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Separated Continuous Linear Programs with Fuzzy Valued Objective Function

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Abstract. *Fuzzy linear programming problems can be used to model a wide variety of practical applications in which all or some decision parameters are stated in an imprecise fashion. These problems have been investigated and expanded by many researchers from various points of view. In this paper, we study a class of infinite-dimensional linear programming problems, so-called separated continuous linear programs with a fuzzy valued objective function. For this class of problem, we develop a strong duality result and present an approximation algorithm. The basic idea is to use the discretization technique to establish a relationship between the problem and an ordinary fuzzy linear programming problem.*

Keywords: *Continuous-time linear programming; Fuzzy linear programming; Discretization; Duality.*

INTRODUCTION

Linear programming is one of the most frequently applied operations research techniques and has applications in a wide range of fields, including economics, computer science, most branches of engineering, manufacturing, scheduling and routing, telecommunication, transportation and logistics etc. A crucial feature of linear programming occurring in real-world applications is that all or some of the parameters may be stated in an imprecise fashion. This characteristic is not captured by classical linear programming and in conventional models, parameters must be precise and well defined. However, in a real world environment, this is not a realistic assumption. Usually the value of many parameters of a linear programming model is estimated by experts. Clearly it cannot be assumed that the knowledge of experts is so precise.

The traditional way to handle the uncertain parameters of a linear programming model is to perform post-optimization analysis or parametric programming. In this approach usually parameters are analyzed separately, which is not suitable for an overall analysis of the effect of imprecision in parameters. Therefore, since the single parameter sensitivity analysis is not appropriate when there are many uncertain parameters, other approaches such as robust optimization or stochastic programming are used in order to investigate the overall effect of all uncertain parameters simultaneously. One practical way is to express the uncertain parameters by fuzzy numbers. In this approach, although, again the knowledge of experts may be utilized, the parameters are not expressed by deterministic data. They are estimated in term of fuzzy numbers, which are more realistic and create a conceptual and theoretical framework for dealing with imprecision and vagueness [1,2]. Many authors have extensively studied different features of fuzzy linear programming since Bellman and Zadeh [3] proposed the notion of fuzzy decision making (see e.g. [1,2,4-20]).

Although fuzzy linear programming has been investigated and expanded for more than two decades by many researchers and from various points of view to best of our knowledge, there is no work on continuous-time linear programming in the framework of fuzzy theory. Bellman [21,22] introduced continuous-time linear programming to model some economic processes. A subclass of continuous-time linear programming is the class of

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Separated Continuous Linear Programs (SCLP):

SCLP:

$$\min \int_0^T c(t)'x(t)dt,$$

s.t.

$$\int_0^t Gx(s)ds + y(t) = a(t), \tag{1}$$

$$Hx(t) \leq b(t), \tag{2}$$

$$x(t) \geq 0, \qquad y(t) \geq 0, \qquad t \in [0,T], \tag{3}$$

where G and H are given fixed $n_2 \times n_1$ and $n_3 \times n_1$ matrices, and $c(t)$, $a(t)$ and $b(t)$ are given n_1, n_2, and n_3 vectors as functions of time, $t \in [0,T]$, respectively. All vectors are as columns, and the superscript, $'$, denotes the transpose operation. The unknown variables are $x(t) = (x_1(t), \cdots, x_{n_1}(t))'$ and $y(t) = (y_1(t), \cdots, y_{n_2}(t))'$ as function of time, $t \in [0,T]$. Here, the description "separated" refers to the fact that the constraints are in two sets: the integral constraints given by Equation 1 and the instantaneous constraints given by Relation 2.

SCLP was first introduced by Anderson [23] as a continuous model for large job shop scheduling problems. SCLP has attracted the most attention in the class of continuous-time linear programs due to its applications. It serves as a useful model for various dynamic network problems, where storage is permitted at the nodes (see [24] for more details). It can also be viewed as a type of optimal control problem with linear dynamics and linear state constraints. Problems of this kind arise in a number of engineering applications (see for example [25,26]).

SCLP has been studied by a number of authors, whose work can be divided into two areas: duality theory and computational methods. The most progress was achieved by Pullan. In a series of papers [27-31], he extensively studied the SCLP, characterized the solution structure, established duality theory, and developed a class of convergence algorithms. Philpott and Craddock [32] proposed an adaptive discretization algorithm for solving a network-based SCLP by using some results of [27]. Fleischer and Sethuraman [33] presented polynomial-time approximation algorithms for a special subclass of SCLP. In contrast to previous approaches [27,30,32,34], their algorithm used a fixed partition of $[0,T]$ specifically designed to meet the accuracy requirement on the solution. Recently, Weiss [26] studied SCLP in a different sense from Pullan's work. Assuming that a non-degeneracy condition holds, he developed a simplex-like algorithm, which always finds an exact optimal solution in finite steps, albeit requiring, typically, a large amount of computations.

Recently, the authors of the present work [35] introduced a class of separated continuous linear programs with fuzzy valued objective functions, since the objective coefficients are usually imprecise and ambiguous in practical applications. In this paper, following our previous work [35], we study in more detail SCLP with a fuzzy valued objective function called for simplicity, "Fuzzy Separated Continuous Linear Program (FSCLP)". In particular, we introduce two different discretizations of the problem to obtain a lower and an upper bound on the optimal value of the original problem. Then, we show that the gap between lower and upper bounds approaches zero when the discretization gets arbitrarily fine. This leads to the development of a strong duality result and an approximation algorithm for FSCLP.

PRELIMINARIES

In this section, preliminaries from the fuzzy set theory needed for the purposes of this paper are presented.

Fuzzy Numbers

Let $\tilde{a}$ be a "fuzzy number" that is a convex normalized fuzzy subset of real line $\mathbb{R}$, whose membership function is piecewise continuous. Denote the set of all fuzzy numbers on real numbers, $\mathbb{R}$, by $\mathcal{FN}(\mathbb{R})$.

An *h-cut* ($0 \leq h \leq 1$) of a fuzzy set, $\tilde{a}$, is defined by $\tilde{a}_h = \{t \in \mathbb{R} | \tilde{a}(t) \geq h\}$ if $h > 0$, and by $\tilde{a}_h = \text{cl}\{t \in \mathbb{R} | \tilde{a}(t) > 0\}$ if $h = 0$ where cl means the closure operator. It is a well-known result that the h-cut of a fuzzy number, $\tilde{a}$, is a closed interval and, hence, is denoted by $\tilde{a}_h = [\tilde{a}_h^L, \tilde{a}_h^R]$ throughout this paper.

Among fuzzy numbers, "Trapezoidal Fuzzy Numbers" (TFNs) are mostly used due to their simplicity of application them (see [36]). We use notation $\tilde{a} = (a^L, a^R, \alpha^L, \alpha^R)$ to represent a TFN. Here, a^L and a^R denote the left and right centers of $\tilde{a}$, respectively, and $\alpha^L > 0$ and $\alpha^R > 0$ denote the left and right spreads taken as real numbers, respectively. The membership function of $\tilde{a}$ is shown in Figure 1 and is given by:

$$\tilde{a}(t) = \begin{cases} 0 & t < a^L - \alpha^L, \\ 1 - \frac{a^L - t}{\alpha^L} & a^L - \alpha^L \leq t < a^L, \\ 1 & a^L \leq t < a^R, \\ 1 - \frac{t - a^R}{\alpha^R} & a^R \leq t < a^R + \alpha^R, \\ 0 & a^R + \alpha^R \leq t. \end{cases}$$

We denote the set of all trapezoidal fuzzy numbers by $\mathcal{TFN}(\mathbb{R})$.

