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Matched pole-zero state-space model and continuous-time properties

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Abstract. The Matched Pole-Zero (MPZ) model is a widely used technique for discrete-time approximation of continuous-time controllers. In this article, a new state-space representation for the (MPZ) model is presented. The new formulation can be used for direct discretization of state-space controllers, and can be easily automated on a digital computer. The most important advantage of the proposed representation is that it preserves the dynamic structure of the original continuous-time realization, i.e. the physical meaning of the states and the direction of eigenvectors remain unchanged. In fact, the new method provides, *exactly*, the same dynamic state equations as the step-invariant model, together with some modifications on the static output state equation.

Up to now, due to the lack of such *eigenstructure-preserving* state-space representations, most of the time domain studies on the effects of discrete approximation of analog controllers were mostly performed using the step-invariant model, although that method is seldom used for actual discretization of controllers. The new formulation paves the way for extending those studies to the case of the more widely used MPZ method.

1. Introduction

Despite the existence of numerous methods and literature on the direct design of sampled-data control systems ([1-7] etc.), there remain motivations to use the more conventional digital redesign methods (indirect methods) in which a predesigned analog controller is approximated by a digital one. These motivations include better physical insight and availability of a wide spectrum of continuous control design methods. There exists a number of well known methods for discretization of finite-dimensional continuous-time controllers/plants [8-10]. Other methods also exist for discretization of infinite-dimensional controllers [11], nonlinear systems [12] and multi-rate controllers [13]. One group of such methods, including the method of Tustin, are based upon numerical approximation of the *integration operator*, which can be used for discretization of controllers. The other group, including the step-invariant model, provides an exact discrete model for a plant with a given hold element. The Matched Pole/Zero (MPZ) discretization technique is yet another method which is not directly motivated by the numerical approximation of integrators or the concept of hold equivalence. In this method, the zeros and poles of the continuous system are mapped by the relation, e^{sh}, where s is a pole or zero, and h is the sampling period. The MPZ technique has been extensively used and proved to be efficient for discrete approximation of continuous-time controllers [8]. This method also plays a central role in a new closed-loop digital redesign method, called the Plant-Input Mapping (PIM) method ([14,15], which guarantees closed-loop stability for *all* nonpathological sampling periods.

Unfortunalety, however, unlike methods such as

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Tustin's and zoh-equivalent models, no state-space algorithms were introduced for the MPZ method until recently. This fact seems to be the main reason for most research to ignore usage of the MPZ method for analytical studies of sampled-data systems.

A number of attempts and studies have been reported on state-space algorithms for the MPZ method. An interesting interpretation of the MPZ model, based on the generalized hold equivalence concept, was proposed in [16,17]. This method, however, involved tedious hand calculations, which could not be easily performed, even for low order systems. A simpler state-space algorithm was introduced in [18]. That method had two drawbacks:

1. It was only applicable to systems with a realization in the observable canonical form;
2. It involved inversion of some controllability matrices which are known to be ill-conditioned for higher order systems ([19], page 105).

Among common discretization methods, the only one used to preserve the structure of original continuous realization was the step-invariant method. In other words, the step-invariant model exceptionally preserves the physical meanings of the states and the direction of system eigenvectors. That is why most time-domain studies on the properties of discretized sampled-data systems are based on controllers which are discretized using the step-invariant method (e.g., see [2,3]). This is despite the fact that the step-invariant method is seldom used for discretization of controllers in practice. The main objective of this paper is to provide a state-space realization, with similar advantages, for the MPZ model. The new formulation should pave the way for further time-domain studies with the more widely used MPZ method, which, compared to the step-invariant method, is much better suited for discretization of controllers.

