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An efficient concept for non-dimensionalizing the duffing equations based on the parameterized perturbation method

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Abstract. In this paper, ongoing studies to investigate nonlinear Ordinary Differential Equations (ODE) are extended by presenting a new concept in non-dimensionalization process. This concept is illustrated with a practical example of nonlinear ODEs, which cannot reliably solve using the numerical methods. In this paper, two perturbation techniques are used to solve the problem. Effect of varying the dimensionless initial displacement on the accuracy of solution is investigated. It is shown that if the process of non-dimensionalizing is done appropriately, the calculated results will be extremely accurate. Moreover, a new concept called "behavior of results" is proposed to find accurate results.

1. Introduction

Mathematical modeling can be used to simulate the behavior of the practical systems and provide a better understanding of them [1]. Theoretical investigations are useful for drawing general conclusions from simple models [2]. In reality, every physical process is a nonlinear system and should be described by nonlinear equations. Kerschen et al. studied the sources of nonlinearity, and classified them [3].

Nowadays, there is a large tendency toward numerical simulation of nonlinear systems [4-6]. The reason for this interest lies in the growth of powerful computers. However, it should be considered that the numerical methods require a considerable number of iterations in order to approach the true solution, and it is necessary to provide initial estimates of the unknowns [7]. Due to the limitations of numerical methods, it is necessary to find analytical solutions for nonlinear problems.

In spite of linear problems, it is very difficult to find analytical solution for nonlinear equations. Consequently, many efforts have been made to develop the methods of studying nonlinear systems. Nayfeh and Mook [8], Verhulst [9] and Rand [10] studied nonlinear equations.

Perturbation methods provide powerful tools to analyze nonlinear problems [11-14]. Ganji et al. applied the perturbation methods to solve nonlinear equations in fluid mechanics problems [15-17]. They also evaluated a nonlinear engineering system, using the so-called amplitude-frequency formulation, which overcomes the difficulty of computing the periodic behavior of such systems [18]. But, as well as other nonlinear techniques, the perturbation methods have several limitations. Nayfeh [19] and O'Malley [20] investigated the restrictions of perturbation methods. Usually, before using the perturbation methods, this question may be arisen: "which technique, out of presented perturbation methods, is better than the

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others?" Finding an answer for this question is not easy, because some special nonlinear problems may be accurately solved using a particular perturbation technique [21,22]. The main motivation of the present work is to find a reasonable answer for the above question.

Consider a plasma tube in which the magnetic field is cylindrical and increases toward the axis in inverse proportion to the radius. A nonlinear equation describes the motion of injected electrons into the discussed plasma tube. In this paper, the nonlinear equation is solved using two perturbation techniques [23]. Parameterized perturbation and iteration perturbation methods are used to solve the nonlinear ODE. It is shown that wavelength of the motion can be exactly approximated using the iteration perturbation method. Finally, effects of the process of non-dimensionalizing on the accuracy of solution, which is obtained using the parameterized perturbation method, are investigated and discussed.

2. Mathematical modeling

The orbit of a charged particle q moving in an electric and magnetic field can be formulated using the Lorentz force equation [24]. This equation can be shown as follows:

$$F = q(E + v \times B), \tag{1}$$

where F is the Lorentz force, E is the electric field, B is the magnetic field and v is the instantaneous velocity of the particle. Consider the uniform magnetic field ($E = 0$) in a plasma tube in which the magnetic field changes cylindrically and increases toward the axis in inverse proportion to the radius. Therefore, the Lorentz force equation can be simplified to the following form:

$$F = qvC/r, \tag{2}$$

where C is a constant, which relates to the magnetic field (B). Variable r is the distance to the center line of the plasma tube. According to the Newton's second law, motion of the charged particle injected into the plasma tube can be given by:

$$mr_{,tt} + Cqvr^{-1} = 0, \tag{3}$$

where m is the mass of the charged particle. For convenience, the above equation can be rewritten as the following relation:

$$r_{,tt} + \alpha r^{-1} = 0. \tag{4}$$

In the above equation $\alpha = Cqv/m$. Consider $u = r/r_0$ and $\tau = \omega_0 t$, where r_0 is the characteristic distance and ω_0 is the natural circular frequency of vibration. Therefore, dimensionless form of the above equation can be written as follows:

$$u_{,tt} = -\beta u^{-1}, \tag{5}$$

where β is equal to $\alpha/\omega_0^2 r_0^2$. In the present study, the initial conditions are assumed to be $u(0) = A$ and $u_{,\tau}(0) = 0$.