There are two important topics in real world applications of the fuzzy set theory: fuzzy arithmetic on

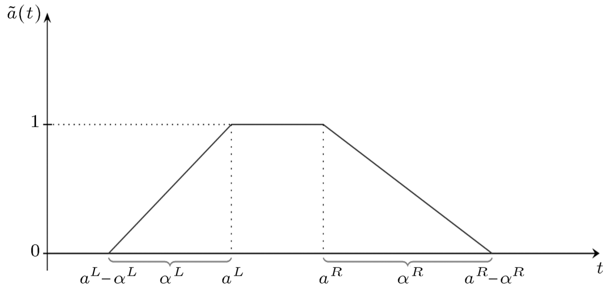

Figure 1. The membership function of TFN $\tilde{a} = (a^L, a^R, \alpha^L, \alpha^R)$.

the fuzzy numbers and comparison of fuzzy numbers. By using an extension principle [37], some of the fuzzy arithmetic of fuzzy numbers could be written in an efficient computational form. Suppose that $\times$ is an algebraic operation on $\mathbb{R}$, the algebraic operation, $\otimes$, on $\mathcal{FN}(\mathbb{R})$ is defined by:

$$\left(\tilde{a} \otimes \tilde{b}\right)(z) := \sup_{x \times y = z} \min\{\tilde{a}(x), \tilde{b}(y)\}.$$

Particularly, when $\times$ is $+$, $-$, and $\cdot$, we can induce operations $\oplus$, $\ominus$ and $\odot$ on $FN(\mathbb{R})$, respectively. Some results of applying fuzzy arithmetic on the TFNs $\tilde{a} = (a^L, a^R, \alpha^L, \alpha^R)$ and $\tilde{b} = (b^L, b^R, \beta^L, \beta^R)$ are as follows:

Scalar multiplication:

$$t.\tilde{a} = (ta^L, ta^R, t\alpha^L, t\alpha^R), \qquad t \in \mathbb{R}, t \geq 0,$$

$$t.\tilde{a} = (ta^R, ta^L, -t\alpha^R, -t\alpha^L), \qquad t \in \mathbb{R}, t < 0.$$

Addition:

$$\tilde{a} \oplus \tilde{b} = (a^L + b^L, a^R + b^R, \alpha^L + \beta^L, \alpha^R + \beta^R).$$

Subtraction:

$$\tilde{a} \ominus \tilde{b} = (a^L - b^R, a^R - b^L, \alpha^L + \beta^R, \alpha^R + \beta^L).$$

However, for comparison of fuzzy numbers, there are many methods (see [38,39] and the references therein), including those which use crisp relations to rank fuzzy numbers. An effective approach for ordering the elements of $\mathcal{FN}(\mathbb{R})$ is to define a "ranking function", $R : \mathcal{FN}(\mathbb{R}) \mapsto \mathbb{R}$, where every fuzzy number is mapped into a point on the real line, where a natural order exists. Then, a ranking on fuzzy numbers can be defined as:

- $\tilde{a} \geq_R \tilde{b}$ if and only if $R(\tilde{a}) \geq R(\tilde{b})$;
- $\tilde{a} =_R \tilde{b}$ if and only if $R(\tilde{a}) = R(\tilde{b})$;
- $\tilde{a} >_R \tilde{b}$ if and only if $R(\tilde{a}) > R(\tilde{b})$,

where $\tilde{a}$ and $\tilde{b}$ are two fuzzy numbers. Also, $\tilde{a} \leq_R \tilde{b}$ if and only if $\tilde{b} \geq_R \tilde{a}$. We shall use notation $\tilde{a} = \tilde{b}$ without subscript R when $\tilde{a}$ and $\tilde{b}$ have the same membership functions and notation $\tilde{a} \geq 0$ when $\tilde{a}(t) = 0$ for every $t < 0$.

A ranking function, R, is said to be "linear" if:

$$R\left(\tilde{a} + k\tilde{b}\right) = R\left(\tilde{a}\right) + kR\left(\tilde{b}\right),$$

for any $\tilde{a}, \tilde{b} \in \mathcal{FN}(\mathbb{R})$ and any $k \in \mathbb{R}$.

The linear ranking functions have been mostly used in solving fuzzy linear programming (see [15,16,38]). In this paper, we restrict our attention to linear ranking functions. Therefore, the results that will be presented later are valid for any arbitrary, but fixed, linearranking function, R.

There are many ranking functions which have been defined by authors according to their requirements (see [38,39]). Some examples are as follows:

1. Yager [40] proposed a ranking function based on the concept of h-cuts. Let $\tilde{a}_h = [\tilde{a}_h^L, \tilde{a}_h^R]$ be the h-cut of $\tilde{a}$. Then, the ranking function proposed by Yager [40] is defined as:

$$R(\tilde{a}) = \frac{1}{2}\int_0^1 (\tilde{a}_h^L + \tilde{a}_h^R)dh,$$

which reduces to $R(\tilde{a}) = \frac{a^L + a^R}{4} + \frac{\alpha^R - \alpha^L}{2}$ for a TFN $\tilde{a} = (a^L, a^R, \alpha^L, \alpha^R)$.

2. Let $\tilde{a} = (a^L, a^R, \alpha^L, \alpha^R)$ be a TFN. The mean of the density function of $\tilde{a}$ is defined by:

$$E[X_{\tilde{a}}] = \frac{1}{3}\Bigg[2(a^L + a^R) + (\alpha^R - \alpha^L) + \frac{a^L(a^L - \alpha^L) - a^R(a^R + \alpha^R)}{2(a^R - a^L) + (\alpha^R + \alpha^L)}\Bigg].$$

Then, a ranking function can be defined by $R(\tilde{a}) = E[X_{\tilde{a}}]$ (see [14]).

Integral of Fuzzy Number Valued Functions

The Lebesgue integral and the Henstock integral of fuzzy number valued functions have been discussed by a number of authors (see [41-44] and the references therein). Here, we define the Lebesgue integral as well as the Lebesgue-Stieltjes integral of fuzzy number valued functions slightly differently from those in the mentioned works.

Let $\tilde{f} : [a, b] \to \mathcal{FN}(\mathbb{R})$ be a fuzzy number valued function and:

$$\tilde{f}_h(t) = [\tilde{f}_h^L(t), \tilde{f}_h^R(t)], \qquad t \in [a, b], h \in (0, 1].$$

We say that $\tilde{f}$ is bounded measurable (Lebesgue-integrable, of bounded variation, monotonic increasing, continuous or continuous from right) on $[a, b]$, if functions $\tilde{f}_h^L$ and $\tilde{f}_h^R$ are both bounded measurable (Lebesgue-integrable, of bounded variation, monotonic increasing, continuous or continuous from right) on $[a,b]$ for any $h \in (0,1]$.

