Research Note

Fuzzy Development of Multivariate Variable Control Charts Using the Fuzzy Likelihood Ratio Test

H. Moheb Alizadeh[1], A.R. Arshadi Khamseh[1] and S.M.T. Fatemi Ghomi[2,*]

Abstract. *This paper is an effort to evolve multivariate variable control charts in a fuzzy environment where each observation in each sample is assumed to be a canonical fuzzy number. To do this, a likelihood ratio test should be exploited in a fuzzy environment, because multivariate variable control charts are constructed using this test. In this way, membership functions of likelihood ratio statistics applied to control the process mean and dispersion are obtained solving four non-linear programming problems. Using these membership functions, membership degrees of in and out of control states of both process mean and dispersion are computed. Hence contrary to the classic multivariate variable control charts categorizing the process into just two states, i.e. in and out of control, the process can be considered in several intermediate states, based on the computed membership degrees, bringing about more flexibility in process analysis.*

Keywords: *Multivariate control charts; Likelihood ratio test; Non-linear programming; Fuzzy numbers; Fuzzy random variables.*

INTRODUCTION

Statistical Process Control (SPC) is a technique applied toward improving the quality of characteristics by monitoring the process under study continuously, in order to detect assignable causes and take required actions as quickly as possible. Control charts are viewed as the most commonly applied SPC tools. A control chart consists of three horizontal lines called; Upper Control Limit (UCL), Center Line (CL) and Lower Control Limit (LCL). The center line in a control chart denotes the average value of the quality characteristic under study. If a point is laid within UCL and LCL, then the process is deemed to be under control. Otherwise, a point plotted outside the control limits can be regarded as evidence representing that the process is out of control and, hence, preventive or corrective actions are necessary in order to find and omit an assignable cause or causes, which

subsequently result in improving quality characteristics [1].

Control charts are classified into two categories, with respect to the number of quality characteristics they monitor: univariate control chart and multivariate control chart. Univariate control charts are exploited to control the processes with only one quality characteristic, whereas multivariate control charts are applied in order to control and monitor any process with two or more correlated quality characteristics. In this case, due to this correlation, using several distinct univariate control charts is not suggested.

Traditional control charts are constructed using precise data. Since these data may be influenced by human judgment, evaluations and decisions, it is better to take into account the variability brought about by human subjectivity, measurement devices or environmental conditions. This variability results in an uncertain situation in which imprecise or linguistic data should be dealt with. For example, an item may be measured by assigning a linguistic term, such as 'a range between 270.15 and 270.3', in a variable-measured production process, or it may be expressed as 'very good', 'good', 'medium', 'bad' and 'very bad' in an attribute-measured production process. This uncertain situation can be modeled by neither a de-

1. *Department of Industrial Engineering, Tarbiat Moallem University, Tehran, P.O. Box 1571914911, Iran.*
2. *Department of Industrial Engineering, Amirkabir University of Technology, Tehran, P.O. Box 15916-34311, Iran.*
*. *Corresponding author. E-mail: fatemi@aut.ac.ir*

terministic model nor a probabilistic model, but the fuzzy set theory, initially proposed by Zadeh [2], is a worthwhile tool to deal with such an uncertainty. In this way, the data obtained from a production process are represented as fuzzy random variables instead of common random variables.

In the literature, there are some attempts to develop fuzzy control charts. Bradshaw [3] used the fuzzy set theory as a basis for interpreting the representation of a graded degree of product conformance. Bradshaw [3] stressed that fuzzy economic control limits would be advantageous over traditional acceptance charts, and that fuzzy economic control charts provide information on the severity, as well as the frequency, of product nonconformance. Raz and Wang [4] proposed an approach based on the fuzzy set theory by assigning a fuzzy set to each linguistic term. Wang and Raz [5] developed two approaches called the fuzzy probabilistic approach and the membership approach. In the fuzzy probabilistic approach, fuzzy subsets associated to the linguistic terms are transformed into their respective representative values using one of the transformation methods. Centerline (CL) corresponds to the arithmetic mean of representative values of the samples initially available. On the other hand, the membership approach is based on the fuzzy set theory to combine all observations in only one fuzzy subset using fuzzy arithmetic. In this approach, the center line is located as the representative value of the aggregate fuzzy subset. Kanagawa et al. [6] introduced an approach aimed at directly controlling the underlying probability distributions of the linguistic data, which were not considered by Wang and Raz [5]. They presented new linguistic control charts for process average and process variability, based on estimation of the probability distribution existing behind linguistic data. They defined the center line as the average mean of the sample cumulants, and then calculated the control limits using the Gram-Charlier series. Statistical and fuzzy control charts were reviewed by Woodall et al. [7] based on categorical data. Taleb and Limam [8] discussed different procedures of constructing control charts for linguistic data based on fuzzy set and probability theories. Using real data and based on average run length and samples under control, they compared fuzzy and probabilistic approaches. Cheng [9] proposed the following approach to deal with expert subjective judgments. Based on the rating scores assigned by individual inspectors to inspected items, fuzzy numbers are constructed to represent the vague outcomes of the process. Then fuzzy control charts are constructed directly from these fuzzy numbers, thereby retaining the fuzziness of the original vague observations. The out of control conditions are formulated using a possibility theory. Engin et al. [10] combined fuzzy sets with genetic algorithms to determine sample sizes in attribute control charts. Gulbay and Kahraman [11] developed a direct fuzzy approach to fuzzy control charts without any defuzzification and then defined unnatural pattern rules based on the defuzzification of crisp rules. They calculated the probability of each fuzzy unnatural pattern using the probability of fuzzy events. Gulbay and Kahraman [12] developed a fuzzy approach to control charts, based on different fuzzy transformation methods, and proposed α-cut approach to determine the tightness of inspection. El-Shal and Morris [13] investigated the use of fuzzy logic to modify statistical process control rules. They aimed at reducing the generation of false alarms and improving the detection and detection-speed of real faults. Rowlands and Wang [14] explored the integration of fuzzy logic and control charts to create and design a fuzzy-SPC evaluation and control method, based on the application of fuzzy logic to the SPC zone rules. Tannock [15] presented an approach to special cause detection for the individual control charts using fuzzy logic. Zarandi et al. [16] developed a new hybrid method, based on a combination of fuzzified sensitivity criteria and fuzzy adaptive sampling rules, to determine the sample size and sample interval of the control charts. Sentruk and Erginel [17] first transformed traditional $\bar{X} - R$ and $\bar{X} - S$ control charts to fuzzy control charts and developed α-cut fuzzy $\tilde{\bar{X}} - \tilde{R}$ and $\tilde{\bar{X}} - \tilde{S}$ control charts using the α-cut approach. To determine the condition of the process, they used α-cut fuzzy midranges of control limits and compared them with α-cut fuzzy midranges of the fuzzy mean and the fuzzy range (or variance) of samples. Amirzadeh et al. [18] proposed a new p-chart controlling the degree of nonconformity indicated by a fuzzy set whose support set is [L,U], where L and U denote lower and upper specification limits for respective quality characteristics, respectively. They showed that the proposed control chart is sensitive not only to changes in the mean of the process, but also to changes in the variance, and is more powerful than the traditional p-chart.

In the case of multivariate control charts, Taleb et al. [19] introduced a multivariate attribute control chart in fuzzy environment, based on the method proposed by Wang and Raz [5], i.e. in the fuzzy approach, they first assigned a membership function to each linguistic term, defuzzified the obtained fuzzy numbers, and then applied a common multivariate control chart to the defuzzified data for monitoring the process. Alipour and Noorossana [20] developed a fuzzy multivariate EWMA control chart in the same way as suggested by Taleb et al. [19]. They applied the multivariate EWMA approach to defuzzified data. However, it should be noted that defuzzification of fuzzy data results in losing precious information.