3. Analytical solution

3.1. Iteration perturbation method

Iteration perturbation method is a relatively new perturbation technique coupling with the Iteration method [25]. The iteration formula for the dimensionless governing equation can be defined as follows:

$$(u_{\tau\tau})_{n+1} + \varepsilon\beta(u)_{n+1}(u)_n^2 = 0, \tag{6}$$

where ε is an introduced artificial parameter and subscript n represents the nth order of solution. Comparing between Eqs. (5) and (6), it can be concluded that, in this section, the artificial parameter (ε) is equal to one. Assume that the initial approximate solution is $u_0 = A\cos\omega t$, where ω is the angular frequency of the oscillation. Therefore, Eq. (6) can be rewritten as follows:

$$u_{,\tau\tau} + 2\beta A^{-2}u + \varepsilon u_{,\tau\tau}\cos(2\omega\tau) = 0, \tag{7}$$

where ε is an introduced artificial parameter. The above equation is a sort of the Mathieu equation. Suppose that:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{8}$$

$$2\beta A^{-2} = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \cdots \tag{9}$$

In the above equation it should be noted that the parameter C_0 is equal to square of the angular frequency ($C_0 = \omega^2$). Substituting Eqs. (8) and (9) into Eq. (7) and equating the coefficients of the same power of ε results in the following differential equation for u_1:

$$u_{1,\tau\tau} + \omega^2 u_1 + \left(C_1 A - \frac{A\omega^2}{2}\right)\cos\omega\tau - \frac{A\omega^2}{2}\cos 3\omega\tau = 0. \tag{10}$$

Vanishing secular term requires $C_1 = \omega^2/2$. Substituting C_1 into Eq. (9) results in:

$$\omega = \frac{2}{A}\sqrt{\frac{\beta}{3}}. \tag{11}$$

Therefore, the first approximate wavelength of the charged particle motion is equal to:

$$\lambda = 5.4414\frac{A}{\sqrt{\beta}}. \tag{12}$$

Neglecting the secular term and solving Eq. (10) with initial conditions $u_1(0) = 0$ and $u_{1,\tau}(0) = 0$ yields the following result:

$$u_1 = \frac{A}{16}\left(\cos\omega\tau - \cos 3\omega\tau\right). \tag{13}$$

Substituting Eq. (13) into Eq. (8) results in the following relation:

$$u_1 = A\cos\omega\tau + \frac{A}{16}(\cos\omega\tau - \cos 3\omega\tau). \tag{14}$$

If the above solution is substituted into Eq. (6) as the 0th order solution, the following relation will be obtained:

$$u_{,\tau\tau} + 256\varepsilon u\beta A^{-2}(17\cos\omega\tau - \cos 3\omega\tau)^{-2} = 0. \tag{15}$$

The above equation can be rewritten as follows:

$$u_{,\tau\tau} + \frac{256}{145}\beta A^{-2}u + \varepsilon\left(\frac{51}{58}\cos 2\omega\tau - \frac{17}{145}\cos 4\omega\tau + \frac{1}{290}\cos 6\omega\tau\right)u_{,\tau\tau} = 0. \tag{16}$$

Therefore, the circular frequency of motion can be expanded, as shown in Eq. (17).

$$\frac{256}{145}\beta A^{-2} = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \cdots \tag{17}$$

where $C_0 = \omega^2$. Substituting Eqs. (8) and (17) into Eq. (16) results in the following differential equation:

$$u_{1,\tau\tau} + \omega^2 u_1 + C_1 u_0 + \left(\frac{51}{58}\cos 2\omega\tau - \frac{17}{145}\cos 4\omega\tau + \frac{1}{290}\cos 6\omega\tau\right)u_{0,\tau\tau} = 0. \tag{18}$$

The 0th order solution, in the above relation, can be replaced by what is shown in Eq. (14). Therefore, Eq. (18) changes to the following equation:

$$u_{1,\tau\tau} + \omega^2 u_1 + \left(\frac{17}{16}AC_1 - \frac{1173}{4640}A\omega^2\right)\cos\omega\tau + \left(\frac{1}{16}AC_1 - \frac{937}{2320}A\omega^2\right)\cos 3\omega\tau + \frac{17A\omega^2}{9280}\left(\frac{357}{1160}\cos 5\omega\tau - \frac{323}{9280}\cos 7\omega\tau + \frac{9}{9280}\cos 9\omega\tau\right) = 0. \tag{19}$$

