

Geometrically Non-linear Analysis of Unsymmetrical Fiber-Reinforced Laminated Annular Sector Composite Plates

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Abstract. The geometrically non-linear behavior of unsymmetrical, fiber-reinforced, laminated, annular sector composite plates is studied. The first order shear deformation theory is applied to the von Karman type non-linear behavior of unsymmetrically, laminated, annular sector composite plates. Five equilibrium equations, five stress-displacement relations, three curvature-displacement relationships, together with eight stress resultants, stress couples and shear force relationships are solved. The nonlinear nature of the problem prohibits the application of a closed form solution method. Consequently, the Dynamic Relaxation (DR) numerical method is chosen for solving the system of 21 simultaneous equations. The in-plane and out-of-plane displacements are reported for different configurations of annular sector plates. Different sector angles, fiber orientations and plate thicknesses are considered. For better observation of the numerical methods, they are illustrated graphically. The correlations of the present results and the corresponding finite element generated results are very satisfactory.

Keywords: Sector plate; Unsymmetric laminates; Dynamic relaxation; Rectilinear orthotropic; Large deflections

INTRODUCTION

Plates, in general, and sector plates in particular have many applications in engineering fields, i.e. aerospace, mechanical and civil engineering as diaphragms, curved bridge decks and end closures of cylindrical vessels. Composite materials have gained many advantages [1-3] over their metal counterparts in engineering applications, in particular aerospace engineering. Fiberreinforced, laminated, composite plates are made of continuous fibers in mainly epoxy resins with different fiber orientations in each layer. The layers can be arranged in a way that the properties of the composite, with respect to the middle plane of the plate, are either symmetrical or nonsymmetrical. The nonsymmetrical properties of the plate are, in particular, useful when thermal loading produces undesired deformations in

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the plate with symmetrical properties. The fiberreinforced, laminated, sector plates with symmetrical properties have been analyzed previously [4], and various parametric studies have been carried out and presented in tabular forms. However, the nonsymmetrical properties of the sector plates where the coupling stiffness matrix is nonzero have received minor or no attention. The types of structure which are close to fiber-reinforced, laminated, annular sector plates with non-symmetrical properties, with respect to the middle plane, which have been studied in the literature, are briefly reviewed. The geometrically non-linear analysis of unsymmetrical, laminated, composite plates with four straight edges is studied in [5]. A type of analytical-numerical solution method is applied to solve the von Karman type plate equations. The nonlinear dynamic analysis of moderately thick, laminated, composite sector plates is studied in [6]. In [6], two and four layer, cross-ply, antisymmetric plates with fibers in radial and circumferential directions are studied. This is only a theoretical fiber orientation, since, in practice, it is extremely difficult to align the fibers in radial and circumferential directions. Even if the fibers are oriented in radial and circumferential directions, the

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Young's modulus of elasticity in the radial direction will vary with the radius of the sector. Consequently, the constant values given in the paper cannot be realistic. However, the module ratios are assumed constant and numerical results are presented for deflections. In [7], the fiber, asymmetric transverse vibration of polar, orthotropic, annular sector plates with the thickness varying parabolically in a radial direction has been studied using a classical plate theory. The plate is, in fact, made up of isotropic material with stiffeners in the radial direction, which produces a polar orthotropic structure. Most literature surveys carried out resulted in the dynamic analysis of annular sector plates with isotropic properties [8-12]. Few static analyses of sectorial plates are mentioned with isotropic layers or completely isotropic properties. A mathematical treatment of laminated, circular, sector plates is presented in [13] by looking at the boundary layer phenomena in the Mindlin-Reissner plate theory. Using the Differential Quadrature Method (DQM) for the vibration analysis of shear deformable annular sector plates [14] is another treatment of sector plates. However, the plates have homogenous and isotropic mechanical properties and the emphasis is put on the application of DQM. The development of the Differential Quadrature Element Method (DQEM) and its application to the analysis of annular sector plates is presented in [15]; the plate is homogenous and isotropic. The solution of the sector plate by the Fourier-Bessel series is presented in [16]. The emphasis is, again, on the method of solution rather than the structure itself. The geometrically nonlinear analysis of thick circular plates with cylindrically orthotropic properties is treated in [17]. Due to the fact that shear deformations are very important in thick plates and, in particular, in fiber reinforced composite plates, their effects have also been examined and taken into account. By looking at the literature survey, it is clear that the treatment of fiber reinforced laminated annular sector plates with symmetric and antisymmetric properties is not treated. The first author has treated isotropic [18-22], as well as fiber reinforced, laminated sector plates [4,23-25] with symmetrical layer arrangements. However, antisymmetrical fiber-reinforced laminated annular sector plates, to the knowledge of the authors are not presented in the literature. This paper, therefore, is an attempt to present new results for the displacements in the radial and circumferential directions and the deflections for this type of plate. The emphasis is on the radial and circumferential displacements for antisymmetric layer arrangements.