The integral of Lebesgue-integrable $\tilde{f}$ from a to b is defined to be a fuzzy set as:

$$\left(\int_a^b \tilde{f}(t)dt\right)(x) := \sup\{h \in (0,1]$$

$$: x \in \left[\int_a^b \tilde{f}_h^L(t)dt, \int_a^b \tilde{f}_h^R(t)dt\right]\right\},$$

$x \in \mathbb{R}$.

Lemma 1

Let $\tilde{f} : [a,b] \to \mathcal{TFN}(\mathbb{R})$ be a Lebesgue-integrable function with:

$$\tilde{f}(t) = (f^L(t), f^R(t), \delta^L(t), \delta^R(t)).$$

If $f^L(t)$, $f^R(t)$, $\delta^L(t)$ and $\delta^R(t)$ are Lebesgue-integrable functions, then:

$$\int_a^b \tilde{f}(t)dt = \left(\int_a^b f^L(t)dt, \int_a^b f^R(t)dt, \int_a^b \delta^L(t)dt,\right.$$

$$\left.\int_a^b \delta^R(t)dt\right).$$

Proof

It is clear that the h-cut of $\int_a^b \tilde{f}(t)dt$ is as follows:

$$\left(\int_a^b \tilde{f}(t)dt\right)_h = \left[\int_a^b \tilde{f}_h^L(t)dt, \int_a^b \tilde{f}_h^R(t)dt\right].$$

Also we have:

$$\tilde{f}_h(t) = [f^L(t) - (1-h)\delta^L(t), f^R(t) + (1-h)\delta^R(t)],$$

for every $t \in [a,b]$. Hence:

$$\left(\int_a^b \tilde{f}(t)dt\right)_h = \left[\int_a^b \left(f^L(t) - (1-h)\delta^L(t)\right)dt,\right.$$

$$\left.\int_a^b \left(f^R(t) + (1-h)\delta^R(t)\right)dt\right]$$

$$= \left[\int_a^b f^L(t)dt - (1-h)\int_a^b \delta^L(t)dt,\right.$$

$$\left.\int_a^b f^R(t)dt + (1-h)\int_a^b \delta^R(t)dt\right],$$

which implies that $\int_a^b \tilde{f}(t)dt$ is a TFN with left center, $\int_a^b f^L(t)dt$, right center, $\int_a^b f^R(t)dt$, left spread, $\int_a^b \delta^L(t)dt$, and right spread, $\int_a^b \delta^R(t)dt$. □

Lemma 2

Let $\tilde{f}$ and $\tilde{g}$ be two TFN valued functions on $[a,b]$, which are Lebesgue-integrable and $\lambda \in \mathbb{R}$. Then:

$$\int_a^b (\tilde{f}(t) + \lambda\tilde{g}(t))dt = \int_a^b \tilde{f}(t)dt + \lambda\int_a^b \tilde{g}(t)dt.$$

Proof

The result follows from the definition of an integral. □

Lemma 3

Let $\tilde{f} : [a,b] \to \mathcal{TFN}(\mathbb{R})$ be Lebesgue-integrable. If $\tilde{f}(t) \geq 0$ for every $t \in [a,b]$, then;

$$\int_a^b \tilde{f}(t)dt \geq 0.$$

Proof

The result follows from Lemma 1. □

We now define the Lebesgue-Stieltjes integral for fuzzy number valued functions.

Definition 1

Let $\tilde{g} : [a,b] \to \mathcal{FN}(\mathbb{R})$ be of bounded variation on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be bounded measurable on $[a,b]$. The Lebesgue-Stieltjes integral of $f(t)$, with respect to $\tilde{g}(t)$, from a to b is defined to be a fuzzy set as:

$$\left(\int_a^b f(t)d\tilde{g}(t)\right)(x) := \sup\left\{h \in (0,1]\right.$$

$$: x \in \left[\int_a^b f(t)d\tilde{g}_h^L(t), \int_a^b f(t)d\tilde{g}_h^R(t)\right]\right\},$$

for every $x \in \mathbb{R}$.

Lemma 4

Let $\tilde{g} : [a,b] \to \mathcal{TFN}(\mathbb{R})$ be of bounded variation on $[a,b]$ with:

$$\tilde{g}(t) = (g^L(t), g^R(t), \delta^L(t), \delta^R(t)),$$

and $f : [a,b] \to \mathbb{R}$ be bounded measurable on $[a,b]$. Then:

$$\int_a^b f(t)d\tilde{g}(t)dt = \left(\int_a^b f(t)dg^L(t), \int_a^b f(t)dg^R(t),\right.$$

$$\left.\int_a^b f(t)d\delta^L(t), \int_a^b f(t)d\delta^R(t)\right).$$

Proof

We can proceed by a similar argument as proof of Lemma 1. □

Theorem 1

If $\tilde{g} : [a,b] \to \mathcal{TFN}(\mathbb{R})$ and $f : [a,b] \to \mathbb{R}$ are continuous and bounded measurable functions, then integration by parts holds, that is:

$$\int_a^b f(t)d\tilde{g}(t) =_R [f(b)\tilde{g}(b) - f(a)\tilde{g}(a)] - \int_a^b \tilde{g}(t)df(t),$$

where $\int_a^b \tilde{g}(t)df(t)$ is a fuzzy set given by:

$$\left(\int_a^b \tilde{g}(t)df(t)\right)(x) := \sup\left\{h \in (0,1] \right.$$

$$\left. : x \in \left[\int_a^b \tilde{g}_h^L(t)df(t), \int_a^b \tilde{g}_h^R(t)df(t)\right]\right\},$$

$x \in \mathbb{R}$.

Proof

The result follows from Lemma 4.

Lemma 5

If $\tilde{g} : [a,b] \to \mathcal{TFN}(\mathbb{R})$ is monotonic increasing on $[a,b]$, and $f : [a,b] \to \mathbb{R}^+$ is a bounded measurable function, then:

$$\int_a^b f(t)d\tilde{g}(t) \geq 0.$$

Proof

The result follows from Lemma 4. □

FUZZY SEPARATED CONTINUOUS LINEAR PROGRAMS

The Separated Continuous Linear Program (SCLP) can be used to model a variety of problems that arise in communications, manufacturing and urban traffic control (see [25,26]). Since the objective coefficients are usually imprecise and ambiguous in real-world problems, in this section we consider an extension of SCLP with a fuzzy valued objective function, called, for simplicity, "Fuzzy Separated Continuous Linear Program (FSCLP)", defined as:

FSCLP:

$$\min \int_0^T (\tilde{\gamma} + t\tilde{c})' x(t)dt,$$

s.t.

$$\int_0^t Gx(s)ds + y(t) = \alpha + ta, \qquad t \in [0,T],$$

$$Hx(t) \leq b, \qquad t \in [0,T],$$

$$x(t) \geq 0, \qquad y(t) \geq 0, \qquad t \in [0,T].$$

Here, $\tilde{\gamma}, \tilde{c} \in \mathcal{TFN}(\mathbb{R})^{n_1}, \alpha, a \in \mathbb{R}^{n_2}, b \in \mathbb{R}^{n_3}, G \in \mathbb{R}^{n_2 \times n_1}, H \in \mathbb{R}^{n_3 \times n_1}$, $x(t) \in L_\infty^{n_1}[0,T]$ and $y(t) \in C^{n_2}[0,T]$. Notice that $\mathcal{TFN}(\mathbb{R})^{n_1}$ denotes the set of all n_1-vectors whose components are trapezoidal fuzzy numbers, $L_\infty^{n_1}[0,T]$ denotes the space of n_1 dimensional vectors whose components are bounded measurable functions over $[0,T]$, and $C^{n_2}[0,T]$ denotes the space of n_2 dimensional vectors whose components are continuous functions over $[0,T]$.