Neglecting the secular term requires $C_1 = 69\omega^2/290$. Regarding the parameter C_1, presented in Eq. (17), second order angular frequency can be calculated as follows:

$$\omega = \frac{1}{A}\sqrt{\frac{512\beta}{359}}. \tag{20}$$

Therefore, the wavelength of motion is equal to:

$$\lambda = 5.2612\frac{A}{\sqrt{\beta}}. \tag{21}$$

For convenience, very small coefficients in Eq. (19) can be neglected. Therefore, this equation can be rewritten as follows:

$$u_{1,\tau\tau} + \omega^2 u_1 \simeq \frac{67}{160}A\omega^2\cos 3\omega\tau - \frac{6069}{10764800}A\omega^2\cos 5\omega\tau. \tag{22}$$

The second iterative solution of Eq. (5), which is calculated using the iteration perturbation method, is given in Eq. (23). For convenience, the second iterative solution of problem is briefly named as "second order" solution.

$$u(\tau) = \frac{17A}{16}\cos\omega\tau - \frac{147A}{1280}\cos 3\omega\tau - \frac{6069A}{258355200}\cos 5\omega\tau. \tag{23}$$

3.2. Parameterized perturbation

Initially, Eq. (5) is multiplied by u^2 , then term $0.u_{,\tau\tau}$, which is equal to zero is added to the left side of this equation. Finally, all right side term of the equation is moved to the left side. Therefore, the new presentation for the dimensionless form of Eq. (5) can be shown as follows:

$$0.u_{,\tau\tau} + u^2 u_{,\tau\tau} + \beta u = 0. \tag{24}$$

Consider linear transformation $u = \varepsilon v$, where ε is an introduced perturbation parameter. Therefore, the above equation can be rewritten as follows:

$$0.\varepsilon v_{,\tau\tau} + \varepsilon^3 v^2 v_{,\tau\tau} + \beta\varepsilon v = 0. \tag{25}$$

Expanded form of the parameters v, β and 0 are given by:

$$v = v_0 + \varepsilon^2 v_1 + \varepsilon^4 v_2 + \cdots \tag{26}$$

$$\beta = \omega^2 + \varepsilon^2\omega_1 + \varepsilon^4\omega_2 + \cdots \tag{27}$$

$$0 = 1 + \varepsilon^2 b_1 + \varepsilon^4 b_2 + \cdots \tag{28}$$

Substituting Eqs. (26), (27) and (28) into Eq. (25) and equating the coefficients of same powers of ε, yields the following relations:

$$v_{0,\tau\tau} + \omega^2 v_0 = 0, \qquad v_0(0) = A\varepsilon^{-1},$$
$$v_{0,\tau}(0) = 0, \tag{29}$$

$$v_{1,\tau\tau} + \omega^2 v_1 + v_0^2 v_{0,\tau\tau} + \omega_1 v_0 + b_1 v_{0,\tau\tau} = 0$$
$$v_1(0) = 0, \qquad v_{1,\tau}(0) = 0, \tag{30}$$

$$v_{2,\tau\tau} + \omega^2 v_2 + (b_2 + 2v_0 v_1) v_{0,\tau\tau} + (b_1 + v_0^2) v_{1,\tau\tau} + \omega_1 v_1 + \omega_2 v_0 = 0,$$
$$v_2(0) = v_{2,\tau}(0) = 0. \tag{31}$$

If v_0 that is calculated by Eq. (29) is substituted into Eq. (30), the following relation will be obtained:

$$\omega_1 = \left(b_1 + \frac{3}{4}A^2\varepsilon^{-2}\right)\omega^2. \tag{32}$$

The parameter β can be achieved by substituting the above equation into Eq. (27).

$$\beta = \omega^2 + \varepsilon^2\left(b_1 + \frac{3}{4}A^2\varepsilon^{-2}\right)\omega^2. \tag{33}$$