In spite of all aforementioned efforts toward considering control charts in a fuzzy environment, there is no study attempting to develop multivariate variable control charts in fuzzy mode. The current paper endeavors to evolve this significant class of control charts in a fuzzy environment. Toward accomplishing this goal, the likelihood ratio test should be applied fuzzily, because multivariate variable control charts are constructed based on this test. In this way, membership functions of the likelihood ratio statistics used to control the mean and dispersion of the process can be computed and, based on these membership functions, it is possible to obtain membership degrees of in and out of control states of the process mean and dispersion. Therefore, contrary to the classic multivariate variable control charts, clustering the process into just two states, i.e. in control and out of control, the process can be considered in several intermediate states, based on the computed membership degrees. Hence, the proposed approach poses two advantages: (1) Contrary to some previous research, it does not need to defuzzify the fuzzy observations and (2) the process state is viewed as a fuzzy set. The latter usefulness leads to more flexibility in the process analysis.

The paper is organized as follows. The following section presents some preliminaries about fuzzy random variable and fuzzy sets. After that, a brief description of multivariate control charts in a crisp environment is given. Subsequently, fuzzy likelihood ratio test is clarified concisely. Then fuzzy multivariate control charts are presented and the general obtained models are specialized to a special case, i.e., a process with two correlated quality characteristics.

PRELIMINARIES

Fuzzy Numbers

If $\tilde{A}$ is a fuzzy set with membership function $\mu_{\tilde{A}}(x)$, and $\tilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\}$ is its α-level cut, the following relationship is known as Resolution Identity [21-23]:

$$\mu_{\tilde{A}}(x) = \sup I_{\tilde{A}_\alpha}(x), \qquad (1)$$

where:

$$I_{\tilde{A}_\alpha}(x) = \begin{cases} 0 & ; x \notin \tilde{A}_\alpha \\ 1 & ; x \in \tilde{A}_\alpha \end{cases} \qquad (2)$$

$\tilde{A}$ is called a bounded fuzzy number, if it is a fuzzy number and the support set of its membership function is bounded. Wu [24] proved that if $\tilde{A}$ is a fuzzy number, then its α-level cut is a closed interval for all $\alpha \in [0,1]$ denoted by $\tilde{A}_\alpha = [\tilde{A}_\alpha^L, \tilde{A}_\alpha^U]$. $\tilde{A}$ is a canonical fuzzy number, if it is a bounded fuzzy number and its membership function is strictly increasing in

interval $[\tilde{A}_0^L, \tilde{A}_1^L]$ decreasing in interval $[\tilde{A}_1^U, \tilde{A}_0^U]$. If $\tilde{A}$ is a canonical fuzzy number, it can be concluded from its strict monotonicity that $g(\alpha)$ and $h(\alpha)$ are both continuous where $g(a) = \tilde{A}_\alpha^L$ and $h(a) = \tilde{A}_\alpha^U$ [24].

Fuzzy Random Variable

Fuzzy random variables represent a well-formulated concept underlying many recent probabilistics and statistics, involving data obtained from a random experiment when these data are assumed to be fuzzy set valued. Fuzzy random variables have been taken into account in the setting of a random experiment in order to model [25]:

• Either a fuzzy observation of a mechanism associating a real value with each experiment outcome.

• Or an essentially fuzzy-valued mechanism, that is a mechanism associating a fuzzy value with each experiment outcome.

Kwakernaak [26,27] introduced a mathematical model for the first situation elaborated later by Kruse and Mayer [28]. On the other hand, Puri and Ralescu [29] gave another approach to the second situation to be modeled.

In this paper, the first situation is applied to model fuzziness of observations in each sample, where a fuzzy random variable is viewed as a fuzzy perception/observation of a classical real-valued random variable, stated as follows:

Definition 1

Given probability space (Ω, A, P), mapping $x : \Omega \to \mathbf{F}(R)$ is said to be fuzzy random variable if for all $\alpha \in [0,1]$, the two real-valued mappings $x_\alpha^L \to R$ and $x_\alpha^U \to R$ are real-valued random variables, where $\tilde{x}_\alpha = [x_\alpha^L, x_\alpha^U]$ and $\mathbf{F}(R)$ is the set of all fuzzy numbers.

Fuzzy Matrix

The term 'fuzzy matrix' has at least two distinct definitions in the literature:

• In the first definition, $A = [a_{ij}]_{m \times n}$ is supposed to be a fuzzy matrix if $a_{ij} \in [0,1]$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. This class of fuzzy matrix emerges with fuzzy relations and initially was introduced by Kim and Roush [30].

• In the second definition, $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$ is called a fuzzy matrix if each of its entries is a fuzzy number [31,32]. Due to mathematical complexity, a few studies have been done on this class of fuzzy matrices.

In the current paper, the second definition of a fuzzy matrix is noticed, where each drawn sample of the process is assumed as a matrix of canonical fuzzy numbers.

MULTIVARIATE QUALITY CONTROL

Multivariate statistical quality control is applied to monitor a process in which p correlated quality characteristics should be simultaneously controlled for each item. The characteristics are assumed to follow p-variable normal distribution with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$.

When there is only one characteristic, the statistical hypothesis, 'the process is in control', is tested by Shewhart control charts in each sampling. If a point falls within the control limits, this hypothesis is accepted and otherwise is rejected. In this case, the regions above UCL and below LCL correspond to the critical region of the likelihood ratio test [33]. This standpoint is exploited to construct multivariate variable control charts.

Multivariate Control Charts for the Mean

Suppose $y_1, \ldots, y_p$ are p quality characteristics following p-variable normal distribution. Using random sample $X_1, \ldots, X_n$, where $X_j; j = 1, \ldots, n$ are $p \times 1$ vectors, the likelihood ratio test rejects null hypothesis '$H : \boldsymbol{\mu} = \boldsymbol{\mu}_0$' against alternative hypothesis '$K : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$' if [34,35]:

$$T^2 \geq T_0^2, \tag{3}$$

where:

$$T^2 = n(\bar{x} - \mu_0)' S^{-1}(\bar{x} - \mu_0), \tag{4}$$

$$T_0^2 = ((n-1)p/(n-p)) F_{p,n-p}(\delta), \tag{5}$$

and $F_{p,n-p}$ is a random variable following F distribution with p and $n-p$ degrees of freedom.

ALT [33] proposed two distinct phases to construct multivariate variable control charts based on the T^2 statistic. In the first phase, using m initial samples of size n, control charts are applied in order to test whether the process was in control when these samples were being drawn. To do this, the process mean vector and variance-covariance matrix should be estimated. If X_t is a $p \times n$ matrix of the observations of the tth sample, $t = 1, \ldots, m$, then unbiased estimators of the process mean vector and variance-covariance matrix are computed, respectively, as follows:

$$\bar{\bar{X}} = \sum_{t=1}^{m} \bar{X}_t/m, \tag{6}$$

$$\bar{S} = \sum_{t=1}^{m} S_t/m, \tag{7}$$

where $\bar{X}_t$ and S_t are the mean vector and variance-covariance matrix of the tth sample, respectively. The process stability in this phase is tested by plotting the values of $T_t^2 = n(\bar{x}_t - \bar{\bar{x}})' \bar{S}^{-1}(\bar{x}_t - \bar{\bar{x}}); t = 1, \cdots, m$, on a control chart with the following limits:

$$\begin{cases} UCL =(p(m-1)(n-1)/n(mn-m-p+1)) \\ \qquad F_{p,mn-m-p+1}(\delta) \\ LCL = 0 \end{cases} \tag{8}$$

The second phase is applied to the future performance of the process. If $\bar{X}_f$ is a $p \times 1$ vector of future sample mean, the process stability is tested by plotting the values of $T_f^2 = n(\bar{x}_f - \bar{\bar{x}})' \bar{S}^{-1}(\bar{x}_f - \bar{\bar{x}})$ on a control chart with the following limits:

$$\begin{cases} UCL =(p(m+1)(n-1)/n(mn-m-p+1)) \\ \qquad F_{p,mn-m-p+1}(\delta) \\ LCL = 0 \end{cases} \tag{9}$$

Multivariate Control Charts for the Process Dispersion

The dispersion of a multivariate variable process can be estimated based on various statistics. One of such statistics is the likelihood ratio statistic to test the null hypothesis '$H : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$' against the alternative hypothesis '$K : \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$'. In this case, Anderson [35] showed the null hypothesis is rejected if:

$$W_t > \chi^2_{p(p+1)/2}, \tag{10}$$

where:

$$W_t = - pn + pn Ln(n) - n Ln\left(|A_t|/|\sum_0|\right)$$

$$+ tr\left(\sum_0^{-1} A_t\right), \tag{11}$$

and $A_t = (n-1)S_t$. S_t is variance-covariance matrix of the tth sample and $\chi^2_{p(p+1)/2}$ is a random variable following χ^2 distribution with a $p(p+1)/2$ degree of freedom. If $\boldsymbol{\Sigma}_0$ is unknown, it can be estimated by Relationship 7. From the control chart's point of view, it can be stated that process dispersion is in control whenever a point is plotted between the following control limits:

$$\begin{cases} UCL = \chi^2_{p(p+1)/2} \\ LCL = 0 \end{cases} \tag{12}$$

To achieve a more comprehensive review on multivariate statistical quality control, the interested reader is referred to [33,36].