2. MPZ technique

2.1. Transfer function version

Consider a strictly stable SISO transfer function:

$$g(s)=\frac{\prod_{i=1}^{m}(s-\alpha_i)}{\prod_{i=1}^{n}(s-\beta_i)}, \qquad m\leq n. \tag{1}$$

The MPZ discrete-time approximation technique is partly motivated by the z-transform method in which the poles of $g(s)$ are mapped to the z-plane, according to the relation $z=e^{hs}$, where h is the sampling period. In fact, the MPZ technique extends such a mapping to the case of zeros as well as poles. In particular, the discrete system, $g_d(z)$, is obtained by the following procedure [9]:

1. All of the poles and finite zeros of $g(s)$ are mapped with the relation $z=e^{hs}$;
2. All but one of the infinite zeros, if any, are mapped to the points $\{-1\}$ (The relative degree of 1 ensures the physical realizability of $g_d(z)$);
3. The dc-gain of the two transfer functions are matched, such that:

$$\lim_{s\to 0} g(s)=\lim_{z\to 1} g_d(z).$$

2.2. Proposed state-space version

Consider a stable SISO continuous-time system with a *minimal* state-space realization, $G:=\{A,b,c,d\}$, where $A\in R^{n\times n}$, $b\in R^{n\times 1}$, $c\in R^{1\times n}$, and d is a scalar. The objective is to find a state-space realization, $G_d:=\{A_d,b_d,c_d,d_d\}$ such that the (transmission) zeros and the eigenvalues of G are mapped according to the procedure described in Section 2.1.

The proposed state-space algorithm for the above problem is introduced below.

3. Proposed state-space algorithm

3.1. Bi-proper system

For the time being, assume that, $g(s)$ is bi-proper, i.e. $m=n$. Let us define the matrix A_d as $A_d=e^{Ah}$. This assures that the eigenvalues (poles) of the discrete system, i.e. $\{\beta_1,\beta_2,\cdots,\beta_n\}$, are mapped by the relation $e^{\beta_i h}$, as desired.

Provision of a state-space formulation for similar mapping of the zeros is more subtle and needs some elaboration. Following the well-known MATLAB syntax, let us define the *tzero* operator as the operator acting on system G and providing the finite (transmission) zeros of G, i.e.:

$$\overline{\alpha}=[\alpha_1,\alpha_2,\cdots,\alpha_n]=tzero(G), \tag{2}$$

where $\overline{\alpha}$ is the vector of finite zeros of G. Robust algorithms exist for computing the zeros of LTI systems ([20], for example). By definition of the MPZ model, the finite zeros of G_d are required to be at the locations given by the vector:

$$\overline{\alpha}_d:=[\exp(\overline{\alpha}h)], \tag{3}$$

where $\exp(\overline{\alpha}h)$ is the element-wise exponential of vector $\overline{\alpha}h$.

It can be shown that the following expression relates the state-space realization of G_d to its transfer function, $g_d(z)$, (see [21], page 651):

$$g_d(z)=\frac{det(zI-e^{Ah}+b_dc_d)+(d_d-1)det(zI-e^{Ah})}{det(zI-e^{Ah})}. \tag{4}$$

Now, by arbitrarily fixing d_d as $d_d=1$, the zero-placement problem is converted into the problem of finding b_d and c_d, such that the eigenvalues of matrix $[e^{Ah} - b_d c_d]$ are placed at the locations given by the elements of vector $\overline{\alpha}_d$. Let us also fix b_d, as $b_d = b$. Now, the problem reduces to finding c_d, such that the eigenvalues of $[e^{Ah} - b_d c_d]$ are placed at $\overline{\alpha}_d$. This is a standard eigenvalue placement problem for which robust algorithms exist [19,22]. Existence of the solution for c_d is guaranteed by controllability of the pair $\{e^{Ah}, b\}$, which is obvious due to the controllability of the pair $\{A, b\}$ and the fact that A and e^{Ah} share the same set of eigenvectors ([21], page 661).

Finally, in order to match the steady-state gains of the two systems, the final vector, c_d, and also d_d are obtained by multiplying the previous c_d and d_d by a gain correction factor, k_d, given by:

$$k_d = \frac{c(-A)^{-1}b + d}{c_d(I_n - A_d)^{-1}b_{d3} + d_d}. \tag{5}$$

Remark 1. Instead of selecting b_d, and then calculating c_d, one may fix c_d, and then calculate b_d accordingly. This, in fact, provides an extra flexibility to the method which can prove useful in practice, as it allows the designer to preserve the structure of the discrete system, from the point of view of actuators or sensors, respectively. In either case, the transfer function of the discrete system remains the same.