In the first order of solution, $b_1 = -\varepsilon^{-2}$. Substituting b_1 into the above relation and simplifying it results in:

$$\omega = \frac{\sqrt{\beta}}{A}\sqrt{\frac{4}{3}}. \tag{34}$$

Therefore, the wavelength of motion is as follows:

$$\lambda = 5.4414\frac{A}{\sqrt{\beta}}. \tag{35}$$

To solve Eq. (31), the variable v_1 can be calculated by Eq. (30). The variable v_1 is equal to:

$$v_1 = A^3\varepsilon^{-3}(\cos\omega\tau - \cos 3\omega\tau). \tag{36}$$

Substituting Eqs. (32) and (36) into Eq. (31) results in:

$$v_{2,\tau\tau} + \omega^2 v_2 + \left(-\frac{A\omega^2 b_2}{\varepsilon} - \frac{A^3\omega^2 b_1}{32\varepsilon^3} + \frac{A^5\omega^2 b_2}{64\varepsilon^5} + \frac{A^3\omega_1}{32\varepsilon^3} + \frac{A\omega_2}{\varepsilon}\right)\cos\omega\tau + \left(\frac{19A^5\omega^2}{128\varepsilon^5} + \frac{9A^3\omega^2 b_1}{32\varepsilon^3} - \frac{A^3\omega_1}{32\varepsilon^3}\right)\cos 3\omega\tau + \left(\frac{11A^5\omega^2}{128\varepsilon^5}\right)\cos 5\omega\tau = 0. \tag{37}$$

Neglecting the secular term, in the above relation, needs to vanish the coefficient of $\cos\omega\tau$. Therefore, the following equation will be constructed:

$$\frac{A\omega_2 - A\omega^2 b_2}{\varepsilon} + \frac{A^3(\omega_1 - \omega^2 b_1)}{32\varepsilon^3} + \frac{A^5\omega^2 b_2}{64\varepsilon^5} = 0. \tag{38}$$

Substituting b_1 and ω (which are obtained previously in Eq. (34)) into the above equation and simplifying it results in:

$$\omega_2 = \omega^2\left(b_2 - \frac{5}{128}\frac{A^4}{\varepsilon^4}\right). \tag{39}$$

Regarding Eq. (27), the following equation can be concluded:

$$\beta = (1 + \varepsilon^2 b_1 + \varepsilon^4 b_2)\omega^2 + \frac{3}{4}A^2\omega^2 - \frac{5}{128}A^4\omega^2. \tag{40}$$

The above relation can be simplified to the following form (Regarding to Eq. (28)):

$$\beta = \omega^2\left(\frac{3}{4}A^2 - \frac{5}{128}A^4\right). \tag{41}$$

The angular frequency of motion can be written as follows:

$$\omega = \frac{\sqrt{\beta}}{A}\frac{2}{\sqrt{3}\sqrt{1 - \frac{5}{96}A^2}}. \tag{42}$$

Therefore, the second order wavelength of motion is equal to:

$$\lambda = 5.4414\frac{A}{\sqrt{\beta}}\sqrt{1 - \frac{5}{96}A^2}. \tag{43}$$

Second order solution of Eq. (5), which is calculated using the parameterized perturbation method, is as follow:

$$u(\tau) = \left(A - \frac{A^3}{128} + \frac{11A^5}{768}\right)\cos\omega\tau - \left(\frac{A^3}{128} - \frac{11A^5}{1024}\right)\cos 3\omega\tau - \left(\frac{11A^5}{3072}\right)\cos 5\omega\tau. \tag{44}$$

4. Results and discussions

A practical nonlinear ODE has been derived and solved using two perturbation techniques. The presented perturbation methods can be employed to solve various kinds of nonlinear problems. Here, this question may be posed: "why is the discussed kind of nonlinear differential equations selected to be solved in this paper?".

The reason for this selection lies in the following facts:

- As shown in Section 2, practical problems can be described using the discussed nonlinear differential equation.
- Numerical methods may wrongly solve the discussed nonlinear ODE. Therefore, it is necessary to find analytical solution.
- Exact wavelength of motion for the discussed problem can be calculated. This will be used to verify the calculated results.

As said, among the above notes, numerical methods may be unable to solve the presented problem. The reason for this inability may lie in the fact that the concavity ($u_{,\tau\tau}$) approaches to infinity when the vibratory particle passes from the origin ($u = 0$). Figure 1 shows the result of using several numerical methods for solving the discussed problem.