PLATE GOVERNING EQUATIONS

The cylindrical coordinate system is used to define the geometry of the plate, as shown in Figure 1. The

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Figure 1. Annular sector plate geometry and fiber orientation.

positive axes, the radii and thickness measurements, are well illustrated. In this study, the fibers are arranged in a regular form and the sector plate is cut from this ply. r_i and r_o are inner and outer radius, respectively, and the sector angle is θ . The analysis is performed in angle Θ_g , and Θ_m is the angle of fiber orientation in each ply (see the Appendix).

Plate governing equations are classified as five equilibrium equations including shear forces, eight kinematic equations, which consist of three in-plane strain-displacement relationships, three curvaturerotation relationships, two out-of-plane shear straindisplacement relationships and eight constitutive equations. In order to carry out the solution procedure, the plate governing equations are supplemented by relevant boundary conditions on displacements and stress resultants and couples. For brevity, the final forms of the resulting equations are given below.

As mentioned in the above paragraph, there are five non-linear definitional equations [26] (Chia). They are obtained by considering the equilibrium of a plate element in the radial, circumferential and transverse directions, and by taking moments about r and θ axes:

$$\frac{\partial N_r}{\partial r} + \frac{\partial N_{r\theta}}{r\partial \theta} + \frac{N_r - N_{\theta}}{r} = 0, \qquad (1a)$$

$$\frac{\partial N_{\theta}}{r\partial \theta} + \frac{\partial N_{r\theta}}{\partial r} + \frac{2}{r}N_{r\theta} = 0, \qquad (1b)$$

$$\frac{Q_r}{r} + \frac{\partial Q_r}{\partial r} + \frac{\partial Q_\theta}{r\partial \theta} + N_r \frac{\partial^2 w}{\partial r^2} \\
+ N_\theta \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) + 2N_{r\theta} \left(\frac{1}{r} \frac{\partial^2 w}{\partial r\partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right) \\
+ q = 0,$$
(1c)

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$$\frac{\partial M_r}{\partial r} + \frac{\partial M_{r\theta}}{r\partial \theta} + \frac{M_r - M_{\theta}}{r} - Q_r = 0, \qquad (1d)$$

$$\frac{\partial M_{\theta}}{r\partial \theta} + \frac{\partial M_{r\theta}}{\partial r} + \frac{2}{r}M_{r\theta} - Q_{\theta} = 0, \qquad (1e)$$

where N_r , N_{θ} and $N_{r\theta}$ are radial, circumferential and shear stress resultants, M_r , M_{θ} and $M_{r\theta}$ are radial, circumferential and twisting stress couples and Q_r , Q_{θ} are transverse shear stress resultants. The underlined part of Equation 1c gives the non-linear part of the equation and, if omitted, small deflection results are found.

Considering the deformations of the plate element, the following relationships are found. The straindisplacement relations given in non-linear terms are similar to those originally obtained by Sanders [27]. Using the assumptions made by Sanders, they are stated as the following:

$$\varepsilon_{r}^{\circ} = \frac{\partial u}{\partial r} + \frac{1}{2} \left(\frac{\partial w}{\partial r}\right)^{2},$$

$$\varepsilon_{\theta}^{\circ} = \frac{\partial v}{r\partial \theta} + \frac{u}{r} + \frac{1}{2} \left(\frac{\partial w}{r\partial \theta}\right)^{2},$$

$$\gamma_{r\theta}^{\circ} = \frac{\partial u}{r\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{\partial w}{\partial r} \frac{\partial w}{r\partial \theta},$$
(2)