3.2. Strictly proper system

When the relative degree of the continuous system is not zero (i.e., when $m < n$), the eigenvalue assignment of $[e^{Ah} - b_d c_d]$ at the locations of $\overline{\alpha}_d$ is not meaningfule, because $\dim(\overline{\alpha}_d) < n$. In order to resolve this problem, Eq. (3) is replaced by:

$$\overline{\alpha}_d := [-1/\epsilon, \underbrace{-1, \cdots, -1}_{n-m-1}, \exp(\overline{\alpha}h)], \tag{6}$$

where ϵ is an arbitrarily selected, very small, positive number. The implication is that with $d_d = 1$ and Eq. (4):

$$g_d(z) = \frac{(z+\frac{1}{\epsilon})(z+1)\cdots(z+1)(z-e^{h\alpha_1})\cdots(z-e^{h\alpha_m})}{det(zI - e^{Ah})}. \tag{7}$$

Also, from Eq. (5):

$$k_d = \frac{g(s)|_{s=0}}{g_d(z)|_{z=1}}, \tag{8}$$

$$= \frac{(\alpha_1)\cdots(\alpha_m)}{(\beta_1)\cdots(\beta_n)} \cdot \frac{(1 - e^{h\beta_1})\cdots(1 - e^{h\beta_n})}{(1+\frac{1}{\epsilon})(1+1)\cdots(1+1)(1-e^{h\alpha_1})\cdots(1-e^{h\alpha_m})} \ll 1. \tag{9}$$

This implies that for the scaled discrete system, $k_d g_d(z)$, it turns out that $d_d \to k_d d_d = k_d \ll 1 \approx 0$, for small enough ϵ, and also $c_d \to k_d c_d$.

The proposed algorithm is summarized as below:

Algorithm 1. (shift form)

1. Set $A_d := e^{Ah}$.
2. Set $b_d := b$.
3. If $n = m$, set $d_d := 1$, otherwise set $d_d = 0$.
4. Find the vector $\overline{\alpha}$ as in Eq. (2).
5. If $n = m$, set the vector $\overline{\alpha}_d$ as in Eq. (3), otherwise as in Eq. (6).
6. Find c_d such that the eigenvalues of $A_d - b_d c_d$ are assigned at $\overline{\alpha}_d$.
7. Set $c_d := k_d c_d$, and $d_d := k_d d_d$ where k_d is given by Eq. (5).

In order to make the discrete representation of the model more similar to its continuous counterpart, and for better numerical properties when h is too small, one may prefer to use the δ operator, where $\delta = \frac{q-1}{h}$, and q is the common shift operator. Clearly, the δ-operator is intuitively closer to a continuous-time derivative than the common q operator [23,24]. That is, we may want to consider the discrete state-space system in the following form:

$$\delta x[kh] = A_\delta x[kh] + b_\delta u[kh],$$

$$y[kh] = c_\delta x[kh] + d_\delta u[kh]. \tag{10}$$

The state-space algorithm for this case can be formulated as below:

Algorithm 2. (δ form)

1. Set $A_\delta := (e^{Ah} - I)/h$.
2. Set $b_\delta := b$.
3. If $n = m$, set $d_\delta := 1$, otherwise set $d_\delta = 0$ (by Remark 1).
4. Find the vector $\overline{\alpha}$ as in Eq. (2).
5. If $n = m$, set the vector $\overline{\alpha}_\delta := [\exp(\overline{\alpha}h - I)/h]$, otherwise, set:
$$\overline{\alpha}_\delta := [-1/\epsilon, \underbrace{-1/h, \cdots, -1/h}_{n-m}, \exp(\overline{\alpha}h - I)/h].$$
6. Find c_δ such that the eigenvalues of $A_\delta - b_\delta c_\delta$ are assigned at $\overline{\alpha}_\delta$.
7. For a very low arbitrary frequencey, ϵ, find k_δ from the following:
$$k_\delta = \frac{c(\epsilon I - A)^{-1}b + d}{c_\delta(\epsilon I - A_\delta)^{-1}b + d_\delta}. \tag{11}$$

8. Set $c_\delta := k_\delta c_\delta$, and $d_\delta := k_\delta d_\delta$.

The following lemma characterizes a connection between the continuous and discrete realizations for small sampling periods.