For the discussed problem, the wavelength of motion can be calculated, analytically. For this reason, consider $u_{,\tau\tau} = u_{,\tau}(u_{,\tau})_{,u}$ and substitute it into Eq. (5). Therefore, the exact wavelength of motion is equal to [22]:

$$\lambda_{\text{exact}} = \sqrt{\frac{2}{\beta}} \int_0^A \frac{du}{\sqrt{\text{Ln}A - \text{Ln}u}} = 5.0133 \frac{A}{\sqrt{\beta}}. \quad (45)$$

Wavelengths of motion, which are achieved by employing the iteration perturbation, are illustrated in Figure 2. It should be noted that the linear wavelength of motion (0th order) is obtained by neglecting the nonlinear term of Eq. (24). In doing so, to find the linear wavelength of motion, term $u^2 u_{,\tau\tau}$ is removed from Eq. (24). With regard to Eq. (28), the equivalent linear form of Eq. (24) can be re-written as follows:

$$u_{,\tau\tau} + \frac{\beta}{1 + \varepsilon^2 b_1 + \varepsilon^4 b_2 + \cdots} u = 0. \quad (46)$$

Note that in the linear problem, the introduced perturbation parameter (ε) can be considered negligible, so the equivalent linear wavelength of motion is equal to $2\pi A \beta^{-1/2}$. A curve that is obtained, using the cubic spline interpolation, is fitted to the calculated results. The spline curve will be used to describe the so-called "behavior of the calculated results", and it is obtained using the curve fitting toolbox in MATLAB software.

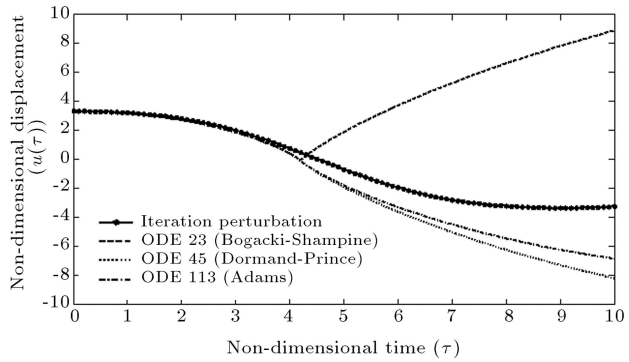

Figure 1. Result of solving the governing equation, using several numerical method and perturbation method.

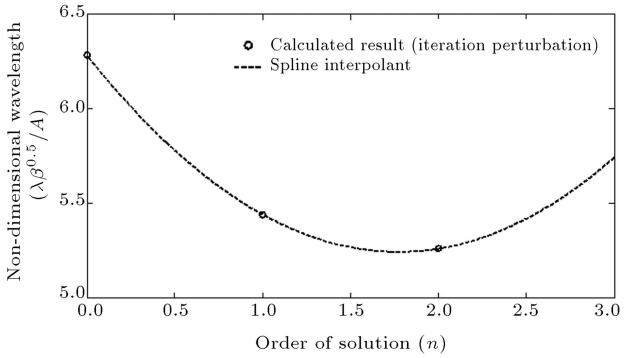

Figure 2. Behavior of the non-dimensional wavelengths with the order of solution.

The calculated wavelengths of motion are compared with the exact solution in Table 1. As shown in this table, increasing the order of solution leads to finding more accurate results.

Wavelengths of motion, which are calculated using the parameterized perturbation method, are presented in Table 2. As shown in this table, in contrast with the iteration perturbation, the error of the second order solution of the parameterized perturbation method is variable with the parameter A.

The second order wavelength calculated, using the parameterized perturbation method, is illustrated in Figure 3. It can be shown that if the dimensionless initial displacement (A) is selected between 1.120 and 2.114, the parameterized perturbation results will be

Table 1. Calculated wavelengths of motion, using the iteration perturbation method.

Order of solution (i)	Wavelength of motion (λ_i)	Error (%)
0	$6.2832A/\sqrt{\beta}$	25.33
1	$5.4414A/\sqrt{\beta}$	8.54
2	$5.2612A/\sqrt{\beta}$	4.94

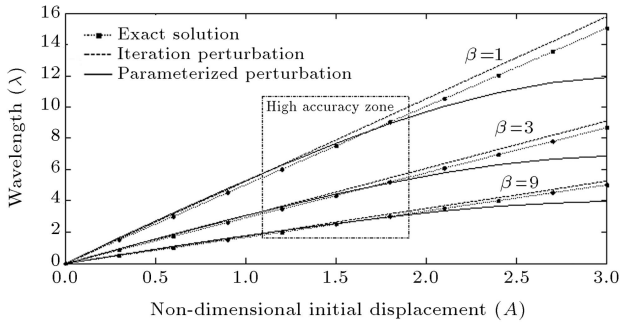

Figure 3. Variation of the wavelength versus initial displacement.

more accurate than the result of the iteration perturbation method. This range, which is simply named "high accuracy zone", can be calculated regarding the exact solution.