FUZZY LIKELIHOOD RATIO TEST

As presented in the previous section, multivariate variable control charts are constructed based on a likelihood ratio test. Thus, to develop them in fuzzy environment, it is necessary to develop the fuzzy likelihood ratio test. In this section, the likelihood ratio test is briefly described in the fuzzy environment, where each observation is assumed to be a canonical fuzzy number.

If fuzzy random sample $\tilde{X}_1, \cdots, \tilde{X}_n$ is applied to test fuzzy null hypothesis '$H : \theta$ belongs approximately to set w' against fuzzy alternative hypothesis '$K : \theta$ does not belong approximately to set w', where each $\tilde{X}_i; i = 1, \cdots, n$ is a canonical fuzzy number, then $\tilde{\Lambda}$, i.e. the fuzzy likelihood ratio statistic, is a fuzzy random variable whose α-level cut can be computed as follows [37]:

$$\tilde{\Lambda}_\alpha = [\min\{ \min_{\alpha \leq \beta \leq 1} \Lambda'(\tilde{x}_\beta^L), \min_{\alpha \leq \beta \leq 1} \Lambda'(\tilde{x}_\beta^U)\},$$

$$\max\{ \max_{\alpha \leq \beta \leq 1} \Lambda'(\tilde{x}_\beta^L), \max_{\alpha \leq \beta \leq 1} \Lambda'(\tilde{x}_\beta^U)\}], \qquad (13)$$

where:

$$\Lambda'(\tilde{x}_\beta^L) = \{\Lambda(\tilde{x}_\beta^L, \theta) | \theta \in (\theta_\alpha^L, \theta_\alpha^U)\}, \qquad (14)$$

$$\Lambda'(\tilde{x}_\beta^U) = \{ \Lambda(\tilde{x}_\beta^U, \theta) | \theta \in (\theta_\alpha^L, \theta_\alpha^U)\}. \qquad (15)$$

Since $\tilde{\Lambda}$ is fuzzy random variable, $\tilde{\Lambda}_\alpha^L$ and $\tilde{\Lambda}_\alpha^U$ are both common random variables with the same density function as Λ for each $\alpha \in [0, 1]$. In a fuzzy environment, critical value k should be determined in such a way that:

$$pr(\tilde{\Lambda} \leq k | \theta \text{ is } H(\theta)) = \delta. \qquad (16)$$

Since an α-level cut of $\tilde{\Lambda}$ results in the crisp interval $[\tilde{\Lambda}_\alpha^L, \tilde{\Lambda}_\alpha^U]$, the above probability is turned out as follows in any specific α-level [37]:

$$p(\tilde{\Lambda}_\alpha^U \leq k) = \delta \text{ when } \theta \in [\theta_\alpha^L, \theta_\alpha^U]. \qquad (17)$$

So, it can be inferred that the critical value of the test in a fuzzy environment is the same as that in a crisp environment. After computing the α-level cuts of $\tilde{\Lambda}$, its membership function can be easily obtained. Without loss of generality, and just for simplicity, suppose the membership function of $\tilde{\Lambda}$ is triangular shaped, as depicted in Figures 1 and 2, where the vertical line specifies the critical value of the test. In each figure, two α-levels have been determined; α_0 and α_k levels on Figure 1 and α_0' and α_k' levels on Figure 2.

In Figure 1, since $\tilde{\Lambda}_{\alpha_0} \leq k$, the null hypothesis is rejected in this level. So, based on the definition of *Resolution Identity*, α_k is the greatest value for which
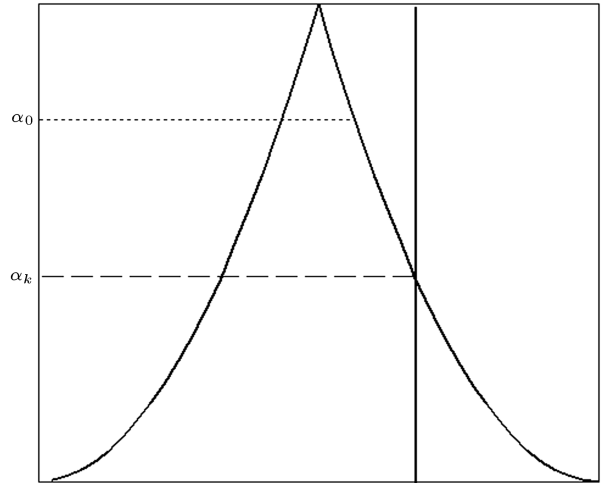

Figure 1. The cross of the critical value with right hand side of the membership function.
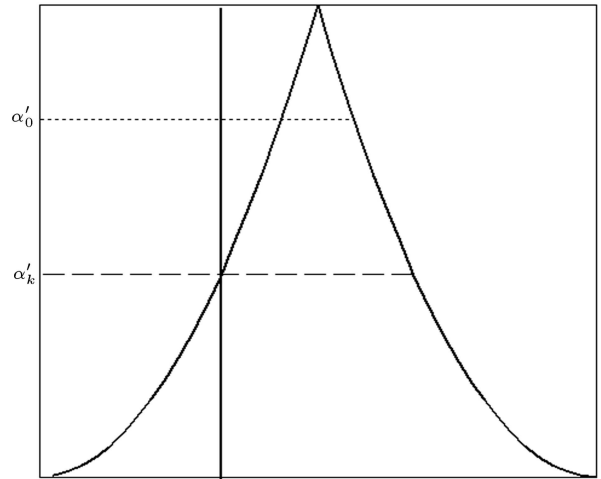

Figure 2. The cross of the critical value with left hand side of the membership function.

the null hypothesis is not rejected, because $\tilde{\Lambda}_\alpha$ is not less than k for $\alpha \in [0, \alpha_k]$. In other words, for any α-level greater than α_k, the null hypothesis is rejected. So whenever the vertical line corresponding to the critical value crosses the right hand side of the membership function of $\tilde{\Lambda}$, the membership degree of null hypothesis acceptance is equal to α_k and the membership degree of null hypothesis rejection (alternative hypothesis acceptance) is equal to 1. Now suppose α_0' and α_k' levels on Figure 2. Since $\tilde{\Lambda}_{\alpha_0'} \geq k$, the null hypothesis is accepted at such a level. So based on the definition of *Resolution Identity*, α_k' is the greatest value of $\alpha \in [0, 1]$ for which the null hypothesis is rejected. In other words, for any α-level greater than α_k', the null hypothesis is accepted. So whenever the vertical line corresponding to the critical value crosses the left hand side of the membership function of $\tilde{\Lambda}$, the membership degree of null hypothesis rejection is equal to α_k', and

the membership degree of null hypothesis acceptance is equal to 1.

As mentioned in the previous section, the likelihood ratio test results in rejecting the null hypothesis, '$H : \boldsymbol{\mu} = \boldsymbol{\mu}_0$', against the alternative hypothesis, '$K : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$' if $T^2 \geq T_0^2$. Hence we should adopt the general approach described in the previous paragraph, which computes membership degrees of the null hypothesis acceptance and rejection, because in the general form of the likelihood ratio test, the null hypothesis is rejected when $\lambda < k$ (note the inequality sign of the likelihood ratio test in the multivariate variable control chart, which is $\geq$, and the inequality sign of the general form of the likelihood ratio test, which is $\leq$). Therefore, in this special case, if the vertical line corresponding to the critical value crosses the right hand side of the membership function of T^2 and W in the fuzzy environment, the membership degree of null hypothesis acceptance, i.e. in control state, is equal to 1, and the membership degree of null hypothesis rejection, i.e. the out of control state, is α_k. On the other hand, when the vertical line of critical value crosses the left hand side of membership functions, membership degrees of null hypothesis acceptance and rejection are equal to α'_k and 1, respectively.