Lemma 1. The discrete realization obtained by the MPZ algorithm (in δ form) converges to its original counterpart, when the sampling period is decreased, i.e.:

1. $\lim_{h\to 0}\ A_\delta \to A$;
2. $\lim_{h\to 0}\ c_\delta \to c$;
3. $\lim_{h\to 0}\ d_\delta \to d$.

Proof

1. Obvious, because $A_\delta = (e^A h - I)/h$;
2. By noting that $b_\delta = b$, and due to the uniqueness of the solution to the eigenvalue assignment problem in Item 6 of Algorithm 2, based on which, c_δ is obtained;
3. Due to Items 1 and 2 above, and Eq. (11), which implies that $\lim_{h\to 0} k_\delta \to d$ and, hence, $d_\delta \to d$. □

4. Time domain properties

Consider the general sampled-data set up shown in Figure 1, with G being a LTI asymptotically stable system, and G_δ the MPZ discrete-time model of G. Here, S and H are synchronized ideal sampling and zero-order hold operators, respectively, and W is a finite-dimensional, linear and time-invariant, stable and strictly causal prefilter. As shown in the figure, Δ is the discretization error operator. In the sequel, $\|.\|_\infty$ denotes the $\mathcal{L}^\infty$ norm of a signal and $\|.\|_{\mathcal{L}^\infty}$ denotes the $\mathcal{L}^\infty$ induced norm of an operator acting on $\mathcal{L}^\infty$ signals.

Inclusion of W in this setup provides a sufficient condition for $\|(I - HS)W\|_{\mathcal{L}^p}$ to be finite for every $1 \le p \le \infty$, and, furthermore, $(I - HS)W$ converges to zero as h tends to zero in the sense of these norms (by Theorem 9.3.3 in [1]).

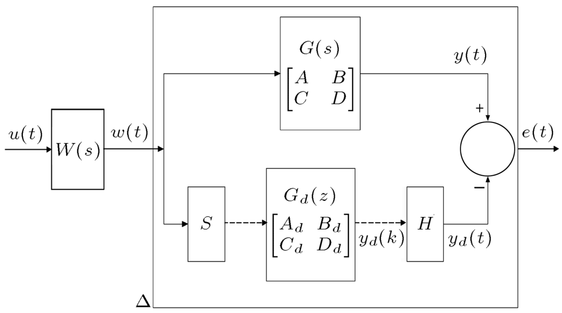

Figure 1. Discretization error in a sampled-data setting.

In this study, we will mostly utilize the *lifting technique* for comparing the time response of a continuous-time system with its discretized counterpart. In a broad sense, the lifting technique is a method for rearranging a continuous-time periodic system, in such a way that its periodicity can be viewed as discrete-time shift invariance [4].

We will limit our discussion to the case of bounded continuous-time signal space, $\mathcal{L}^\infty[0,\infty)$.

We also define $\ell_{\mathcal{L}^\infty[0,h]}$ to be the space of all sequences that take their values in the Banach space $\mathcal{L}^\infty[0,h]$. Next, we define $\ell^\infty_{\mathcal{L}^\infty[0,h]}$ as the subspace of bounded sequences in $\ell_{\mathcal{L}^\infty[0,h]}$.

We will use the notation $L_h : \mathcal{L}^\infty[0,\infty) \to \ell_{\mathcal{L}^\infty[0,h]}$ to denote the *norm preserving* lifting operator. Suppose G is a stable linear continuous-time operator: $\mathcal{L}^\infty[0,\infty) \to \mathcal{L}^\infty[0,\infty)$. The lifted version of G, noted as $\underline{G}$, is the linear discrete-time system acting on $l_{\mathcal{L}^\infty[0,h]}$, i.e., $\underline{G} : l_{\mathcal{L}^\infty[0,h]} \to l_{\mathcal{L}^\infty[0,h]}$.

Considering the norm preserving property of the lifting operator, we will convert the setup of Figure 1 into an equivalent setup, as shown in Figure 2.