where $\varepsilon_r^{\circ}, \varepsilon_{\theta}^{\circ}$ and $\gamma_{r\theta}^{\circ}$ are the radial, circumferential and shear strains of the plate mid-plane, respectively. u and v are radial and circumferential in-plane displacements and w is the deflection. The Sanders' assumptions are also used to determine the curvature-rotation relationships:

$$\kappa_r^{\circ} = \frac{\partial \phi_r}{\partial r},$$

$$\kappa_{\theta}^{\circ} = \frac{\phi_r}{r} + \frac{\partial \phi_{\theta}}{r\partial \theta},$$

$$\kappa_{r\theta}^{\circ} = \frac{\partial \phi_{\theta}}{\partial r} + \frac{\partial \phi_r}{r\partial \theta} - \frac{\phi_{\theta}}{r},$$
(3)

where $\kappa_r^{\circ}, \kappa_{\theta}^{\circ}$ and $\kappa_{r\theta}^{\circ}$ are mid-plane curvatures, ϕ_r and ϕ_{θ} are rotations.

These relationships stem from the fact that the shear deformations are accounted for and, in terms of the differentials of displacements, they are stated below:

$$\gamma^{\circ}_{\theta z} = \frac{\partial w}{r \partial \theta} + \phi_{\theta},$$

$$\gamma^{\circ}_{rz} = \frac{\partial w}{\partial r} + \phi_{r},$$
 (4)

where γ_{rz}° and $\gamma_{\theta z}^{\circ}$ are plate mid-plane shear strains. General forms of the unsymmetrically laminated plate constitutive equation are given here. They include the coupling between in-plane strains and curvatures in the stress resultant equations, and the coupling between curvatures and strains in the stress couple equations. Consequently, in general form, they are stated as follows:

$$N_{\alpha} = \int_{-h/2}^{h/2} \sigma_{\alpha} dz \quad \text{and} \quad M_{\alpha} = \int_{-h/2}^{h/2} \sigma_{\alpha} z dz, \quad (5)$$

where $\alpha = r, \theta, r\theta$ and, in expanded form, they are as follows:

$$\begin{cases} N_{r} \\ N_{\theta} \\ N_{r\theta} \\ N_{r\theta} \end{cases} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{cases} \varepsilon_{r}^{\circ} \\ \varepsilon_{\theta}^{\circ} \\ \gamma_{r\theta}^{\circ} \end{cases} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{cases} \kappa_{r}^{\circ} \\ \kappa_{\theta}^{\circ} \\ \kappa_{r\theta}^{\circ} \end{cases} ,$$

$$\begin{cases} M_{r} \\ M_{\theta} \\ M_{r\theta} \end{cases} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{cases} \varepsilon_{r}^{\circ} \\ \varepsilon_{\theta}^{\circ} \\ \gamma_{r\theta}^{\circ} \end{cases} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{cases} \kappa_{r}^{\circ} \\ \kappa_{\theta}^{\circ} \\ \kappa_{r\theta}^{\circ} \end{cases} .$$

$$(6a)$$

The extensional, coupling, bending stiffness matrices are defined in terms of the reduced stiffness matrix, $\bar{\mathbf{Q}} = \bar{\mathbf{S}}^{-1}$, and the location of ply, z, as follows:

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{k} (z_{k} - z_{k-1}) \qquad i, j = 1, 2, 6,$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \bar{Q}_{ij}^{k} (z_{k}^{2} - z_{k-1}^{2}) \qquad i, j = 1, 2, 6,$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{Q}_{ij}^{k} (z_{k}^{3} - z_{k-1}^{3}) \qquad i, j = 1, 2, 6, \qquad (6b)$$

The transverse shear stress resultants are as stated below:

$$\begin{cases} Q_r \\ Q_\theta \end{cases} = \begin{bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{bmatrix} \begin{cases} \gamma_{rz}^{\circ} \\ \gamma_{\theta z}^{\circ} \end{cases},$$
 (6c)

where:

$$A_{ij} = \sum_{k=1}^{N} K_i K_j Q_{ij}^k (z_k - z_{k-1}) \qquad i, j = 4, 5.$$
 (6d)

 K_1 and K_2 are shear correction factors of FSDT.