Any pair of $(x(t), y(t))$ with $x(t) \in L_\infty^{n_1}[0,T]$, and $y(t) \in C^{n_2}[0,T]$ which satisfies the set of Constraints 1-3, is called a feasible solution for FSCLP. Let S be the set of all feasible solutions for FSCLP. We shall say that $(x^*(t), y^*(t)) \in S$ is an optimal feasible solution, if:

$$V[\text{FSCLP}, (x^*(t), y^*(t))] \leq_R V[\text{FSCLP}, (x(t), y(t))],$$

for all $(x(t), y(t)) \in S$.

Here and subsequently notation $V[OP, x]$ is used to denote the objective function value of an Optimization Problem (OP) for a given feasible solution, x. Also notation $V[OP]$ will be used to denote the optimal value of an OP where it is infinity if OP is an infeasible minimization problem and $-\infty$ if OP is an infeasible maximization problem.

Before we proceed, we introduce some more definitions and notations. Let f be a real-valued function defined on the time interval, $[0,T]$, and $P = \{t_0, \cdots, t_m\}$ be a partition of $[0,T]$, that is: $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T$. Function f is said to be "piecewise constant (linear)", with respect to the partition P, if it is constant (linear) on $[t_{k-1}, t_k)$ for $k = 1, \cdots, m$. We say that f is piecewise constant (linear) on $[0,T]$, if it is piecewise constant (linear), with respect to some partition of $[0,T]$. The "breakpoints" of a piecewise linear or piecewise constant function are the discontinuity points in the function and its derivatives.

It is shown [45] that if the feasible region for SCLP is bounded and nonempty, then there exists an optimal solution for SCLP for which the components of $x(t)$ are piecewise constant. The same result is true for FSCLP. This leads to the following assumption.

Assumption 1

The feasible region for FSCLP is bounded and nonempty.

It is clear that if $||x(t)|| \leq M$ for some constant M and any feasible solution $x(t)$, then the feasible region for FSCLP is bounded.

Following Pullan [27], the dual problem of FSCLP can be defined as follows:

FSCLP*:

$$\max -\int_0^T (\alpha + ta)d\tilde{\pi}(t)' - \int_0^T \tilde{\eta}(t)'bdt,$$

s.t.

$$\tilde{\gamma} + t\tilde{c} - G'\tilde{\pi}(t) + H'\tilde{\eta}(t) \geq_R 0,$$

with variables $\tilde{\eta} : [0,T] \rightarrow \mathcal{TFN}(\mathbb{R})^{n_3}$ whose components are Lebesgue-integrable functions on $[0,T]$ and $\tilde{\pi} : [0,T] \rightarrow \mathcal{TFN}(\mathbb{R})^{n_2}$, where $\tilde{\pi}(t)$ is monotonic increasing and right continuous on $[0,T]$ with $\tilde{\pi}(T) = 0$, in the sense that each component of $\tilde{\pi}(t)$ is monotonic increasing and right continuous. Here, the following expression:

$$\int_0^T (\alpha + at)d\tilde{\pi}(t)',$$

is understood to be a Lebesgue-Stieltjes integral.

The following weak duality result can be concluded for FSCLP.

Theorem 2

Weak duality holds between FSCLP and FSCLP*, i.e.:

$$V[\text{FSCLP}^*] \leq_R V[\text{FSCLP}].$$

Proof

Consider any two feasible solutions, $(x(t), y(t))$ and $(\tilde{\pi}(t), \tilde{\eta}(t))$ for FSCLP and FSCLP*, respectively. Then by Lemmas 2, 3, 5 and Theorem 1, we have:

$$\int_0^T (\gamma' + t\tilde{c}')x(t)dt - \left(-\int_0^T (\alpha + ta)d\tilde{\pi}(t)' - \int_0^T \tilde{\eta}(t)'dt\right) =_R \int_0^T (\tilde{\gamma} + t\tilde{c})'x(t)dt + \int_0^T \left(\int_0^t Gx(s)ds + y(t)\right) d\tilde{\pi}(t)' + \int_0^T \tilde{\eta}(t)'bdt =_R \int_0^T (\tilde{\gamma} + t\tilde{c} - G'\tilde{\pi}(t) + H'\tilde{\eta}(t))x(t)dt + \int_0^T y(t)d\tilde{\pi}(t)' + \int_0^T \tilde{\eta}(t)'(b - Hx(t))dt \geq_R 0. \qquad \square$$

The weak duality result motivates the notion of complementary slackness optimality conditions given in the following corollary.

Corollary 1

Strong duality holds between FSCLP and FSCLP* if, and only if, there are feasible solutions, $(x(t), y(t))$ and $(\tilde{\pi}(t), \tilde{\eta}(t))$ for FSCLP and FSCLP*, respectively, which satisfy the following complementary slackness conditions:

$$\int_0^T (\tilde{\gamma} + t\tilde{c} - G'\tilde{\pi}(t) + H'\tilde{\eta}(t))\, x(t)dt =_R 0,$$

$$\int_0^T y(t)d\tilde{\pi}(t)' =_R 0,$$

$$\int_0^T \tilde{\eta}(t)'(b - Hx(t))dt =_R 0.$$

DISCRETE APPROXIMATIONS

In this section, two discrete approximations of FSCLP are introduced followed by a discussion of their properties. We first introduce the standard and natural discretization of FSCLP.

Given a partition, $P = \{t_0, \cdots, t_m\}$ of $[0,T]$, the standard discrete approximation of FSCLP, so-called FDP(P), is defined as follows:

FDP(P):

$$\min \sum_{k=1}^{m} (t_k - t_{k-1}) \left(\tilde{\gamma}' + \left(\frac{t_k + t_{k-1}}{2}\right) \tilde{c}'\right) \hat{x}(t_{k-1}+),$$

s.t.

$$\hat{y}(t_0) = \alpha,$$

$$(t_1 - t_0)G\hat{x}(t_0+) + \hat{y}(t_1) = \alpha + t_1 a,$$

$$(t_k - t_{k-1})G\hat{x}(t_{k-1}+) + \hat{y}(t_k) - \hat{y}(t_{k-1}) = (t_k - t_{k-1})a,$$

$$k = 2, \cdots, m,$$

$$H\hat{x}(t_{k-1}+) \leq b, \qquad k = 1, \cdots, m,$$

$$\hat{x}(t_{k-1}+), \hat{y}(t_k) \geq 0, \qquad k = 1, \cdots, m.$$

Notice that FDP(P) is fuzzy linear programming and it can be efficiently solved by the methods presented in [15,16]. As with the notation in [27], the labeling of the variables in FDP(P) is for convenience and does not mean that they explicitly refer to a function but rather in an implicit way as shown in the following lemma.