For the linear continuous-time system, G: $\{A,B,C,D\}$, it can be shown [1] that the lifted system is given by $\underline{G}$: $\{\underline{A},\underline{B},\underline{C},\underline{D}\}$, where:

$$\underline{A} : \underline{A}x = e^{Ah}x,$$

$$\underline{B} : \underline{B}u_k = \int_0^h e^{A(h-\tau)}B\underline{u}_k(\tau)d\tau,$$

$$\underline{C} : (\underline{C}x)(t) = Ce^{At}x,$$

$$\underline{D} : (\underline{D}u_k)(t) = D\underline{u}_k(t) + \int_0^t Ce^{A(t-\tau)}B\underline{u}_k(\tau)d\tau. \tag{12}$$

The kth component of the lifted output of the lifted system is given by [1]:

$$\hat{y}[kh+t] = \sum_{l=0}^{k-1}\left[\underline{C}\underline{A}^{k-1-l}\underline{B}w[kh+t]\right] + \underline{D}w[kh+t],$$

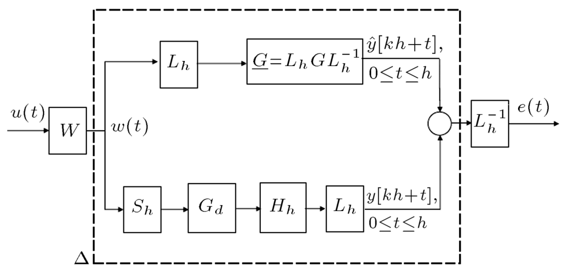

Figure 2. Equivalent discretization problem in a lifted setting.

where:

$$0 \le t < h. \tag{13}$$

Now, consider the discretized MPZ model, G_d. In order to make the discrete-time behavior of the MPZ model comparable with its continuous counterpart, we prefer to use the δ operator. Using the results specified in Algorithm 2, the discrete-time output of G_d, for a given input $w[kh]$, can be written as:

$$y_d[kh] = \sum_{l=0}^{k-1} \left[c_\delta(hA_\delta + I)^{k-1-l}(hb_\delta)w(lh)\right] + d_\delta w(kh). \tag{14}$$

If $y_d[kh]$ is passed through a zero order hold element, the output would be a staircase (semi) continuous signal, which, in lifted form, can be represented as:

$$y[kh+t] = \sum_{l=0}^{k-1} \left[(c_\delta)(hA_\delta + I)^{k-1-l}h(b_\delta)w[lh]\right] + (d_\delta)w[kh], \qquad (0 \le t < h). \tag{15}$$

Consider the setup of Figure 1, and a bounded continuous input, $u(t) \in \mathcal{L}^\infty$. Define $w[kh+\tau]$ as the kth component of the discrete vector obtained by lifting of $w(t)$. Also, define $\delta w[kh+\tau]$ as the deviation of $w[kh+\tau]$ from its staircase equivalent, i.e.:

$$w[kh+\tau] = w[kh] + \delta w[kh+\tau], \quad 0 \le \tau < h. \tag{16}$$

Clearly:

$$\lim_{h\to 0} \delta w[kh+\tau] = 0. \tag{17}$$

Lemma 2. Let us define the kth component of the lifted error signal as:

$$e[kh+t] \triangleq \hat{y}[kh+t] - y[kh+t], \quad 0 \le t < h. \tag{18}$$

Under the above assumptions, the following property holds:

$$\lim_{h\to 0} \frac{\|e[kh+t]\|_\infty}{\|w[kh+t]\|_\infty} = 0. \tag{19}$$

Proof (This proof closely follows the method of [25], which is partially motivated by [26].) Since the lifting operator, L_h, is norm preserving, we will work with the equivalent discretization problem in a lifted setting (Figure 2). Using Eqs. (13) and (15), the lifted outputs of G and HG_dS can be, respectively, written as:

$$\hat{y}[kh+t] = \sum_{l=0}^{k-1}\left[ce^{At}(e^{Ah})^{k-1-l}\int_0^h e^{A(h-s)}bw[lh+s]ds\right] + \int_0^t ce^{A(t-s)}bw[kh+s]ds + dw[kh+t],$$

$$(0 \le t < h),$$

$$y[kh+t] = \sum_{l=0}^{k-1}\left[(c+\delta c)(h(A+\delta A)+I)^{k-1-l}h(b)w[lh]\right] + (d+\delta d)w[kh],$$