FUZZY MULTIVARIATE CONTROL CHARTS

This section develops fuzzy multivariate variable control charts using an adopted fuzzy likelihood ratio test under the supposition that the observations in each sample are canonical fuzzy numbers with triangular membership functions, i.e. a $p \times n$ matrix with entries shown as canonical fuzzy numbers with triangular membership functions is obtained in each sampling. For instance, $\tilde{X}_t = [(x_{ij1t}, x_{ij2t}, x_{ij3t})]$ for $i = 1, \ldots, p$ and $j = 1, \ldots, n$, shows the tth sample, $t = 1, \ldots, m$, of size n drawn from a p-variable process, where each observation is represented as a triangular fuzzy number. In this case,

$$\tilde{\bar{X}}_t = [\tilde{\bar{x}}_{it}]_{p \times 1} = [(\bar{x}_{i1t}, \bar{x}_{i2t}, \bar{x}_{i3t})]_{p \times 1}, \tag{18}$$

is the $p \times 1$ fuzzy mean vector of the tth sample where:

$$\bar{x}_{irt} = \sum\nolimits_{j=1}^{n} x_{ijrt}/n,$$

$$i = 1, \cdots, p \quad r = 1, 2, 3. \tag{19}$$

On the other hand:

$$\tilde{S}_t = [\tilde{s}_{ikt}]_{p \times p} = [(s_{ik1t}, s_{ik2t}, s_{ik3t})]_{p \times p};$$

$$i = 1, \cdots, p, \quad k = 1, \cdots, p, \tag{20}$$

is the $p \times p$ fuzzy variance-covariance matrix of the tth

sample, $t = 1, \ldots, m$, where if $i = k$, then:

$$\tilde{s}_{iit} = \frac{1}{n-1} \sum\nolimits_{j=1}^{n} \left[(x_{ij1t}, x_{ij2t}, x_{ij3t}) \right.$$
$$\left. - (\bar{x}_{i1t}, \bar{x}_{i2t}, \bar{x}_{i3t}) \right]^2. \tag{21}$$

And, otherwise (if $i \neq k$):

$$\tilde{s}_{ikt} = \frac{1}{n-1} \sum\nolimits_{j=1}^{n} [(x_{ij1t}, x_{ij2t}, x_{ij3t})$$
$$- (\bar{x}_{i1t}, \bar{x}_{i2t}, \bar{x}_{i3t})][(x_{kj1t}, x_{kj2t}, x_{kj3t})$$
$$- (\bar{x}_{k1t}, \bar{x}_{k2t}, \bar{x}_{k3t})]. \tag{22}$$

In this case, based on the study of Wang [38], fuzzy unbiased estimators of the process mean vector and variance-covariance matrix can be computed, respectively, as follows:

$$\tilde{\bar{\bar{X}}} = [\tilde{\bar{\bar{x}}}_i]_{p \times 1} = [(\bar{\bar{x}}_{i1}, \bar{\bar{x}}_{i2}, \bar{\bar{x}}_{i3})]_{p \times 1}, \tag{23}$$

$$\tilde{\bar{S}} = [\tilde{\bar{s}}_{ik}]_{p \times p} = [(\bar{s}_{ik1}, \bar{s}_{ik2}, \bar{s}_{ik3})]_{p \times p}, \tag{24}$$

where:

$$\bar{\bar{x}}_{ir} = \sum\nolimits_{t=1}^{m} \bar{x}_{irt}/m, \quad r = 1, 2, 3, \tag{25}$$

$$\bar{s}_{ikr} = \sum\nolimits_{t=1}^{m} s_{ikrt}/m, \quad r = 1, 2, 3. \tag{26}$$

Since the following relationship holds true in a crisp environment:

$$T_t^2 = n(\bar{x}_t - \bar{\bar{x}})' \bar{S}^{-1} (\bar{x}_t - \bar{\bar{x}})$$
$$= \sum\nolimits_{i=1}^{p} \sum\nolimits_{k=1}^{p} (\bar{x}_i - \bar{\bar{x}}_i) \bar{s}'_{ik} (\bar{x}_k - \bar{\bar{x}}_k), \tag{27}$$

where $\bar{s}'_{ik}$ is the (i, k) entry in the inverse matrix of $\bar{S}$, i.e. $\bar{S}^{-1} = [\bar{s}'_{ik}]_{p \times p}$. Based on what was mentioned previously, the following four non-linear programming problems should be firstly optimized for $\alpha \in [0, 1]$ in order to compute the α-level cuts of T_t^2 in the fuzzy environment:

I) $\min \sum\limits_{i=1}^{p} \sum\limits_{k=1}^{p} (\bar{x}_{i1t} + \beta(\bar{x}_{i2t} - \bar{x}_{i1t}) - \bar{\bar{x}}_i)$

$$\bar{s}'_{ik}(\bar{x}_{k1t} + \beta(\bar{x}_{k2t} - \bar{x}_{k1t}) - \bar{\bar{x}}_k), \tag{28}$$

s.t.

$\alpha \leq \beta \leq 1,$

$(\tilde{\bar{\bar{x}}}_i)_\alpha^L \leq \bar{\bar{x}}_i \leq (\tilde{\bar{\bar{x}}}_i)_\alpha^U, \quad i = 1, \cdots, p,$

$(\tilde{\bar{\bar{x}}}_k)_\alpha^L \leq \bar{\bar{x}}_k \leq (\tilde{\bar{\bar{x}}}_k)_\alpha^U, \quad k = 1, \cdots, p,$

$(\tilde{\bar{s}}_{ik})_\alpha^L \leq \bar{s}_{ik} \leq (\tilde{\bar{s}}_{ik})_\alpha^U.$

II) $\min \sum_{i=1}^{p} \sum_{k=1}^{p} (\bar{x}_{i3t} - \beta(\bar{x}_{i3t} - \bar{x}_{i2t}) - \bar{\bar{x}}_i)\bar{s}'_{ik}$

$$(\bar{x}_{k3t} - \beta(\bar{x}_{k3t} - \bar{x}_{k2t}) - \bar{\bar{x}}_k), \qquad (29)$$

s.t.

$\alpha \le \beta \le 1,$

$(\bar{\bar{x}}_i)_\alpha^L \le \bar{\bar{x}}_i \le (\bar{\bar{x}}_i)_\alpha^U; \quad i = 1, \cdots, p,$

$(\bar{\bar{x}}_k)_\alpha^L \le \bar{\bar{x}}_k \le (\bar{\bar{x}}_k)_\alpha^U; \quad k = 1, \cdots, p,$

$(\bar{\bar{s}}_{ik})_\alpha^L \le \bar{s}_{ik} \le (\bar{\bar{s}}_{ik})_\alpha^U.$

III) Substitute 'min' with 'max' in the problem I,

$$(30)$$

IV) Substitute 'min' with 'max' in the problem II.

$$(31)$$

Eventually, the α-level cuts of T_t^2 and consequently its membership function in the fuzzy environment, can be easily obtained using Relationship 13. Then based on the critical value, $(p(m-1)(n-1)/n(mn-m-p+1))F_{p,mn-m-p+1}(\delta)$, and whether this value crosses the right or left hand side of the membership function, the membership degrees of in control and out of control states of the process mean in phase I can be computed.

To calculate the membership degrees of in control and out of control states of the process mean in phase II, i.e. for future samples, the membership function of T_f^2 in the fuzzy environment should be firstly obtained applying the above four non-linear programming problems and Relationship 13. Then, these membership degrees are computed based on the critical value, $(p(m+1)(n-1)/n(mn-m-p+1))F_{p,mn-m-p+1}(\delta)$, and whether this critical value crosses the right or left hand side of the membership function.