Variation of error of the second order solution with varying the non-dimensional initial displacement (A) is shown in Table 3. As shown in this table, unlike the iteration perturbation method, error of solution in the parameterized perturbation method varies with changing the non-dimensional initial displacement (A). As said before, in the high accuracy range, the error of second order solution of the parameterized perturbation method becomes lower than the error of second order solution of the iteration perturbation method (4.94%). Furthermore, in this table it is shown that if $A = 1.7036$, the calculated wavelength of motion, which is achieved using the parameterized perturbation method, is equal to the exact solution.

Here, this question may be arisen: "can the so-called high accuracy range be obtained without finding the exact solution?".

Finding an answer for the above question is really important, because the exact solution cannot be found for all nonlinear problems. To find the relatively exact solution, it should be noted that the difference between the higher order wavelengths of motion (which are obtained using the perturbation method) should be smaller than the lower order solutions. In other words, the following conditions should be fulfilled:

(I) $|\lambda_0 - \lambda_1| > |\lambda_1 - \lambda_2|$

(II) $|\lambda_2 - \lambda_3| < |\lambda_1 - \lambda_2|$

Therefore, the difference between the second order solution and the first order solution should be smaller than $0.8418A\beta^{-1/2}$ (Condition (I)). This range, which is simply named "acceptable range for non-dimensional initial displacement" is shown in Figure 4.

Table 2. Calculated wavelengths of motion, using the parameterized perturbation method.

Order of solution (i)	Wavelength of motion (λ_i)	Error (%)
0	$6.2832A/\sqrt{\beta}$	25.33
1	$5.4414A/\sqrt{\beta}$	8.54
2	$\left(5.4414\sqrt{1-5A^2/96}\right)A/\sqrt{\beta}$	$108.54\sqrt{1-5A^2/96}-100$

Table 3. Error of using the second order solution with varying non-dimensional initial displacement.

		Parameterized perturbation method		Iteration perturbation method	
	Non-dimensional initial displacement (A)	Non-dimensional wavelength ($\lambda\sqrt{\beta}/A$)	Error of solution (%)	Non-dimensional wavelength ($\lambda\sqrt{\beta}/A$)	Error of solution (%)
High Accuracy Range (Error< 4.94%)	1.0	5.2978	5.67	5.2612	4.94
	1.1	5.2671	5.06	5.2612	4.94
	1.2	5.2334	4.39	5.2612	4.94
	1.3	5.1964	3.65	5.2612	4.94
	1.4	5.1562	2.85	5.2612	4.94
	1.5	5.1126	1.98	5.2612	4.94
	1.6	5.0657	1.05	5.2612	4.94
	1.7	5.0152	0.04	5.2612	4.94
	1.7036	5.0133	0.00	5.2612	4.94
	1.8	4.9611	1.04	5.2612	4.94
	1.9	4.9032	2.19	5.2612	4.94
	2.0	4.8415	3.43	5.2612	4.94
	2.1	4.7758	4.74	5.2612	4.94
	2.2	4.7058	6.13	5.2612	4.94
	2.3	4.6315	7.62	5.2612	4.94

Table 4. Difference between the calculated dimensionless wavelengths.

(A)	$\lvert\lambda_0-\lambda_1\rvert\sqrt{\beta}/A$	$\lvert\lambda_1-\lambda_2\rvert\sqrt{\beta}/A$	$\lvert\lambda_2-\lambda_3\rvert\sqrt{\beta}/A$	Note (I)	Note (II)
1.0	0.8418	0.1436	0.5508	Satisfied	-
1.5	**0.8418**	**0.3288**	**0.0398**	**Satisfied**	**Satisfied**
2.0	0.8418	0.5999	0.7807	Satisfied	-
2.5	0.8418	0.9726	1.1148	-	-

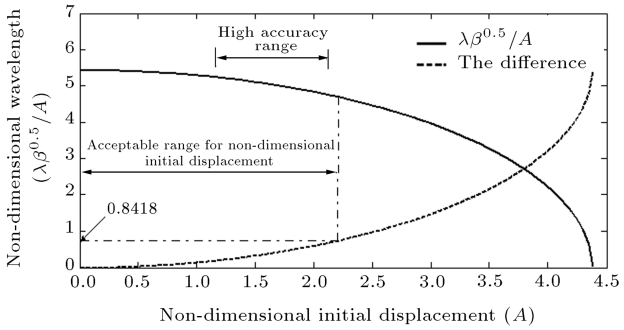

Figure 4. Variation of the difference and the non-dimensional wavelength versus initial displacement.