	Along the Radial Edge		Along the Circumferential Edge	
Edge Condition	Simply Supported	Clamped	Simply Supported	Clamped
In-plane fixed	u = v = w = 0 $\phi_r = 0$ $M_\theta = 0$	$u = v = w = 0$ $\phi_r = \phi_\theta = 0$	u = v = w = 0 $\phi_{\theta} = 0$ $M_r = 0$	$u = v = w = 0$ $\phi_r = \phi_\theta = 0$
In-plane free	w = 0 $\phi_r = 0$ $N_{\theta} = N_{r\theta} = M_{\theta} = 0$	w = 0 $\phi_r = \phi_\theta = 0$ $N_\theta = N_{r\theta} = 0$	w = 0 $\phi_{\theta} = 0$ $N_r = N_{r\theta} = M_r = 0$	w = 0 $\phi_r = \phi_\theta = 0$ $N_r = N_{r\theta} = 0$

Table 1. Boundary conditions of the plate radial and circumferential edges.

Two types of boundary condition are used in the present study. They are simply supported and clamped edge constraints, each with fixed and moving boundaries. The boundary conditions are such defined that both edge conditions can be applied to different edges of a plate. The mathematical expressions in Table 1 can be used to define the above edge constraints along the two radial and two circumferential edges.

PLATE GOVERNING EQUATIONS NUMERICAL SOLUTION

There are several numerical procedures that may be chosen for the solution of non-linear partial differential equations, i.e. plate equilibrium equations. Among them, the most suitable one, considering the first author's experience, is the Dynamic Relaxation (DR) iterative method. This method is accompanied by the finite difference discretization technique. The DR method is only applied to the plate equilibrium equations, and the finite difference discretization technique is used to discretize the partial differential terms in the plate governing equations. The DR algorithm is a time stepping initial value iterative procedure. Consequently, it cannot be used directly to solve the boundary value problem given by Equations 1-5. Therefore, Equations 1 must first be transformed to an initial value format. This is achieved by introducing damping and inertia terms to the right hand side of Equations 1 and, then, the following finite difference approximations for velocity and accelerations can be applied:

$$\dot{\alpha} = 0.5[\dot{u}^a + \dot{u}^b],$$

$$\ddot{\alpha} = \delta t^{-1}, [\dot{u}^a - \dot{u}^b],$$
(7)

where α may be one of the following displacements or rotations: u, v, w, ϕ_r or ϕ_{θ} . The superscripts a and brefer to the values of the velocities after and before the time increment, δt , respectively. With the application of the above procedure, from the five equilibrium equations, the following linear and rotational velocity equations may be obtained.

$$\begin{split} \dot{u}^{a} &= \frac{1}{1+C_{u}^{*}} \Biggl\{ \left(1-C_{u}^{*}\right) \dot{u}^{b} \\ &+ \frac{\delta t}{\rho_{u}} \left[\frac{\partial N_{r}}{\partial r} + \frac{\partial N_{r\theta}}{r\partial \theta} + \frac{N_{r} - N_{\theta}}{r} \right] \Biggr\}, \\ \dot{v}^{a} &= \frac{1}{1+C_{v}^{*}} \Biggl\{ \left(1-C_{v}^{*}\right) \dot{v}^{b} \\ &+ \frac{\delta t}{\rho_{v}} \left[\frac{\partial N_{\theta}}{r\partial \theta} + \frac{\partial N_{r\theta}}{\partial r} + \frac{2N_{r\theta}}{r} \right] \Biggr\}, \\ \dot{w}^{a} &= \frac{1}{1+C_{w}^{*}} \Biggl\{ \left(1-C_{w}^{*}\right) \dot{w}^{b} \\ &+ \frac{\delta t}{\rho_{w}} \left[\left(\frac{Q_{r}}{r} + \frac{\partial Q_{r}}{\partial r} + \frac{\partial Q_{\theta}}{r\partial \theta} \right) + N_{r} \frac{\partial^{2} w}{\partial r^{2}} \\ &+ N_{\theta} \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^{2}} \frac{\partial^{2} w}{\partial \theta^{2}} \right) \\ &+ 2N_{r\theta} \left(\frac{1}{r} \frac{\partial^{2} w}{\partial r\partial \theta} - \frac{1}{r^{2}} \frac{\partial w}{\partial \theta} \right) + q \Biggr] \Biggr\}, \\ \dot{\phi}^{a}_{r} &= \frac{1}{1+C_{\phi_{r}}^{*}} \Biggl\{ \left(1-C_{\phi_{r}}^{*}\right) \dot{\phi}^{b}_{r} \\ &+ \frac{\delta t}{\rho_{\phi_{r}}} \left[\frac{\partial M_{r}}{\partial r} + \frac{\partial M_{r\theta}}{r\partial \theta} + \frac{M_{r} - M_{\theta}}{r} - Q_{r} \right] \Biggr\}, \end{aligned}$$