On the other hand, since $A_t = (n-1)S_t$ and $\bar{S}$ is an unbiased estimator of Σ_0 in a crisp environment, Relationship 11 can be rewritten as follows:

$W_t = -pn + pnLn(n/(n-1))$

$$-n(Ln|S_t| - Ln|\bar{S}|) + (n-1)\sum_{i=1}^{p} \bar{s}'_{ii}s_{iit}, \quad (32)$$

where s_{iit} is the (i,i) entry of the variance-covariance matrix of the tth sample, and $\bar{s}'_{ii}$ is the (i,i) entry in the inverse matrix of $\bar{S}$.

In a fuzzy environment, the fuzzy variance-covariance matrix of the tth sample and unbiased estimator of the fuzzy variance-covariance matrix of the process are computed using Relationships 20 and

24, respectively. Now, based on what was mentioned previously, the following four non-linear programming problems should be firstly optimized for $\alpha \in [0,1]$ in order to obtain α-level cuts of W_t in the fuzzy environment:

I) $\min -pn + pnLn(n/(n-1)) - n[Ln|(\tilde{S}_\beta^L)_t| - Ln|\bar{S}|]$

$$+ (n-1)\sum_{i=1}^{p} \bar{s}'_{ii}(s_{ii1t} + \beta(s_{ii2t} - s_{ii1t})), \quad (33)$$

s.t.

$\alpha \le \beta \le 1,$

$(\bar{\bar{s}}_{ik})_\alpha^L \le \bar{s}_{ik} \le (\bar{\bar{s}}_{ik})_\alpha^U,$

$i = 1, \cdots, p, \quad k = 1, \cdots, p.$

II) $\min -pn + pnLn(n/(n-1)) - n[Ln|(\tilde{S}_\beta^U)_t| - Ln|\bar{S}|]$

$$+ (n-1)\sum_{i=1}^{p} \bar{s}'_{ii}(s_{ii3t} - \beta(s_{ii3t} - s_{ii2t})), \quad (34)$$

s.t.

$\alpha \le \beta \le 1,$

$(\bar{\bar{s}}_{ik})_\alpha^L \le \bar{s}_{ik} \le (\bar{\bar{s}}_{ik})_\alpha^U;$

$i = 1, \cdots, p, \qquad k = 1, \cdots, p.$

III) Substitute 'min' with 'max' in the problem I,

$$(35)$$

IV) Substitute 'min' with 'max' in the problem II.

$$(36)$$

If the β-level cut of the fuzzy matrix $\tilde{S}_t$ is represented as $(\tilde{S}_t)_\beta$, that is:

$(\tilde{S}_t)_\beta = \big[[s_{ik1t} + \beta(s_{ik2t} - s_{ik1t}), s_{ik3t}$

$$- \beta(s_{ik3t} - s_{ik2t})]\big]_{p \times p}, \qquad (37)$$

for $i = 1, \ldots, p, k = 1, \ldots, p$ and $t = 1, \ldots, m$, then:

$$(\tilde{S}_\beta^L)_t = [s_{ik1t} + \beta(s_{ik2t} - s_{ik1t})]_{p \times p}, \qquad (38)$$

$$(\tilde{S}_\beta^U)_t = [s_{ik3t} - \beta(s_{ik3t} - s_{ik2t})]_{p \times p}. \qquad (39)$$

It is worth noting that the β-level cut of the fuzzy matrix $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$, is obtained as $\tilde{A}_\beta = [(\tilde{a}_{ij})_\beta]_{m \times n}$. Eventually, the α-level cuts of W_t are computed applying Relationship 13. Using these α-level cuts, the membership function of W_t in the fuzzy environment can be easily obtained. Membership degrees of in control and out of control states of the process dispersion are computed based on the critical value $\chi^2_{p(p+1)/2}$ and whether this critical value crosses the right or left hand side of the membership function.

NUMERICAL EXAMPLE

This section considers a bivariate process in the fuzzy environment and formulizes non-linear programming problems given in the previous section for such a process. To do this, the 2×1 fuzzy mean vector of the tth sample is computed applying Relationships 18 and 19, as follows:

$$\tilde{\bar{X}}_t = \begin{bmatrix} (\bar{x}_{11t}, \bar{x}_{12t}, \bar{x}_{13t}) \\ (\bar{x}_{21t}, \bar{x}_{22t}, \bar{x}_{23t}) \end{bmatrix}. \tag{40}$$

Using Relationships 20, 21 and 22, the 2×2 fuzzy variance-covariance matrix of such a sample is obtained as follows:

$$\tilde{S}_t = \begin{bmatrix} (s_{111t}, s_{112t}, s_{113t}) & (s_{121t}, s_{122t}, s_{123t}) \\ (s_{211t}, s_{212t}, s_{213t}) & (s_{221t}, s_{222t}, s_{223t}) \end{bmatrix}. \tag{41}$$

Now, applying Relationships 23 and 24, unbiased estimators of the fuzzy mean vector and fuzzy variance-covariance matrix of the process are computed, respectively, as follows:

$$\tilde{\bar{\bar{X}}} = \begin{bmatrix} (\bar{\bar{x}}_{11}, \bar{\bar{x}}_{12}, \bar{\bar{x}}_{13}) \\ (\bar{\bar{x}}_{21}, \bar{\bar{x}}_{22}, \bar{\bar{x}}_{23}) \end{bmatrix}, \tag{42}$$

$$\tilde{\bar{S}} = \begin{bmatrix} (\bar{s}_{111}, \bar{s}_{112}, \bar{s}_{113}) & (\bar{s}_{121}, \bar{s}_{122}, \bar{s}_{123}) \\ (\bar{s}_{211}, \bar{s}_{212}, \bar{s}_{213}) & (\bar{s}_{221}, \bar{s}_{222}, \bar{s}_{223}) \end{bmatrix}. \tag{43}$$

On the other hand, if the crisp matrix $\tilde{S}$ is defined as:

$$\bar{S} = [\bar{s}_{ik}]_{2 \times 2} = \begin{bmatrix} \bar{s}_{11} & \bar{s}_{12} \\ \bar{s}_{21} & \bar{s}_{22} \end{bmatrix}. \tag{44}$$

Then its inverse is as follows:

$$\bar{S}^{-1} = [\bar{s}'_{ik}]_{2 \times 2} = \begin{bmatrix} \bar{s}'_{11} & \bar{s}'_{12} \\ \bar{s}'_{21} & \bar{s}'_{22} \end{bmatrix}, \tag{45}$$

where $\bar{s}'_{11} = (\bar{s}_{22}/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}))$, $\bar{s}'_{12} = (-\bar{s}_{12}/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}))$, $\bar{s}'_{21} = (-\bar{s}_{21}/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}))$ and $\bar{s}'_{22} = (\bar{s}_{11}/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}))$. In this case, the four non-linear programming Problems 28-31, presented in the previous section to compute the α-level cuts of T_t^2 in the fuzzy environment, are rewritten as follows:

I) $\min(\bar{x}_{11t} - \beta(\bar{x}_{12t} - \bar{x}_{11t}) - \bar{\bar{x}}_1)^2$

$(\bar{s}_{22}/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}))$

$+(\bar{x}_{21t} + \beta(\bar{x}_{22t} - \bar{x}_{21t}) - \bar{\bar{x}}_2)^2$

$(\bar{s}_{11}/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}))$

$-(\bar{x}_{11t} - \beta(\bar{x}_{12t} - \bar{x}_{11t}) - \bar{\bar{x}}_1)$

$(\bar{x}_{21t} + \beta(\bar{x}_{22t} - \bar{x}_{21t}) - \bar{\bar{x}}_2)$

$(\bar{s}_{12} + \bar{s}_{21})/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}), \tag{46}$

s.t.