Lemma 6

For any partition P, we have:

$$V[\text{FSCLP}] \leq_R V[\text{FDP}(P)].$$

Proof

It is easy to see that any feasible solution for FDP(P) can be used to construct a feasible solution for FSCLP with the same objective function value. Specifically if $(\hat{x}, \hat{y})$ is a feasible solution for FDP(P), then:

$$x(t) = \begin{cases} \hat{x}(t_{k-1}+), & t \in [t_{k-1}, t_k), \quad k = 1, \cdots, m, \\ \hat{x}(t_{m-1}+), & t = T, \end{cases} \tag{4}$$

$$y(t) = \left(\frac{t_k - t}{t_k - t_{k-1}}\right) \hat{y}(t_{k-1}) + \left(\frac{t - t_{k-1}}{t_k - t_{k-1}}\right) \hat{y}(t_k),$$

$$t \in [t_{k-1}, t_k], \qquad k = 1, \cdots, m, \tag{5}$$

form the desired feasible solution for FSCLP. □

In the following, we introduce another discrete approximation of FSCLP for a given partition, $P = \{t_0, \cdots, t_m\}$, named FAP(P), as follows:

FAP(P):

$$\min \sum_{k=1}^{m} \left(\frac{t_k - t_{k-1}}{2}\right) ((\tilde{\gamma}' + \tilde{c}' t_{k-1})\hat{x}(t_{k-1}+)$$

$$+ (\tilde{\gamma}' + \tilde{c}' t_k)\, \hat{x}(t_k-)),$$

s.t.

$$\hat{y}(t_0) = \alpha,$$

$$\left(\frac{t_1 - t_0}{2}\right) G\hat{x}(t_0+) + \hat{y}\left(\frac{t_1 + t_0}{2}\right) = \alpha + \left(\frac{t_1 + t_0}{2}\right) a,$$

$$\left(\frac{t_k - t_{k-1}}{2}\right) G\hat{x}(t_k-) + \hat{y}(t_k) - \hat{y}\left(\frac{t_k + t_{k-1}}{2}\right)$$

$$= \left(t_k - \frac{t_k + t_{k-1}}{2}\right) a,$$

$$k = 1, \cdots, m,$$

$$\left(\frac{t_k - t_{k-1}}{2}\right) G\hat{x}(t_{k-1}+) + \hat{y}\left(\frac{t_k + t_{k-1}}{2}\right) - \hat{y}(t_{k-1})$$

$$= \left(\frac{t_k + t_{k-1}}{2} t_{k-1}\right) a,$$

$$k = 2, \cdots, m,$$

$$H\hat{x}(t_{k-1}+) \leq b, \qquad k = 1, \cdots, m,$$

$$H\hat{x}(t_k-) \leq b, \qquad k = 1, \cdots, m,$$

$$\hat{x}(t_{k-1}+), \hat{x}(t_k-), \hat{y}(t_k), \hat{y}\left(\frac{t_k + t_{k-1}}{2}\right) \geq 0,$$

$$k = 1, \cdots, m.$$

This is a fuzzy variation of the second discretization in Pullan [27] for SCLP.

Notice that the feasible set of FAP(P) is the same as the feasible set of FDP($\bar{P}$) where:

$$\bar{P} = \left\{t_0, \frac{t_0 + t_1}{2}, t_1, \frac{t_1 + t_2}{2}, t_2, \cdots, \frac{t_{m-1} + t_m}{2}, t_m\right\},$$

and we identify $\hat{x}(t_k-)$ in FAP(P) with $\hat{x}([(t_k + t_{k-1})/2]+)$ in FDP($\bar{P}$). As a consequence, any solution for FAP(P) can be turned into a feasible solution for FSCLP, but unlike FDP(P), not with the same objective function value.

In the following, some properties of discretization FAP(P) that are needed for the purposes of this paper are stated.

Lemma 7

Let P be an arbitrary partition. Then, FSCLP is feasible if, and only if, FAP(P) is feasible.

Proof

Let $P = \{t_0, \cdots, t_m\} \in \mathcal{P}$ and $(\hat{x}, \hat{y})$ be a feasible solution for FAP(P). It is clear that this solution forms a feasible solution, $(x(t), y(t))$, to FSCLP defined by:

$$x(t) = \begin{cases} \hat{x}(t_{k-1}+), & t \in \left[t_{k-1}, \frac{t_k - t_{k-1}}{2}\right), \\ & k = 1, \cdots, m, \\ \hat{x}(t_k-), & t \in \left[\frac{t_k - t_{k-1}}{2}, t_k\right), \\ & k = 1, \cdots, m, \\ \hat{x}(t_m-), & t = T, \end{cases} \tag{6}$$

with $y(t)$ derived from Equation 4. Now assume that $(x(t), y(t))$ is a feasible solution for FSCLP. Define $(\hat{x}, \hat{y})$ by:

$$\hat{x}(t_{k-1}+) = \frac{2}{t_k + t_{k-1}} \int_{t_{k-1}}^{\frac{t_{k-1} + t_k}{2}} x(t)dt,$$

$$k = 1, \cdots, m, \tag{7}$$

$$\hat{x}(t_k-) = \frac{2}{t_k + t_{k-1}} \int_{\frac{t_{k-1} + t_k}{2}}^{t_k} x(t)dt,$$

$$k = 1, \cdots, m, \tag{8}$$

$$\hat{y}(t_k) = y(t_k), \qquad k = 1, \cdots, m, \tag{9}$$

$$\hat{y}\left(\frac{t_k + t_{k-1}}{2}\right) = y\left(\frac{t_k + t_{k-1}}{2}\right),$$

$$k = 1, \cdots, m. \tag{10}$$

Then, $(\hat{x}, \hat{y})$ is a feasible solution for FAP(P). □

Theorem 3

Let P be any arbitrary partition and suppose that FAP(P) has an optimal solution. Then:

$$V[\text{FAP}(P)] \leq_R V[\text{FSCLP}].$$

Proof

The dual problem FAP$^*(P)$ for FAP(P) can be written as:

FAP$^*(P)$:

$$\max \tilde{\hat{\pi}}'(t_0+)\alpha$$

$$+ \sum_{k=1}^{m} \left(\frac{t_k - t_{k-1}}{2}\right) a' \left(\tilde{\hat{\pi}}(t_{k-1}+) + \tilde{\hat{\pi}}(t_k-)\right)$$

$$- \sum_{k=1}^{m} \left(\frac{t_k - t_{k-1}}{2}\right) b^T \left(\tilde{\hat{\eta}}(t_{k-1}+) + \tilde{\hat{\eta}}(t_k-)\right),$$

s.t.