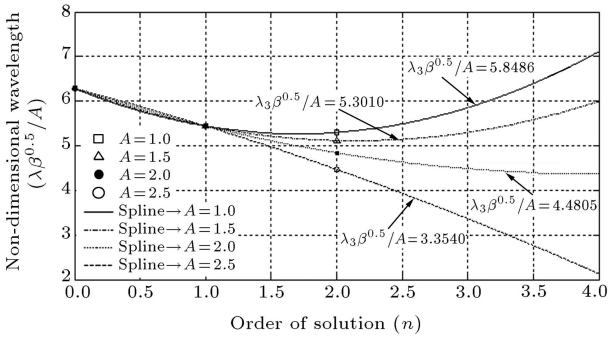

Figure 5. Behavior of the non-dimensional wavelengths with varying the parameter A.

Regarding the above two conditions, to find the so-called high accuracy range for the non-dimensional initial displacement (A) can be calculated. Effect of varying the variable A on the behavior of the parameterized perturbation results is illustrated in Figure 5.

Difference between the calculated dimensionless wavelengths of motion is presented in Table 4. As shown in this table, if $A = 1.5$, both of the two presented notes are satisfied. Therefore, error of the results that is obtained using the second order solution of the parameterized perturbation method, is equal to 1.98%. This solution is more accurate than the results which are obtained using the second order solution of the iteration perturbation method (4.94%).

Variation of the dimensionless distance (u) versus dimensionless time (τ), which is calculated using the second order solution of Eq. (5), is shown in Figure 6. In this figure, the initial dimensionless distance (A) is equal to 1.7, and the iteration perturbation and parameterized perturbation solutions are respectively plotted according to Eqs. (23) and (44).

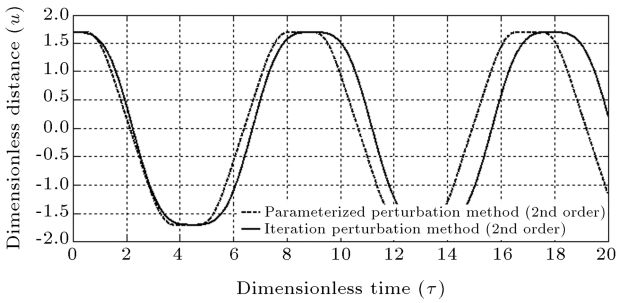

Figure 6. Variation of the dimensionless distance with the dimensionless time $(A = 1.7)$.

5. Conclusion

This paper studies a practical nonlinear ordinary differential equation that describes the behavior of a particle, which is excited in a plasma tube. The discussed nonlinear equation cannot reliably be solved using numerical methods. For this reason, the governing equation has been solved analytically, using two perturbation techniques.

Both of the first and the second order solutions of the iteration perturbation method provide accurate results. Simplicity and accuracy of the iteration perturbation method makes it appropriate for practical applications. However, it has been shown that the so-called "behavior of results" in the iteration perturbation method, presented in this paper, is not acceptable.

Unlike the iteration perturbation method, accuracy of the parameterized perturbation result is variable with the dimensionless initial displacement. It is shown that in a special range of the dimensionless initial displacements, the parameterized perturbation method provides extremely accurate results. This range is approximated using the concept of results behavior.

Nomenclature

A	Initial dimensionless distance
B	Magnetic field
C	Constant
E	Electric field

F	Lorentz force
P	Coefficient
B	Coefficient
M	Mass of the charged particle
N	Order of solution
Q	Charged particle
S	Independent variable
T	Time
R	Displacement of the charged particle
U	Dimensionless distance to the center line of the plasma tube (r/r_0)
V	Instantaneous velocity of the particle
X	Independent variable
R	Distance to the center line of the plasma tube
r_0	Characteristic distance

Greek symbols

α	Cqv/m
β	$\alpha/\omega_0^2 r_0^2$
ε	Coefficient
Λ	wave length of motion
T	Dimensionless time $(\tau = \omega_0 t)$
Ω	Circular frequency
ω_0	Natural frequency of system