$\alpha \le \beta \le 1,$

$\bar{\bar{x}}_{11} + \alpha(\bar{\bar{x}}_{12} - \bar{\bar{x}}_{11}) \le \bar{\bar{x}}_1 \le \bar{\bar{x}}_{13} - \alpha(\bar{\bar{x}}_{13} - \bar{\bar{x}}_{12}),$

$\bar{\bar{x}}_{21} + \alpha(\bar{\bar{x}}_{22} - \bar{\bar{x}}_{21}) \le \bar{\bar{x}}_2 \le \bar{\bar{x}}_{23} - \alpha(\bar{\bar{x}}_{23} - \bar{\bar{x}}_{22}),$

$\bar{s}_{111} + \alpha(\bar{s}_{112} - \bar{s}_{111}) \le \bar{s}_{11} \le \bar{s}_{113} - \alpha(\bar{s}_{113} - \bar{s}_{112}),$

$\bar{s}_{121} + \alpha(\bar{s}_{122} - \bar{s}_{121}) \le \bar{s}_{12} \le \bar{s}_{123} - \alpha(\bar{s}_{123} - \bar{s}_{122}),$

$\bar{s}_{211} + \alpha(\bar{s}_{212} - \bar{s}_{211}) \le \bar{s}_{21} \le \bar{s}_{213} - \alpha(\bar{s}_{213} - \bar{s}_{212}),$

$\bar{s}_{221} + \alpha(\bar{s}_{222} - \bar{s}_{221}) \le \bar{s}_{22} \le \bar{s}_{223} - \alpha(\bar{s}_{223} - \bar{s}_{222}),$

II) $\min(\bar{x}_{13t} - \beta(\bar{x}_{13t} - \bar{x}_{12t}) - \bar{\bar{x}}_1)^2$

$(\bar{s}_{22}/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}))$

$+(\bar{x}_{23t} - \beta(\bar{x}_{23t} - \bar{x}_{22t}) - \bar{\bar{x}}_2)^2$

$(\bar{s}_{11}/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}))$

$-(\bar{x}_{13t} - \beta(\bar{x}_{13t} - \bar{x}_{12t}) - \bar{\bar{x}}_1)$

$(\bar{x}_{23t} - \beta(\bar{x}_{23t} - \bar{x}_{22t}) - \bar{\bar{x}}_2)$

$(\bar{s}_{12} + \bar{s}_{21})/(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}), \tag{47}$

s.t.

$\alpha \le \beta \le 1,$

$\bar{\bar{x}}_{11} + \alpha(\bar{\bar{x}}_{12} - \bar{\bar{x}}_{11}) \le \bar{\bar{x}}_1 \le \bar{\bar{x}}_{13} - \alpha(\bar{\bar{x}}_{13} - \bar{\bar{x}}_{12}),$

$\bar{\bar{x}}_{21} + \alpha(\bar{\bar{x}}_{22} - \bar{\bar{x}}_{21}) \le \bar{\bar{x}}_2 \le \bar{\bar{x}}_{23} - \alpha(\bar{\bar{x}}_{23} - \bar{\bar{x}}_{22}),$

$\bar{s}_{111} + \alpha(\bar{s}_{112} - \bar{s}_{111}) \le \bar{s}_{11} \le \bar{s}_{113} - \alpha(\bar{s}_{113} - \bar{s}_{112}),$

$\bar{s}_{121} + \alpha(\bar{s}_{122} - \bar{s}_{121}) \le \bar{s}_{12} \le \bar{s}_{123} - \alpha(\bar{s}_{123} - \bar{s}_{122}),$

$\bar{s}_{211} + \alpha(\bar{s}_{212} - \bar{s}_{211}) \le \bar{s}_{21} \le \bar{s}_{213} - \alpha(\bar{s}_{213} - \bar{s}_{212}),$

$\bar{s}_{221} + \alpha(\bar{s}_{222} - \bar{s}_{221}) \le \bar{s}_{22} \le \bar{s}_{223} - \alpha(\bar{s}_{223} - \bar{s}_{222}),$

III) Substitute 'min' with 'max' in the problem I, $\tag{48}$

IV) Substitute 'min' with 'max' in the problem II. $\tag{49}$

Based on the definition of $\tilde{S}_t$, the crisp matrices $\tilde{S}_\beta^L$ and $\tilde{S}_\beta^U$ of the tth sample are computed as follows:

$$(\tilde{S}_{\beta}^{L})_t = \begin{bmatrix} s_{111t} + \beta(s_{112t} - s_{111t}) \\ s_{211t} + \beta(s_{212t} - s_{211t}) \end{bmatrix}$$

$$\begin{matrix} s_{121t} + \beta(s_{122t} - s_{121t}) \\ s_{221t} + \beta(s_{222t} - s_{221t}) \end{matrix}\Bigg], \qquad (50)$$

$$(\tilde{S}_{\beta}^{U})_t = \begin{bmatrix} s_{113t} - \beta(s_{113t} - s_{112t}) \\ s_{213t} - \beta(s_{213t} - s_{212t}) \end{bmatrix}$$

$$\begin{matrix} s_{123t} - \beta(s_{123t} - s_{122t}) \\ s_{223t} - \beta(s_{223t} - s_{222t}) \end{matrix}\Bigg]. \qquad (51)$$

Therefore, to compute the α-level cuts of W_t in the fuzzy environment, the four non-linear programming Problems 33-36, given in the previous section, are rewritten as follows:

I) $\min - pn + pn\,Ln(n/(n-1)) - n[Ln|(\tilde{S}_{\beta}^{U})_t|$

$\qquad - Ln(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21})]$

$\qquad + (n-1)(\bar{s}_{22}(s_{111t} + \beta(s_{112t} - s_{111t})$

$\qquad + \bar{s}_{11}(s_{221t} + \beta(s_{222t} - s_{221t}))$

$\qquad /(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}), \qquad (52)$

s.t.

$\bar{s}_{111} + \alpha(\bar{s}_{112} - \bar{s}_{111}) \le \bar{s}_{11} \le \bar{s}_{113} - \alpha(\bar{s}_{113} - \bar{s}_{112}),$

$\bar{s}_{121} + \alpha(\bar{s}_{122} - \bar{s}_{121}) \le \bar{s}_{12} \le \bar{s}_{123} - \alpha(\bar{s}_{123} - \bar{s}_{122}),$

$\bar{s}_{211} + \alpha(\bar{s}_{212} - \bar{s}_{211}) \le \bar{s}_{21} \le \bar{s}_{213} - \alpha(\bar{s}_{213} - \bar{s}_{212}),$

$\bar{s}_{221} + \alpha(\bar{s}_{222} - \bar{s}_{221}) \le \bar{s}_{22} \le \bar{s}_{223} - \alpha(\bar{s}_{223} - \bar{s}_{222}),$

II) $\min - pn + pn\,Ln(n/(n-1)) - n[Ln|(\tilde{S}_{\beta}^{U})_t|$

$\qquad - Ln(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21})]$

$\qquad + (n-1)(\bar{s}_{22}(s_{113t} - \beta(s_{113t} - s_{112t})$

$\qquad + \bar{s}_{11}(s_{223t} - \beta(s_{223t} - s_{222t}))$

$\qquad /(\bar{s}_{11}\bar{s}_{22} - \bar{s}_{12}\bar{s}_{21}), \qquad (53)$

s.t.

$\bar{s}_{111} + \alpha(\bar{s}_{112} - \bar{s}_{111}) \le \bar{s}_{11} \le \bar{s}_{113} - \alpha(\bar{s}_{113} - \bar{s}_{112}),$

$\bar{s}_{121} + \alpha(\bar{s}_{122} - \bar{s}_{121}) \le \bar{s}_{12} \le \bar{s}_{123} - \alpha(\bar{s}_{123} - \bar{s}_{122}),$

$\bar{s}_{211} + \alpha(\bar{s}_{212} - \bar{s}_{211}) \le \bar{s}_{21} \le \bar{s}_{213} - \alpha(\bar{s}_{213} - \bar{s}_{212}),$

$\bar{s}_{221} + \alpha(\bar{s}_{222} - \bar{s}_{221}) \le \bar{s}_{22} \le \bar{s}_{223} - \alpha(\bar{s}_{223} - \bar{s}_{222}),$

III) Substitute 'min' with 'max' in the problem I, $\qquad (54)$

IV) Substitute 'min' with 'max' in the problem II. $\qquad (55)$

Now, suppose a quality engineer is going to control a chemical process with two correlated quality characteristics using fuzzy samples of size 4. Table 1 presents fuzzy mean vectors and fuzzy variance-covariance matrices of 20 initial samples drawn in order to estimate the process mean vector and variance-covariance matrix. The fuzzy mean vector and fuzzy variance-covariance matrix of each sample are computed by Relationships 40 and 41, respectively.