$$\tilde{\gamma} + t_k\tilde{c} - G'\tilde{\hat{\pi}}(t_k-) + H'\tilde{\hat{\eta}}(t_k-) \geq_R 0,$$

$$k = 1, \cdots, m,$$

$$\tilde{\gamma} + t_{k-1}\tilde{c} - G'\tilde{\hat{\pi}}(t_{k-1}+) + H'\tilde{\hat{\eta}}(t_{k-1}+) \geq_R 0,$$

$$k = 1, \cdots, m,$$

$$\tilde{\hat{\eta}}(t_k-), \tilde{\hat{\eta}}(t_{k-1}+) \geq_R 0, \qquad k = 1, \cdots, m,$$

$$\tilde{\hat{\pi}}(t_k-) - \tilde{\hat{\pi}}(t_{k-1}+) \geq_R 0, \qquad k = 1, \cdots, m,$$

$$\tilde{\hat{\pi}}(t_k+) - \tilde{\hat{\pi}}(t_k-) \geq_R 0, \qquad k = 1, \cdots, m-1,$$

$$\tilde{\hat{\pi}}(t_m-) \leq_R 0.$$

Now, suppose that $(\hat{x}, \hat{y})$ is an optimal solution for FAP(P). From the duality theory for fuzzy linear programming [15,16], there is some $(\tilde{\hat{\pi}}, \tilde{\hat{\eta}})$ that solves FAP$^*(P)$ with:

$$V[\text{FAP}(P), (\hat{x}, \hat{y})] =_R V[\text{FAP}^*(P), (\tilde{\hat{\pi}}, \tilde{\hat{\eta}})]. \tag{11}$$

Now, let:

$$\tilde{\pi}(t) =$$

$$\begin{cases} \tilde{\hat{\pi}}(t+), & t = t_0, t_1, \cdots, t_{m-1}, \\ 0, & t = T, \\ \left(\frac{t_k - t}{t_k - t_{k-1}}\right)\tilde{\hat{\pi}}(t_{k-1}+) + \left(\frac{t - t_{k-1}}{t_k - t_{k-1}}\right)\tilde{\hat{\pi}}(t_k-), & \\ & t \in (t_{k-1}, t_k), \quad k = 1, \cdots, m, \end{cases} \tag{12}$$

$$\tilde{\eta}(t) =$$

$$\begin{cases} \tilde{\hat{\eta}}(t+), & t = t_0, t_1, \cdots, t_{m-1}, \\ 0, & t = T, \\ \left(\frac{t_k - t}{t_k - t_{k-1}}\right)\tilde{\hat{\eta}}(t_{k-1}-) + \left(\frac{t - t_{k-1}}{t_k - t_{k-1}}\right)\tilde{\hat{\eta}}(t_k+), & \\ & t \in (t_{k-1}, t_k), \quad k = 1, \cdots, m. \end{cases} \tag{13}$$

It is not difficult to check that $(\tilde{\pi}(t), \tilde{\eta}(t))$ is a feasible solution for FSCLP* and:

$$V[\text{FSCLP}^*(P), (\tilde{\pi}(t), \tilde{\eta}(t))] =_R V[\text{FAP}^*(P), (\tilde{\hat{\pi}}, \tilde{\hat{\eta}})].$$

Thus, we have:

$$V[\text{FAP}^*(P), (\tilde{\hat{\pi}}, \tilde{\hat{\eta}})] \leq_R V[\text{FSCLP}^*]. \tag{14}$$

The result now immediately follows from Equations 11 and 14, and Theorem 2. □

Combining Lemma 6 and Theorem 3 leads to the following important result.

Corollary 2

For any two partitions, P and Q, we have:

$$V[\text{FAP}(Q)] \leq_R V[\text{FSCLP}^*] \leq_R V[\text{FSCLP}]$$

$$\leq_R V[\text{FDP}(P)].$$

STRONG DUALITY

In this section, we first show that the optimal values of discretizations, FDP(P) and FAP(P), close up to the same value as the norm of partition P tends to zero. Then, we establish a strong duality result.

Suppose that $P = \{t_0, t_1, \cdots, t_m\}$ is a partition of interval $[0, T]$, and $(\hat{x}, \hat{y})$ is an optimal solution for FAP(P). Let $(x(t), y(t))$ be the corresponding feasible solution for FSCLP constructed from $(\hat{x}, \hat{y})$ by Equation 6. We define:

$$\alpha[\hat{x}, \hat{y}] := \int_0^T (\tilde{\gamma} + \tilde{c}t)'x(t)dt$$

$$- \sum_{k=1}^{m} \frac{(t_k - t_{k-1})}{2}\{(\tilde{\gamma} + \tilde{c}t_{k-1})'\hat{x}(t_{k-1}+)$$

$$+ (\tilde{\gamma} + \tilde{c}t_k)'\hat{x}(t_k-)\}. \tag{15}$$

The value $\alpha[\hat{x},\hat{y}]$ gives the difference in objective function values yielded by $(x(t),y(t))$ for FSCLP and $(\hat{x},\hat{y})$ for FAP(P). By Theorem 3, it can be seen that $\alpha[x,y] \geq_R 0$, and $\alpha[\hat{x},\hat{y}] =_R 0$ implies that $(x(t),y(t))$ is optimal for FSCLP. Other properties of $\alpha[\hat{x},\hat{y}]$ are as follows.

Lemma 8

The value of $\alpha[\hat{x},\hat{y}]$ can be computed by the following formula:

$$\alpha[\hat{x},\hat{y}] = \sum_{k=1}^{m} \frac{(t_k - t_{k-1})^2}{8} \tilde{c}' \left\{ \hat{x}(t_k-) - \hat{x}(t_{k-1}+) \right\}.$$

Proof

The result follows after simple integration and algebra. □

The norm of partition $P = \{t_0, \cdots, t_m\}$ denoted by $||P||$ can be defined by:

$$||P|| := \max_k (t_k - t_{k-1}).$$

Then, we can establish the following result.

Lemma 9

There is a constant $\mathcal{K}$ such that for any partition P, the following inequality holds:

$$\alpha[\hat{x}_P,\hat{y}_P] \leq_R ||P||\mathcal{K}, \tag{16}$$

where $(\hat{x}_P,\hat{y}_P)$ is an optimal solution for FAP(P).

Proof

By Assumption 1, $||x(t)|| \leq M$ for any feasible solution, $x(t)$. Moreover, suppose that $||R(\tilde{c})|| \leq C$ for any $\tilde{c} \in \mathrm{TFN}(\mathbb{R})$. Let $P = \{t_0, t_1, \cdots, t_m\}$ be an arbitrary partition. Then:

$$\alpha[\hat{x}_P,\hat{y}_P] = \sum_{k=1}^{m} \frac{(t_k - t_{k-1})^2}{8} \tilde{c}' \{ \hat{x}(t_k-) - \hat{x}t_{k-1}+) \}$$

$$\leq_R \frac{MC}{4} \sum_{k=1}^{m} (t_k - t_{k-1})^2 \leq \frac{MC||P||T}{4}.$$

The result follows by setting:

$$\mathcal{K} = \frac{MCT}{4}. \tag{17}$$

Corollary 3

Let $\{P_n\}_{n=1}^{\infty}$ be any sequence of partitions such that $\lim_{n\to\infty} ||P_n|| = 0$, and $(\hat{x}_n,\hat{y}_n)$ be an optimal solution for FAP(P_n). Then:

$$\lim_{n\to\infty} \alpha[\hat{x}_n,\hat{y}_n] =_R 0.$$

Corollary 3 implies that the optimal values of discretizations, FDP$(\bar{P}_n)$ and FAP(P_n), close up to the same value, as the norm of sequence $\{P_n\}$ tends to zero value. This fact leads to the strong duality theorem.