Subscripts

0	Initial conditions
Exact	Exact solution

References

1. Timmons, D.L., Johnson, C.W. and McCook, S.M., *Fundamentals of Algebraic Modeling: An Introduction to Mathematical Modeling with Algebra and Statistics*, 5th Ed. Brooks/Cole, Belmont, Calif. (2010).
2. Bender, E.A., *An Introduction to Mathematical Modeling*, Dover Publications, Mineola, N.Y. (2000).
3. Kerschen, G., Worden, K., Vakakis, A.F. and Golinval, J. "Past, present and future of nonlinear system identification in structural dynamics", *Mechanical Systems and Signal Processing*, **20**, pp. 505-592 (2006).
4. Afsharfard, A. and Farshidianfar, A. "An efficient method to solve the strongly coupled nonlinear differential equations of impact dampers", *Archive of Applied Mechanics*, **82**(7), pp. 977-984 (2012). doi:10.1007/s00419-011-0605-1
5. Kerschen, G. and Golinval, J.C. "A model updating strategy of non-linear vibrating structures", *International Journal for Numerical Methods in Engineering*, **60**(13), pp. 2147-2164 (2004). doi:10.1002/nme.1040
6. Steinbach, O., *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements*, Springer, New York (2008).
7. Ayyub, B.M. and McCuen, R.H., *Numerical Methods for Engineers*, Prentice Hall, Upper Saddle River, N.J. (1996).
8. Nayfeh, A.H. and Mook, D.T. "Nonlinear oscillations", *Pure and Applied Mathematics*, Wiley, New York (1979).
9. Verhulst, F., *Nonlinear Differential Equations and Dynamical Systems*, 2nd Rev. and Expanded Ed. Universitext., Springer, Berlin, New York (1996).
10. Rand, R.H., *Lecture Notes on Nonlinear Vibrations*, Cornell University (2005).
11. Murdock, J.A., *Perturbations Theory and Methods*, John Wiley & Sons Inc., (1991).
12. Nayfeh, A.H., *Perturbation Methods*, Wiley classics library Ed., John Wiley & Sons, New York (2000).
13. Rand, R.H. and Armbruster, D. "Perturbation methods, bifurcation theory, and computer algebra", *Applied Mathematical Sciences*, **65**, Springer-Verlag, New York (1987).
14. Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Annotated Ed., Parabolic Press, Stanford, Calif. (1975).
15. Ganji, D.D. "The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer", *Physics Letters A*, **355**(4-5), pp. 337-341 (2006). doi:http://dx.doi.org/10.1016/j.physleta.2006.02.056
16. Ganji, D.D. and Rafei, M. "Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation by homotopy perturbation method", *Physics Letters A*, **356**(2), pp. 131-137 (2006). doi:http://dx.doi.org/10.1016/j.physleta.2006.03.039
17. Ganji, D.D. and Rajabi, A. "Assessment of homotopy-perturbation and perturbation methods in heat radiation equations", *International Communications in Heat and Mass Transfer*, **33**(3), pp. 391-400 (2006). doi:http://dx.doi.org/10.1016/j.icheatmasstransfer.2005.11.001
18. Ganji, S.S., Ganji, D.D., Babazadeh, H. and Sadoughi, N. "Application of amplitude-frequency formulation to nonlinear oscillation system of the motion of a rigid rod rocking back", *Mathematical Methods in the Applied Sciences*, **33**(2), pp. 157-166 (2010). doi:10.1002/mma.1159
19. Nayfeh, A.H., *Introduction to Perturbation Techniques*, Wiley, New York (1981).
20. O'Malley, R.E. "Singular perturbation methods for ordinary differential equations", *Applied Mathematical Sciences*, **89**, Springer-Verlag, New York (1991).
21. Farshidianfar, A. and Nickmehr, N. "Some new analytical techniques for duffing oscillator with very strong nonlinearity", *Iranian Journal of Mechanical Engineering*, **10**(1), pp. 37-54 (2008).

22. He, J.H. "Some asymptotic methods for strongly nonlinear equations", *International Journal of Modern Physics*, **B 20**(10), pp. 1141-1199 (2006).

23. He, J.-H., *Perturbation Methods: Basic and Beyond*, Elsevier, Boston (2006).

24. Reitz, J.R. and Milford, F.J., *Foundations of Electromagnetic Theory*, Addison-Wesley Publishing Company (1967).

25. He, J.H. "Iteration perturbation method for strongly nonlinear oscillations", *Vibration Control*, **7**(5), pp. 631-642 (2001).

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