac{\varepsilon_0 L^2 V^2 \psi^3}{d_0^4}\left(\frac{4b}{d_0}+\beta\right)\Bigg)dx,$$

$$P = -\frac{\varepsilon_0 V^2}{2Md_0^7}\left(\frac{7b}{d_0}+\beta\right)\int_0^L \psi^7 dx,$$

$$J = -\frac{\varepsilon_0 V^2}{2Md_0^6}\left(\frac{6b}{d_0}+\beta\right)\int_0^L \psi^6 dx.$$

## Appendix B

Coefficients required for determining Eq. (27) are presented as follows:

$$\begin{aligned}
13440\Omega^{14}c_1 =& 134640PQ^6 + 26880Q\Omega^{12} - 13440Q\Omega^{10}\gamma + 17920RQ^2\Omega^8 \\
& - 27300SQ^3\Omega^6 + 44688GQ^4\Omega^4 - 76440JQ^5\Omega^2,
\end{aligned}$$

$$\begin{aligned}
13440\Omega^{14}c_2 =& 2240RQ^2\Omega^8 + 69300P\Omega^6 - 6720SQ^3\Omega^6 - 33600JQ^5\Omega^2 \\
& + 15680GQ^4\Omega^4,
\end{aligned}$$

$$13440\Omega^{14}c_3 = 4725JQ^5\Omega^2 - 1680GQ^4\Omega^4 - 11550PQ^6 + 420SQ^3\Omega^6,$$

$$13440\Omega^{14}c_4 = -560JQ^5\Omega^2 + 112GQ^4\Omega^4 + 1848PQ^6,$$

$$13440\Omega^{14}c_5 = 35JQ^5\Omega^2 - 210PQ^6,$$

$$13440\Omega^{14}c_6 = 12PQ^6,$$

$$\begin{aligned}
13440\Omega^{14}c_7 =& -26880Q\Omega^{12} - 58800GQ^4\Omega^4 + 105840JQ^5\Omega^2 - 194040PQ^6 \\
& - 20160RQ^2\Omega^8 + 33600SQ^3\Omega^6 + 13440Q\Omega^{10}\gamma.
\end{aligned}$$

## Appendix C

A higher order approximation for the frequency based on Eq. (27), is given as:

$$\begin{aligned}
&10Jc_1^5 + (5Jc_5 + 25Jc_3)c_1^4 + (70Jc_2^2 + (80Jc_4 \\
&+ 160Jc_7 + 32G)c_2 + 60Jc_3^2 + 80Jc_3c_5 \\
&+ 60Jc_4^2 + (40Jc_7 + 8G)c_4 + 48Gc_7 + 60Jc_5^2 \\
&+ 120Jc_7^2 + 12S)c_1^3 + ((150Jc_3 + 90Jc_5)c_2^2 \\
&+ ((360Jc_7 + 240Jc_4 + 72G)c_3 + 240Jc_4c_5 \\
&+ (24G + 120Jc_7)c_5)c_2 + 30Jc_3^3 + 90Jc_3^2c_5 \\
&+ (60Jc_4^2 + (360Jc_7 + 72G)C_4 + 48Gc_7 \\
&+ 60Jc_5^2 + 120Jc_7^2 + 12S)c_3 + 30Jc_4^2c_5 \\
&+ (360Jc_7 + 72G)c_5c_4)c_1^2 + (30Jc_2^4 + (24G \\
&+ 120Jc_7 + 80Jc_4)c_2^3 + (150Jc_3^2 + 180Jc_3c_5
\end{aligned}$$

$$+ 120Jc_4^2 + (240Jc_7 + 48G)c_4 + 96Gc_7$$
$$+ 240Jc_7^2 + 120Jc_5^2 + 24S)c_2^2 + ((48G$$
$$+ 240Jc_7 + 240Jc_4)c_3^2 + (360Jc_4c_5 + (480Jc_7$$
$$+ 96G)c_5)c_3 + 60Jc_4^3 + (240Jc_7 + 48G)c_4^2$$
$$+ (96Gc_7 + 240Jc_7^2 + 120Jc_5^2 + 24S)c_4$$
$$+ (240Jc_7 + 48G)c_5^2 + 160Jc_7^3 + 16R + 48Sc_7$$
$$+ 96Gc_7^2)c_2 + 30Jc_3^4 + 60Jc_3^3c_5 + (150Jc_4^2$$
$$+ (24G + 120Jc_7)c_4 + 96Gc_7 + 240Jc_7^2$$
$$+ 120Jc_5^2 + 24S)c_3^2 + (240Jc_4^2c_5 + (240Jc_7$$
$$+ 48G)c_5c_4 + 60Jc_5^3 + (96Gc_7 + 240Jc_7^2$$
$$+ 24S)c_5)c_3 + 30Jc_4^4 + (96Gc_7 + 240Jc_7^2$$
$$+ 150Jc_5^2 + 24S)c_4^2 + 30Jc_5^4 + (96Gc_7$$
$$+ 240Jc_7^2 + 24S)c_5^2 + 32Rc_7 + 80Jc_7^4$$
$$+ 48Sc_7^2 + 64Gc_7^3)c_1 + (20Jc_3 + 20Jc_5)c_2^4$$
$$+ ((24G + 120Jc_7 + 80Jc_4)c_3 + 60Jc_4c_5$$
$$+ (40Jc_7 + 8G)c_5)c_2^3 + (30Jc_3^3 + 120Jc_3^2c_5$$
$$+ (90Jc_4^2 + (240Jc_7 + 48G)c_4 + 48Gc_7 + 60Jc_5^2$$
$$+ 120Jc_7^2 + 12S)c_3 + 90Jc_4^2c_5 + (240Jc_7$$
$$+ 48G)c_5c_4 + 30Jc_5^3 + (48Gc_7 + 120Jc_7^2$$
$$+ 12S)c_5)c_2^2 + ((24G + 120Jc_7 + 60Jc_4)c_3^3$$
$$+ (180Jc_4c_5 + (24G + 120Jc_7)c_5)c_3^2 + (60Jc_4^3$$
$$+ (240Jc_7 + 48G)c_4^2 + (180Jc_5^2 + 24S + 240Jc_7^2$$
$$+ 96Gc_7)c_4 + (240Jc_7 + 48G)c_5^2 + 160Jc_7^3 + 16R$$
$$+ 48Sc_7 + 96Gc_7^2)c_3 + 60Jc_4^3c_5 + (24G$$
$$+ 120Jc_7)c_5c_4^2 + (60Jc_5^3 + (96Gc_7 + 240Jc_7^2$$
$$+ 24S)c_5)c_4)c_2 + 20Jc_3^4c_5 + (10Jc_4^2 + (24G$$
$$+ 120Jc_7)c_4 + 10Jc_5^2)c_3^3 + (60Jc_4^2c_5 + (240Jc_7$$
$$+ 48G)c_5c_4 + 30Jc_5^3 + (48Gc_7 + 120Jc_7^2$$
$$+ 12S)c_5)c_3^2 + ((24G + 120Jc_7)c_4^3 + 30Jc_4^2c_5^2$$
$$+ ((240Jc_7 + 48G)c_5^2 + 160Jc_7^3 + 16R + 48Sc_7$$
$$+ 96Gc_7^2)c_4)c_3 + (24G + 120Jc_7)c_5c_4^3$$
$$+ ((24G + 120Jc_7)c_5^3 + (160Jc_7^3 + 16R$$
$$+ 48Sc_7 + 96Gc_7^2)c_5)c_4 = 0.$$

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