Theorem 4

If FSCLP has an optimal solution, then, strong duality holds between FSCLP and FSCLP* with respect to an arbitrary linear ranking function, R.

Proof

Let P be an arbitrary partition and $(x(t),y(t))$ be an optimal solution for FSCLP. By Lemma 7, this solution can be turned into a feasible solution for FAP(P), and, as a consequence, for FDP$(\bar{P})$. Furthermore, by Lemma 6, the optimal value of FSCLP is a lower bound on the optimal value of FDP$(\bar{P})$. Thus, FDP$(\bar{P})$ has an optimal solution. On the other hand, the optimal value of FSCLP is an upper bound on the optimal value of the maximization problem FAP$^*(P)$. Since FAP(P) is feasible, and the objective function value of its dual is bounded by the duality theory in fuzzy linear programming [15,16] both FAP(P) and FAP$^*(P)$ have optimal solutions.

Now, let $\{P_n\}_{n=1}^{\infty}$ be any sequence of partitions with $\lim_{n\to\infty} ||P_n|| = 0$. From the above argument, all three problems, FDP$(\bar{P}_n)$, FAP(P_n) and FAP$^*(P_n)$ have optimal solutions for any n. Let $(\hat{x}_n,\hat{y}_n)$, $(\hat{\bar{x}}_n,\hat{\bar{y}}_n)$, and $(\hat{\pi}_n,\hat{\bar{\eta}}_n)$ represent optimal solutions for FDP$(\bar{P}_n)$, FAP(P_n) and FAP$^*(P_n)$, respectively, and:

$$a_n = V[\mathrm{FDP}(\bar{P}_n)], \qquad b_n = V[\mathrm{FAP}^*(P_n)],$$

$$\text{for} \quad n = 1,2,\cdots.$$

By Lemma 6, and the fact that $\alpha[\hat{\bar{x}}_n,\hat{\bar{y}}_n]$ given by Equation 15, is an upper bound on the difference between a_n and b_n, we have:

$$0 \leq_R b_n - a_n \leq_R \alpha[\hat{\bar{x}}_n,\hat{\bar{y}}_n], \qquad \text{for} \quad n = 1,2,\cdots. \tag{18}$$

Corollary 3 and Relation 18 imply that:

$$\lim_{n\to\infty} (b_n - a_n) =_R 0.$$

On the other hand, it follows from Corollary 2 that $b_n - a_m \geq_R 0$ for all m and n. Thus, both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist (see Lemma 3.8 in [27]) and:

$$\lim_{n\to\infty} a_n =_R \lim_{n\to\infty} b_n.$$

On the other hand, by Corollary 2 we have:

$$V[\mathrm{FAP}^*(P_n)] \leq_R V[\mathrm{FSCLP}^*] \leq_R V[\mathrm{FSCLP}]$$

$$\leq_R V[\mathrm{FDP}(\bar{P}_n)], \qquad \text{for} \quad n = 1,2,\cdots.$$

Therefore, we can deduce:

$$\lim_{n\to\infty} V[\text{FAP}^*(P_n)]$$

$$=_R \lim_{n\to\infty}\left\{\int_0^T(\alpha+ta)d\tilde{\pi}_n(t)^T-\int_0^T\tilde{\tilde{\eta}}_n(t)^T bdt\right\}$$

$$=_R V[\text{FSCLP}^*]=_R V[\text{FSCLP}]$$

$$=_R \lim_{n\to\infty}\int_0^T(\tilde{\gamma}+t\tilde{c})^T x_n(t)dt$$

$$=_R \lim_{n\to\infty} V[\text{FDP}(\bar{P}_n)],$$

where $(x_n(t),y_n(t))$ is obtained from $(\hat{x}_n,\hat{y}_n)$ by Equations 4 and 5, and $(\tilde{\pi}_n(t),\tilde{\eta}_n(t))$ from $(\tilde{\hat{\pi}}_n,\tilde{\hat{\eta}}_n)$ by Equations 12 and 13. The above relation shows that strong duality holds between FSCLP and FSCLP*. □

APPROXIMATION ALGORITHM FOR FSCLP

In this section, we present an algorithm based on a sequence of discrete approximations to the FSCLP problem. First, some needed prerequisites are stated and then the algorithm is given.

Corollary 3 suggests that FSCLP can be solved by solving a sequence of discrete approximations to the problem. Let P be a fixed partition of time interval $[0,T]$. Then, FDP(P) and FAP(P) are two ordinary fuzzy linear programming problems. Recently, two methods for solving fuzzy linear programming problems have been proposed in [15,16], based on the concept of linear ranking functions. In these methods, a crisp model of the same size is constructed which is equivalent with fuzzy linear programming and the optimal solution of this equivalent model is considered as the desired one. Thus, FDP(P) and FAP(P) can be solved easily. Then, by Corollary 2, optimal values of FDP(P) and FAP(P) yield an upper bound and a lower bound on the optimal value of FSCLP, respectively. Furthermore, the explicit error bound in Relation 16 shows that any required accuracy can be achieved by a partition P of $[0,T]$ into a sufficiently large number of equal intervals as the following lemma suggests:

Lemma 10

Let $\epsilon>0$ be the required accuracy and:

$$m=\left\lceil\frac{T\mathcal{K}}{\epsilon}\right\rceil, \tag{19}$$

where $\mathcal{K}$ is given by Equation 17. Then, the error bound is guaranteed to be no greater than ϵ. Moreover, the error bound approaches to zero as ϵ tends to zero.

Proof

By the use of Corollary 2 and Lemma 9, we can write:

$$\epsilon\geq\frac{T\mathcal{K}}{m}=||P||\mathcal{K}\geq\alpha[\hat{x}_P,\hat{y}_P]$$

$$\geq V[\text{FDP}(P)]-V[\text{AP}(P)]$$

$$\geq[\text{FDP}(P)]-V[\text{FSCLP}],$$

which establishes the lemma. □

In practical applications, $\mathcal{K}$ could be very huge and consequently the number of breakpoints in partition P, i.e. the value m given by Equation 19, could be very large. Choosing m can be avoided by using the Discretization algorithm described as follows. Starting from some initial partition, P, discretizations, FDP(P) and FAP(P), are solved. Then, the error bound can be estimated. If the error bound is not small enough, the number of breakpoints is doubled with a new breakpoint added at the midpoint of the current partition. A formal description of the Discretization Algorithm is as follows:

Discretization Algorithm

Step 0 Let $P_1=\{0,T\}$ be an initial partition. Set $i=1$.

Step 1 Solve FDP(P_i) to produce $(\hat{x}_i,\hat{y}_i)$.