$$(8)$$

in which $C^*_{\alpha} = \frac{1}{2}C_{\alpha}\delta t\rho_{\alpha}^{-1}$, α and superscripts *a* and *b* are as defined above.

The finite difference discretization technique is applied to the right hand side of Equations 1 in the square brackets. Using the following simple integration relations, the displacement and rotations are calculated at the end of each time step, δt :

$$\alpha^a = \alpha^b + \delta t \dot{\alpha}^a,\tag{9}$$

in which α and superscripts *a* and *b* are the same as those defined in Equations 7. Consequently, Equations 8, 9, 2 and 5, together with the appropriate boundary conditions, Equations 6, in their finite difference form, constitute the set of equations suitable for the DR algorithm, which briefly is outlined in Figure 2.

STABILITY OF DR ALGORITHM

The stability and convergence of the DR algorithm is heavily dependent on the correct selection of fictitious densities and damping factors. The densities are automatically computed, using the procedure originally



Figure 2. Flowchart for DR procedure.

developed by Cassel and Hobbs and presented in [28]. Based on the procedure in [28], the pseudo-densities, ρ_{α} , are set to one quarter of the absolute values of the largest rowsum of the stiffness matrix implicit in the finite difference discretization. So, the time step, δt , is set to one.

The damping factors which also control the convergence of the iterations have been computed from the method proposed in [29]. To calculate the critical time step, the vibration of the dynamical system with zero damping factors is considered. Following the variation of kinetic energies, the number of iterations to reach the first true maximum of total kinetic energies for each direction of displacement is calculated. These are equivalent to one quarter of a cycle of the fundamental mode. If the kinetic energy is the maximum after N_{α} iterations, then, the critical damping factor is given by:

$$C_{\alpha} = \pi / N_{\alpha} \delta t. \tag{10}$$

This procedure has been successfully employed in previous studies.

NUMERICAL RESULTS

Graphical numerical results are useful in terms of observing variations of a parameter and making the appropriate design or other decisions. Fiber-reinforced laminated composite plates have many different parameters, which may be studied. Consequently, the number of examples had to be limited, as well as the output results.

A set of load-deflection results are presented for correlation purposes and the new results are presented for the first time.

A comprehensive set of results for symmetrical laminated sector plates has been presented previously [4]. In the following subsections, graphical results for unsymmetrical Graphite/Epoxy laminates have been presented for annular sector plates.

The numerical results are illustrated along the radial line of symmetry for deflections, radial displacements and rotations about the *r*-axis for different sector angles, fiber orientations and thicknesses, for both simply supported (SSSS) and clamped edge (CCCC) constraints with in-plane fixed and free boundaries. A similar set of results is presented along the midradius circumferential direction for deflections, circumferential displacements and rotations about the θ -axis.

This subsection is divided into three subsections for better description of the present results. A comparison with FE results is presented for simply supported and clamped edge plates.



Figure 3. Load-center deflection for fiber arrangement [60/-60/90/10/-45/45/30/10], $r_o/r_i = 2$, $\theta = 90^\circ$.

Different Loads

In order to verify the accuracy of the computational procedure outlined above, and to illustrate the nonlinear behavior of the plate, a set of load-center deflection results is presented for simply supported and clamped annular sector plates in Figure 3. The fiber arrangement is [60/-60/90/10/-45/45/30/10], $r_o/r_i = 2$, $\theta = 90^{\circ}$ and two thicknesses for SSSS and CCCC edges under dimensionless uniform loads, $\bar{q}(=qr_o^4/E_2h^4)$, are assumed.

On the same graph, the non-linear Finite Element Method (FEM) results are presented for comparison purposes. The FE results are generated by the ANSYS commercial code using the nonlinear 8 noded shell element [30]. The results show that the FE predicts slightly higher values for deflections at high pressures. However, for smaller values of pressure, the correlations are very good.