$$(0 \le t < h),$$

where $\{\delta A, \delta c, \delta d\} \to 0$, when $h \to 0$, by Lemma 1. With repeated use of the triangle inequality, and also, addition/subtraction of terms to allow suitable factorizations, it is tedious yet straightforward to show that the $\mathcal{L}^\infty$ norm of the error is:

$$\frac{\|\hat{y}[kh+t] - y[kh+t]\|_\infty}{\|w[kh+t]\|_\infty} = \max_{t\in[0,h)} \Bigg\{ \int_0^t \left|ce^{As}b\right| ds + |d|\frac{\|\delta w[kh+t]\|_\infty}{\|w[kh+t]\|_\infty} + |\delta d| + \sum_{l=0}^{+\infty}\Bigg[\left|c(e^{Ah})^l h(b) - c(h(A+\delta A)+I)^l h(b)\right| + \left|\delta c(h(A+\delta A)+I)^l h(b)\right| + \int_0^h \left|c(e^{At}-I)(e^{Ah})^l e^{As}b\right| ds + \left|c(e^{Ah})^l\left(\int_0^h e^{As}bds - h(b)\right)\right| + \int_0^h \left|c(e^{Ah})^l e^{As}b\right| ds \frac{\|\delta w[kh+t]\|_\infty}{\|w[kh+t]\|_\infty}\Bigg]\Bigg\}. \tag{20}$$

Now, we need to prove that every term in the above equation converges to zero, when $h \to 0$. Since G is asymptotically stable, all the eigenvalues of A are in the open left half of the s plane, hence, $\int_0^h e^{As}ds < h$ by the mean-value theorem. Also, all the eigenvalues of

e^{Ah} belong to the open unit circle in the z plane and, hence, $h\sum_{l=0}^{+\infty}\|e^{Ahl}\|$ is finite. Therefore:

$$\max_{t\in[0,h)}\left\{\int_0^t \left|ce^{As}b\right| ds\right\}\leq\|c\|\left(\int_0^h \left\|e^{As}\right\| ds\right)\|b\|\xrightarrow{h\to 0} 0.$$

As mentioned before, W provides a sufficient condition for $\|(I-HS)W\|_{\mathcal{L}^p}$ to be finite for every $1\leq p\leq\infty$, and, furthermore, $(I-HS)W$ converges to zero as h tends to zero in the sense of these norms. The implication is that:

$$\sup_{\|w[kh+t]\|_\infty\neq 0}\frac{\|\delta w[kh+t]\|_\infty}{\|w[kh+t]\|_\infty}$$

$$=\sup_{\|w[kh+t]\|_\infty\neq 0}\frac{\|(I-HS)w[kh+t]\|_\infty}{\|w[kh+t]\|_\infty}$$

$$=\sup_{\|Wu\|_\infty\neq 0}\frac{\|(I-HS)Wu\|_\infty}{\|Wu\|_\infty}\xrightarrow{h\to 0} 0.$$

Therefore:

$$\max_{t\in[0,h)}\left\{|d|\frac{\|\delta w[kh+t]\|_\infty}{\|w[kh+t]\|_\infty}\right\}=|d|\frac{\|\delta w[kh+t]\|_\infty}{\|w[kh+t]\|_\infty}\xrightarrow{h\to 0} 0.$$

Also:

$$\max_{t\in[0,h)}\{|\delta d|\}=|\delta d|\xrightarrow{h\to 0} 0,$$

and:

$$\max_{t\in[0,h)}\left\{\sum_{l=0}^{+\infty}\int_0^h|c(e^{At}-I)(e^{Ah})^l e^{As}b|ds\right\}$$

$$\leq\|c\|\left\|e^{At}-I\right\|\left(\sum_{l=0}^{+\infty}\left\|e^{Ahl}\right\|\right)h\,\|b\|\xrightarrow{h\to 0} 0,$$

$$\max_{t\in[0,h)}\left\{\sum_{l=0}^{+\infty}|c(e^{Ah})^l\left(\int_0^h e^{As}bds-h(b)\right)|\right\}$$