Now, using Relationships 42 and 43, unbiased estimators of the fuzzy mean vector and fuzzy variance-covariance matrix of the process are computed as follows:

$$\bar{\bar{\tilde{X}}} = \begin{bmatrix} (262.43, 262.50, 262.55) \\ (437.00, 437.05, 437.11) \end{bmatrix}, \qquad (56)$$

$$\bar{\bar{\tilde{S}}} = \begin{bmatrix} (94.056, 119.342, 142.162) \\ (38.499, 55.207, 68.918) \end{bmatrix}$$

$$\begin{matrix} (38.499, 55.207, 68.918) \\ (70.825, 82.530, 94.589) \end{matrix}\Bigg]. \qquad (57)$$

Now, to calculate the α-level cuts of the T_t^2 statistic in the fuzzy environment, the non-linear programming Problems 46-49 should be firstly solved for $\alpha \in [0,1]$. These problems are solved for sample 10, as an example, to show how the α-level cuts of T_{10}^2 are obtained solving these problems and using Relationship 13.

Table 2 gives the optimal solution of these problems for various values of α obtained by Lingo 8 software. The last two columns of this table present end points of the α-level cuts of T_{10}^2 at the corresponding α-level obtained by Relationship 13.

The membership function of T_{10}^2 in the fuzzy environment can be easily obtained using these end points, as depicted by Figure 3, where a vertical line shows the critical value, $(2(19)(3)/(80 - 20 - 2 + 1))F_{2,(80-20-2+1)}(0.05) = 6.068$.

Now, based on what was mentioned previously, the membership degree of the in control state of the process mean is equal to 1, whereas the membership degree of the out of control state is 0.48. Following this procedure for all 20 samples, membership degrees of the in control and out of control states of the process mean can be obtained in each sampling. These membership degrees are presented in Table 3. In this table, M.D. stands for membership degree.

Suppose that according to a managerial decision, when the membership degree of the in control state of process mean is equal to 1, the process mean is assumed

Table 1. Fuzzy mean vectors and variance-covariance matrixes of 20 initial samples.

Sample No.	Fuzzy Mean Vector	Fuzzy Variance-Covariance Matrix	
1	(270.16, 270.22, 270.28)	(35.233, 66.649, 73.169)	(-12.723, 8.569, 27.067)
	(445.90, 445.99, 446.06)	(-12.723, 8.569, 27.067)	(20.985, 30.487, 57.548)
2	(259.33, 259.39, 259.44)	(102.953, 103.692, 29.012)	(69.566, 76.972, 77.666)
	(433.41, 433.46, 433.51)	(69.566, 76.972, 77.666)	(58.756, 59.293, 67.333)
3	(249.69, 249.73, 249.76)	(112.357, 171.261, 228.015)	(-28.037, 10.15, 36.015)
	(427.39, 427,44, 427.49)	(-28.037, 10.15, 36.015)	(78.584, 93.073, 102.368)
4	(270.05, 270.11, 270.16)	(58.998, 61.398, 76.126)	(49.942, 56.317, 60.005)
	(443.83, 443.89, 443.95)	(49.942, 56.317, 60.005)	(60.991, 64.541, 73.688)
5	(251.60, 251.67, 251.71)	(39.026, 68.296, 83.629)	(-2.374, 14.482, 25.647)
	(431.16, 431.21, 431.30)	(-2.374, 14.482, 25.647)	(56.319, 70.687, 84.584)
6	(259.59, 259.67, 259.72)	(143.866, 191.573, 208.111)	(58.398, 71.561, 86.684)
	(448.60, 448.64, 448.70)	(58.398, 71.561, 86.684)	(93.486, 136.7, 151.988)
7	(271.80, 271.91, 271.93)	(81.811, 138.33, 192.455)	(-16.333, 7.585, 45.338)
	(442.97, 443.01, 443.06)	(-16.333, 7.585, 45.338)	(27.33, 34.871, 51.367)
8	(266.91, 266.96, 267.01)	(78.998,86.322,96.114)	(71.395,84.867,90.199)
	(439.17, 439.23, 439.27)	(71.395,84.867,90.199)	(103.255,108.866,115.655)
9	(270.89, 270.93, 270.97)	(51.988, 60.504, 79.736)	(36.285, 45.293, 58.366)
	(434.00, 434.04, 434.11)	(36.285, 45.293, 58.366)	(67.342,81.305,94.876)
10	(250.37, 250.44, 250.49)	(119.555, 157.568, 180.111)	(-7.369, 15.989, 38.174)
	(428.96, 429.01, 429.06)	(-7.369, 15.989, 38.174)	(12.222, 26.793, 43.333)
11	(269.94, 270.00, 270.03)	(103.933, 112.729, 139.222)	(65.388, 88.026, 91.566)
	(436.47, 436.51, 436.58)	(65.388, 88.026, 91.566)	(80.998, 88.245, 101.355)
12	(256.12, 256.20, 256.25)	(138.333, 185.394, 202.366)	(46.364, 65.461, 90.366)
	(425.38, 425.43, 425.50)	(46.364, 65.461, 90.366)	(83.988, 102.207, 127.011)
13	(261.88, 261.92, 261.95)	(39.399, 87.43, 101.222)	(-2.365, 18.526, 42.842)
	(445.92, 445.95, 445.97)	(-2.365, 18.526, 42.842)	(63.111, 82.704, 97.355)
14	(263.00, 263.04, 263.09)	(171.333, 196.692, 221.385)	(11.958, 38.05, 42.088)
	(438.09, 438.14, 438.20)	(11.958, 38.05, 42.088)	(10.366, 14.549, 16.358)
15	(269.55, 269.60, 269.63)	(51.969, 56.073, 72.555)	(15.945, 23.13, 26.071)
	(438.37, 438.42, 438.45)	(15.945, 23.13, 26.071)	(13.924, 17.968, 22.196)
16	(259.75, 259.81, 259.86)	(94.995, 95.895, 108.943)	(95.954, 102.406, 103.01)
	(445.01, 445.08, 445.14)	(95.954, 102.406, 103.01)	(111.942, 112.542, 123.333)
17	(256.74, 256.82, 256.89)	(102.588, 124.51, 153.333)	(78.345, 88.06, 93.756)
	(430.90, 430.97, 431.05)	(78.345, 88.06, 93.756)	(86.369, 96.807, 102.888)
18	(261.70, 261.78, 261.83)	(138.999, 160.302, 191.563)	(88.411, 102.114, 129.655)
	(430.25, 430.34, 430.43)	(88.411, 102.114, 129.655)	(153.333, 177.14, 189.367)
19	(279.72, 279.81, 279.89)	(128.471, 139.995, 149.388)	(101.836, 116.32, 127.711)
	(445.76, 445.83, 445.90)	(101.836, 116.32, 127.711)	(127.557, 130.322, 136.887)
20	(249.86, 249.95, 250.01)	(86.322, 122.224, 156.781)	(49.396, 70.255, 85.955)
	(428.37, 428.41, 428.45)	(49.396, 70.255, 85.955)	(105.644, 121.5, 132.285)

to be 'completely in control' if the membership degree of the out of control state is less than 0.4. It is assumed to be 'relatively in control' if the membership degree of the out of control state is between 0.4 and 0.8, otherwise, it is assumed to be 'slightly in control'.

On the other hand, when the membership degree of the out of control state of process mean is equal to 1, the process mean is assumed to be 'slightly out of control' if the membership degree of the in control state is greater than 0.7, otherwise, it is supposed to be 'completely out of control'. These results are shown in Table 3.