Step 2 Solve FAP(P_i) to give $(\bar{\hat{x}}_i,\hat{\hat{y}}_i)$.

Step 3 Calculate the current error bound, i.e.:

$$\delta_n=V[\text{FDP}(P_i)]-V[\text{FAP}(P_i)].$$

If $\delta_i=_R 0$, then stop as $(\hat{x}_i,\hat{y}_i)$ yields an optimal solution for FSCLP. Otherwise, construct a new partition, P_{i+1}, with a new breakpoint added at the midpoints of P_i.

Step 4 Set $i=i+1$ and return to Step 1.

Lemma 11

The Discretization Algorithm terminates after a finite number of iterations at an optimal solution to FSCLP, or both $V[FDP(P_i)]$ and $V[FAP(P_i)]$ converge to $V[FSCLP]$, i.e.:

$$\lim_{|P_i|\to 0}V[\text{FAP}(P_i)]=\lim_{|P_i|\to 0}V[\text{FDP}(P_i)]=V[\text{FSCLP}].$$

Proof

The result easily follows from Lemma 9. □

Now, assume that $(\hat{x}_i,\hat{y}_i)$ is an optimal solution for FDP(P_i) generated at iteration i of the Discretization Algorithm, and $(x_i(t),y_i(t))$ denotes the associated solution for FSCLP. Usually, the $x_i(t)$ is identical in some consecutive intervals of P_n and, as a consequence, some breakpoints are redundant. Specifically, breakpoint t_k in P_i is said to be redundant if:

$$\frac{\hat{x}_i(t_{k-1}+)}{t_k - t_{k-1}} = \frac{\hat{x}_i(t_k+)}{t_{k+1} - t_k}.$$

It is clear that if t_k is redundant, then it can be removed from P_i without increasing the objective function value. Thus, it is desirable to remove the redundant breakpoints as they increase the size of the subproblems to be solved at each iteration. It is worthwhile to mention that the idea of removing redundant breakpoints is due to Philpott and Craddock [32].

AN ILLUSTRATIVE EXAMPLE

In this section, one simple example is solved using the proposed algorithm and Yager's method [40] is used as the linear ranking function, R, for comparison of fuzzy numbers.

Consider a given network with four nodes (numbered from 1 to 4) and four arcs (numbered from 1 to 4) connecting those nodes as shown in Figure 2. Arc j has a transit capacity, $b_j(t)$, given by:

$$b_1(t) = 0.6, \qquad b_2(t) = 0.8,$$

$$b_3(t) = 0.8, \qquad b_4(t) = 1.6.$$

In fact, $b_j(t)$ is an upper bound of the flow that can enter in arc j at time t. Moreover, each arc, j, has an associated transit cost, $c_j(t)$, which gives the cost for sending one unit of flow through arc (i, j) at time t. An initial storage of 8 units must be routed from node 1 to node 4 over the time interval $[0, 10]$, such that the transit capacity constraints are satisfied and the total cost is minimized. This problem can be formulated as an instance of an SCLP problem by putting a constant demand of 1.6 per unit time at node 4 during $(5, 10]$. In terms of the SCLP problem, G is the node-arc incidence matrix of the network and H is an identity matrix.

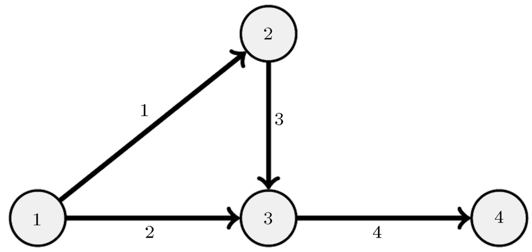

Figure 2. The given network.

More specifically:

$$T = 10, \qquad n_1 = n_2 = n_3 = 4,$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $a_i(t)$ is given by:

$$a_1(t) = 8, \qquad a_2(t) = a_3(t) = 0,$$

$$a_4(t) = \begin{cases} 0, & t \in [0, 5], \\ -1.6(t-5), & t \in [5, 10]. \end{cases}$$

Assume that transit costs are subjected to uncertainty and they are expressed by TFNs as follows:

$$\tilde{c}_1(t) = (0.3, 1.5, 0.5, 0.8) - (0.2, 0.8, 0.2, 0.6)t,$$

$$\tilde{c}_2(t) = (0.5, 2, 1.5, 0.5) + (0.8, 2.2, 1, 1.2)t,$$

$$\tilde{c}_3(t) = (10, 12.5, 2, 1) - t,$$

$$\tilde{c}_4(t) = (5.2, 6.5, 0.8, 0.5),$$

where $(a^L, a^R, \alpha^L, \alpha^R)$ denotes a TFN.

The Discretization Algorithm is applied for solving this example. At each iteration of the algorithm, redundant breakpoints are identified and removed. The results of the first seven iterations are given in Table 1, including optimal values of FDP(P_i) and FAP(P_i), error bound, δ_i, defined as $V[\text{FDP}(P_i)] - V[\text{FAP}(P_i)]$, the number of breakpoints at partition P_i (denoted by "# BP"), and the number of breakpoints after removing redundant ones on the optimal solution of FDP(P_i) (denoted by "# BPR").

The results of Table 1 show that the gap between lower and upper bounds approaches to zero when the discretization gets finer. In particular, after seven iterations, an approximation solution is found with breakpoints in:

$$P = \{0, 2.5, 2.6563, 2.7344, 2.8125, 3.4375, \\ 3.5938, 3.75, 5, 10\},$$

such that the error bound is guaranteed to be less than 0.0017.

Table 1. Results of the discretization algorithm.

i	$R(V[\mathrm{FDP}(P_i)])$	$R(V[\mathrm{FAP}(P_i)])$	$R(\delta_i)$	# BP	# BPR
1	101.0000	94.6625	6.3375	3	2
2	99.4187	97.5187	1.9000	5	4
3	98.6047	98.4187	0.1859	7	5
4	98.5656	98.4765	0.0929	9	6
5	98.5405	98.5045	0.0360	11	7
6	98.5239	98.5198	0.0041	13	9
7	98.5233	98.5215	0.0017	17	10

CONCLUSIONS

In this paper, a class of a separated continuous linear program with a fuzzy valued objective function, so-called FSCLP was introduced. A duality notion for FSCLP was established via a discretization of the time interval $[0,T]$ into a finite number of subintervals. In particular, we have shown that the strong duality between the FSCLP and its dual is concluded by two related fuzzy linear programming problems. As a by-product, an algorithm that computes, or at least converges to an optimal value of FSCLP, was derived. Although discussion of this paper was confined to FSCLP with constant data (i.e. the vectors $\tilde{\gamma}, \tilde{c}, \alpha, a, b$), still, all results can be readily generalized to cases in which the data are piecewise constants over $[0,T]$.

In model FSCLP, although the objective function coefficients, $\tilde{\gamma}, \tilde{c}$ are fuzzy numbers, matrices G and H and the right hand side vectors, α, a, and b, must be well defined and precise. Thus, an interesting problem is to study generalization of FSCLP in cases where entries of matrices G and H and the entries of vectors α, a, and b are also fuzzy numbers.

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