$$\leq\|c\|\left(\sum_{l=0}^{+\infty}\left\|e^{Ahl}\right\|\right)\|hb-h(b)\|\xrightarrow{h\to 0} 0,$$

$$\max_{t\in[0,h)}\left\{\left(\sum_{l=0}^{+\infty}\int_0^h|c(e^{Ah})^l e^{As}b|ds\right)\frac{\|\delta w\|_\infty}{\|w\|_\infty}\right\}$$

$$\leq\|c\|\left(\sum_{l=0}^{+\infty}\left\|e^{Ahl}\right\|\right)h\|b\|\frac{\|\delta w\|_\infty}{\|w\|_\infty}\xrightarrow{h\to 0} 0,$$

$$\max_{t\in[0,h)}\left\{\sum_{l=0}^{+\infty}|c(e^{Ah})^l h(b)-c(h(A+\delta A)+I)^l h(b)|\right\}$$

$$\leq\|c\|\left(\sum_{l=0}^{+\infty}\left\|e^{Ahl}-(h(A+\delta A)+I)^l\right\|\right)h\|b\|$$

$$\xrightarrow{h\to 0} 0,$$

$$\max_{t\in[0,h)}\left\{\sum_{l=0}^{+\infty}|\delta c(h(A+\delta A)+I)^l h(b)|\right\}$$

$$\leq\|\delta c\|\left(\sum_{l=0}^{+\infty}\left\|(h(A+\delta A)+I)^l\right\|\right)h\|b\|\xrightarrow{h\to 0} 0.$$

Now, since for all $t\in[0,h)$, all the individual terms in Eq. (20) approach zero when $h\to 0$, the desired result is proven.

Taking the stability of G and G_d, the continuous error signal, $e(t)\in\mathcal{L}^\infty[0,\infty)$, and its lifted version, $\underline{e}\in\ell_{\mathcal{L}^\infty[0,h]}$, is defined as a sequence with values in $\mathcal{L}^\infty[0,h]$, denoted by$\{e_k\}$, where for each k, we have:

$$\underline{e}(k)\triangleq e[kh+t],\qquad 0\leq t<h. \tag{21}$$

Furthermore:

$$\|\{e_k\}\|_\infty\triangleq\sup_k\|e_k\|_\infty,\qquad \{e_k\}\in\ell_{\mathcal{L}^\infty[0,h]}. \tag{22}$$

Considering the norm preserving property of the lifting operator, it can be deduced that:

$$\lim_{h\to 0}\|e(t)\|_\infty\to 0. \tag{23}$$

Now, since $u(t)\in\mathcal{L}^\infty$ is arbitrary, then the following holds:

$$\lim_{h\to 0}\|\Delta W\|_{\mathcal{L}^\infty}\triangleq\lim_{h\to 0}\sup_{\|u\|_\infty\neq 0}\left\{\frac{\|e\|_\infty}{\|u\|_\infty}:u(t)\in\mathcal{L}^\infty\right\}\to 0. \tag{24}$$

This property is of practical importance when an analog system is implemented digitally, because it assures that a better performance can be attained by using a smaller sampling period. It is interesting to note that not all of the modern digital redesign methods provide such a practically important property; for instance, see the method in [27] and examples thereof. □

Example 1. With $h=0.01$ sec, find the MPZ discrete-time model of the continuous-time system:

$$A=\begin{bmatrix}-3 & -0.5 & -0.125\\ 8 & 0 & 0\\ 0 & 2 & 0\end{bmatrix},\qquad b=\begin{bmatrix}1\\1\\0\end{bmatrix}, \tag{25}$$

$$c=\begin{bmatrix}0 & 0.1818 & 0.909\end{bmatrix},\qquad d=0. \tag{26}$$

We follow the steps described in Algorithm 2:

Step 1:

$$A_\delta = (e^{Ah} - I)/h$$

$$= \begin{bmatrix} -2.9751 & -0.4938 & -0.1231 \\ 7.8807 & -0.0198 & -0.005 \\ 0.0792 & 1.9999 & 0 \end{bmatrix}. \quad (27)$$

Step 2: Set $b_\delta = b$;

Step 3: Set $d_\delta = 0$, because of the nonzero relative degree of the system;

Step 4: $\overline{\alpha} = [-11, -1]$;

Step 5: $\overline{\alpha_\delta} = [-1000, -10.9397, -0.9995]$;

Step 6: $c_\delta = [11.9349, 997.0048, 497.7565]$;

Step 7: $k_\delta = 1.8267e - 004$;

Step 8: Set new $c_\delta := k_\delta c_\delta = [0.0022, 0.1821, 0.0909]$.