The procedure of computing the membership function of W_t in the fuzzy environment is exactly similar to that of T_t^2 with this exception that the non-linear programming Problems 52-55 should be

Table 2. The optimal solution of non-linear Problems 48-51 for sample 10 and the end points of α-level cuts.

| α	Optimal Solutions of the Problem				The End Points of α-Level Cuts	
	i	ii	iii	$iiii$		
0.01	4.363	4.322	7.014	6.933	4.322	7.014
0.05	4.395	4.355	6.969	6.892	4.355	6.969
0.1	4.435	4.397	6.817	6.746	4.397	6.817
0.15	4.476	4.44	6.713	6.646	4.44	6.713
0.2	4.518	4.483	6.611	6.549	4.483	6.611
0.25	4.56	4.528	6.512	6.454	4.528	6.512
0.3	4.603	4.573	6.415	6.363	4.573	6.415
0.35	4.647	4.619	6.322	6.273	4.619	6.322
0.4	4.692	4.666	6.23	6.189	4.666	6.23
0.45	4.738	4.713	6.142	6.102	4.713	6.142
0.5	4.784	4.761	6.055	6.02	4.761	6.055
0.55	4.831	4.811	5.971	5.939	4.811	5.971
0.6	4.879	4.861	5.889	5.861	4.861	5.889
0.65	4.929	4.912	5.809	5.785	4.912	5.809
0.7	4.979	4.965	5.731	5.71	4.965	5.731
0.75	5.03	5.018	5.654	5.638	5.018	5.654
0.8	5.081	5.072	5.58	5.567	5.072	5.58
0.85	5.135	5.127	5.508	5.498	5.127	5.508
0.9	5.189	5.184	5.437	5.43	5.184	5.437
0.95	5.244	5.241	5.368	5.364	5.241	5.368
1	5.295	5.295	5.295	5.295	5.295	5.295
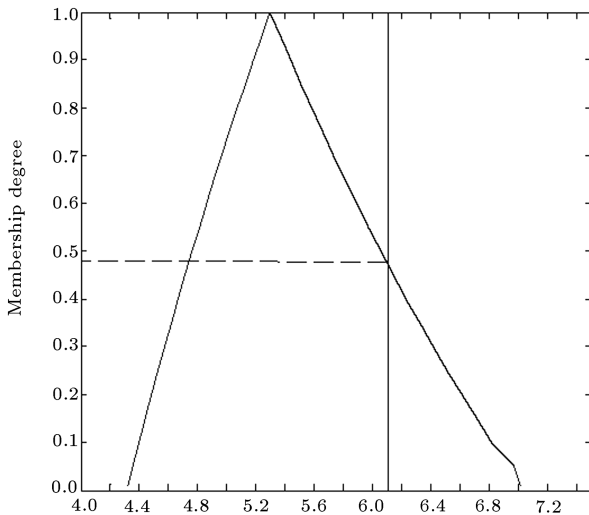
Figure 3. Membership function of T_{10}^2 in fuzzy environment.

optimized for $\alpha \in [0,1]$. Table 4 presents membership degrees of both in control and out of control states of process dispersion, where these membership degrees are computed using the critical value $\chi_3^2(0.05) = 7.81$. In this case, when the membership degree of the in control state of process dispersion is equal to 1, it is said that the process dispersion is 'completely in control', if the membership degree of the out of control state is, for example, less than 0.2. It is also said that process dispersion is 'relatively in control' if the membership degree of the out of control state is between 0.2 and 0.7, otherwise, process dispersion is said to be 'slightly in control'. On the other hand, when the membership degree of the out of control state is equal to 1, it is said that process dispersion is 'slightly out of control' if the membership degree of the in control state is greater than 0.8, otherwise, it is assumed that process dispersion is 'completely out of control'.

Finally, it is worth comparing the results of this paper with those obtained from the classic multivariate variable control charts. Table 5 presents values of the statistics T_t^2 and W_t, $t = 1,\ldots,20$, in the crisp environment, i.e. when $\alpha = 1$. The in control and out of control states of process mean and dispersion are determined using the critical values 6.068 and 7.81, respectively. As is obvious, the process state is classified to either being in control or out of control in the crisp environment, i.e. there is no difference between, for example, the samples 9 and 18 when controlling the process mean, whereas, in the fuzzy environment process, the mean is slightly in control

Table 3. Membership degrees of in control and out of control states of process mean.

Sample No.	M.D. of in Control State	M.D. of Out of Control State	State of Process Mean
1	1	< 0.01	Completely in control
2	1	< 0.01	Completely in control
3	0.75	1	Slightly out of control
4	1	< 0.01	Completely in control
5	1	< 0.01	Completely in control
6	< 0.01	1	Completely out of control
7	1	< 0.01	Completely in control
8	1	< 0.01	Completely in control
9	1	0.92	Slightly in control
10	1	0.48	Relatively in control
11	1	0.26	Completely in control
12	0.55	1	Completely out of control
13	1	0.95	Slightly in control
14	1	< 0.01	Completely in control
15	1	< 0.01	Completely in control
16	0.96	1	Slightly out of control
17	1	< 0.01	Completely in control
18	1	0.14	Completely in control
19	< 0.01	1	Completely out of control
20	1	0.84	Slightly in control

Table 4. Membership degrees of the in control and out of control states of process dispersion.

Sample No.	M.D. of in Control State	M.D. of Out of Control State	State of Process Dispersion
1	1	0.03	Completely in control
2	0.5	1	Completely out of control
3	1	0.41	Relatively in control
4	0.83	1	Slightly out of control
5	1	0.1	Completely in control
6	1	0.58	Relatively in control
7	1	0.27	Relatively in control
8	1	0.95	Slightly in control
9	1	0.24	Relatively in control
10	1	0.26	Relatively in control
11	1	0.89	Slightly in control
12	1	0.5	Relatively in control
13	1	0.19	Completely in control
14	0.86	1	Slightly out of control
15	0.97	1	Slightly out of control
16	0.22	1	Completely out of control
17	1	0.54	Relatively in control
18	1	0.74	Slightly in control
19	1	0.98	Slightly in control
20	1	0.45	Relatively in control

Table 5. The values of T_t^2 and W_t in crisp environment.

Sample No.	T_t^2	W_t	State of Process Mean	State of Process Dispersion
1	4.024	3.312	In control	In control
2	0.650	14.859	In control	Out of control
3	6.420	2.054	Out of control	In control
4	2.714	8.540	In control	Out of control
5	3.973	2.057	In control	In control
6	11.947	3.949	Out of control	In control
7	3.149	2.595	In control	In control
8	0.666	7.698	In control	In control
9	5.737	4.238	In control	In control
10	5.295	3.605	In control	In control
11	3.015	7.566	In control	In control
12	6.648	3.357	Out of control	In control
13	5.910	1.787	In control	In control
14	0.059	8.511	In control	Out of control
15	1.949	7.956	In control	Out of control
16	6.283	16.132	Out of control	Out of control
17	1.919	5.765	In control	In control
18	2.871	5.575	In control	In control
19	10.077	7.724	Out of control	In control
20	5.837	3.639	In control	In control

when taking sample 9 and is completely in control when drawing sample 18.

CONCLUSION

This paper develops multivariate variable control charts in fuzzy environment, i.e. it was assumed that each observation in each sample is a canonical fuzzy number with a triangular membership function. Since multivariate variable control charts are constructed using a likelihood ratio test, this test was introduced in a fuzzy environment, leading to compute membership degrees of both in control and out of control states of process mean and dispersion. Therefore, contrary to the classic multivariate variable control charts, one is able to consider the process in several intermediate states, such as 'completely in (out of) control', 'relatively in (out of) control' and 'slightly in (out of) control'. Moreover, it does not need to defuzzify data in the proposed approach. Hence, the approach presented in this paper brought about more flexibility in process analysis.

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BIOGRAPHIES

Hadi Moheb Alizadeh received a M.S. degree in Industrial Engineering from Tarbiat Moallem University, Tehran, in 2010. His research interests are: Statistical Quality Control, Design of Experiments, Measurement System Analysis and Supply Chain Management.

Ali Reza Arshadi Khamseh received a Ph.D. degree in Industrial Engineering from Amirkabir University of Technology, Tehran, in 2006. He is currently an Assistant Professor in the Department of Industrial Engineering at Tarbiat Moallem University in Tehran. His research and teaching interests are: Statistical Quality Control, Sampling Plans, Factory Design and Material Handling Problems.

Seyed Mohammad Taghi Fatemi Ghomi received a Ph.D. degree in Industrial Engineering from the University of Bradford, England, in 1980. He is currently a Professor in the Department of Industrial Engineering at Amirkabir University of Technology in Tehran. His research and teaching interests are: Stochastic Activity Networks, Production Planning and Control, Scheduling, Queuing Systems and Statistical Quality Control.