Resemblance between the discrete state-space realization $\{A_\delta, b_\delta, c_\delta, d_\delta\}$ and the original continuous counterpart $\{A, b, c, d\}$ is obvious from the above which shows the advantage of the proposed algorithm compared with the transfer function approach.

Example 2. Consider the RLC network shown in Figure 3. It can be shown that the following realization describes the dynamical behavior of this system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It can be easily seen that this system is unobservable, therefore, the algorithm proposed in [18] is not applicable. The MPZ model of this system using Algorithm 1

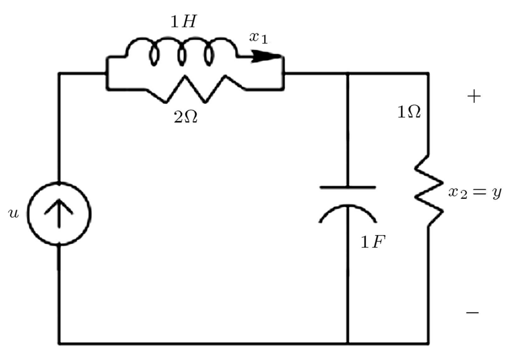

Figure 3. An unobservable RLC network.

can be obtained for $h = 0.1$ sec, as below:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.7408 & 0.0820 \\ 0 & 0.9048 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 20.2 \\ 0.1 \end{bmatrix} u(k),$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

It should be noted that, unlike the δ form, the realization in the shift form lacks the proprty of resemblance between the discrete and continuous state-space matrices. Another point is that, in this example, use was made of Remark 1 to preserve c_d instead of b_d, so that the sensor connection of the realization could be preserved.

Example 3. Consider the following continuous system:

$$g(s) = \frac{1}{s^2 + s + 1}.$$

In order to study the effect of the sampling period on the continuous-time performance of the discretized system, a range of sampling period $h = \{0.1, 0.2, \cdots 1\}$ is considered. The input signal is considered as a unit step function, $u(t) = 1(t)$, which has the property $u(t) \in \mathcal{L}^\infty[0, \infty)$. In order to guarantee uniform convergence a strictly causal prefilter is selected:

$$W(s) = \frac{1}{0.5s + 1}.$$

Table 1 shows the values of continuous-time error signal $\|e(t)\|_\infty$ over the range of pre-specified sampling periods. Uniform convergence of the error norm can be observed in the table.

5. Concluding remarks

A new state-space algorithm is introduced for the matched pole/zero discretization technique. Unlike previously existing methods, the new algorithm is not limited to any specific realization form and can be automated by existing software packages. The robustness properties of the algorithm have been improved considerably, compared with the algorithm in [18], and does not include the inversion of ill-conditioned matrices that are usually encountered with high order systems and with very small sampling periods.

It is also shown in this paper that the continuous-time response of the discretized system converges to

Table 1. Norm of the error signal versus sampling perio. Uniform convergence is shown here.

h	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\|e(t)\|_\infty$	0.025	0.054	0.17	0.25	0.33	0.38	0.44	0.47	0.63	0.68

that of the original continuous system in the $\mathcal{L}^{\infty}$-induced norm sense. This means that the maximum difference in the time responses of the continuous and discretized systems, subject to *any* low-pass filtered and bounded continuous-time signal, approaches zero, when the sampling period is decreased. This property is of considerable practical importance, because it assures the designer that by reducing the sampling period, the resulting discretization error is reduced as well. It can be shown that most of the classical discretization techniques enjoy such a useful property, while there are some advanced (global) discretization methods which lack such a property. In other words, while, for a fixed h, the performance of most global methods could be better than that of the local discretization methods, they may not provide the practically important assurance of improvement in performance when the sampling period is reduced. The method is limited to the SISO case. Extension to the MIMO case is not obvious and is left for future work.

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Biography

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