Different Plate Thicknesses

The next set of results corresponds to $[\pm 60/\mp 45]_2$

fiber orientation, $\theta = 90^{\circ}$ sector angle, two uniform pressure and two different boundary conditions (see Figure 4). Although the FEM predicts slightly higher deflections than the DR method for thicker plates, they correlate well for thinner plates (see Figure 4a). The correlation for the clamped edge plate is very good (Figure 4b).

Different Fiber Angle

Different fiber angles, from 0° to 90°, are chosen for a single layer composite annular sector plate with $r_o/r_i = 2$, $\theta = 120^\circ$ and $h = 0.1(r_o - r_i)$. Deflections along the radial and circumferential lines of symmetry with inplane free and in-plane fixed boundary conditions are illustrated in Figures 5 and 6, respectively. The SSSS plates are under a uniform pressure of $\bar{q} = 100$ and CCCC plates are under a uniform pressure of $\bar{q} = 300$. From the experience gained in isotropic sector plate analysis [20], the deflections along the radial line of symmetry of all angles of 0° to 90° fiber orientation are symmetric at about the mid-radial point (Figures 5a and 5c). However, this is not true for the deflections

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along the mid-radius at different angles (Figures 5b and 5d). The deflections exhibit the expected behavior, i.e. larger deflections for 0° sector angles.

Different Fiber Arrangements

The final set of results is presented for the following fiber arrangements:

- I. $(0/90)_4$ fixed edge,
- II. $(45/-45)_4$ fixed edge,
- III. $(0/90)_4$ free edge,
- IV. $(45/-45)_4$ free edge,

with $r_o/r_i = 5$, $\theta = 90^\circ$, $h = 0.1(r_o - r_i)$ and uniform pressure $\bar{q} = 200$ for simply supported plates and $\bar{q} = 400$ for clamped plates. The simply supported



Figure 4. Center deflection for different thicknesses of fiber arrangement $[\pm 60/\mp 45]_2$, $r_o/r_i = 2$, $\theta = 90^\circ$.



Figure 5. Deflections for different laminae with in-plane fixed boundary.

and clamped plate deflections, in-plane displacements and rotations are illustrated in Figures 7 and 8, respectively, for in-plane free and fixed conditions. The results include dimensionless radial displacements, $\bar{u}(=\frac{ur_o}{h^2})$, circumferential displacements, $\bar{v}(=\frac{vr_o}{h^2})$, deflections, $\bar{w}(=\frac{w}{h})$ and rotations about the radial axis, $\bar{\phi_r}(=\frac{\phi_r r_o}{h})$. The radial displacements for in-plane fixed cases I and II are small in comparison to those for free in-plane edge conditions, cases III and IV, of about 15 times, as shown in Figure 7a, and of about 7 times, as shown in Figure 8a. There is almost no difference in transverse displacements (or deflections) for cases I and II, as shown in Figures 7c and 7d, which means that the fiber orientations have no effect on the plate behavior in terms of displacements. A similar behavior is observed for cases III and IV for simply supported plates, as shown in Figures 7c and 7d. However, the deflections for cases I and II and III and IV are the same, as shown in Figures 8c and 8d. This means that in clamped plates, the fiber orientations have similar responses, in terms of displacements under uniform loading conditions. However, case I is more suitable than the other two cases, in terms of exhibiting smaller deflections for the same conditions of the plate. The rotation results for the four cases are shown in Figures 7e and 7f for simply supported plates, and Figures 8e and 8f for clamped plates. The rotations for cases I and III are nearly the same, whereas those for cases II and IV are very different. This means that the in-plane fixed or in-plane free edge conditions for fiber orientations of $(0/90)_4$ do not make a significant difference to the rotations about the radial axis, whereas they make a lot of difference to the $(45/-45)_4$ fiber arrangements, $\overline{\phi}_r$.

CONCLUDING REMARKS

A linear and non-linear, non-axisymmetric formulation for symmetrically, as well as unsymmetrically, fiberreinforced, laminated, thick annular sector plates is developed. The loading is uniform pressure with a simply supported and clamped edge with fixed or moving



Figure 6. Deflections for different laminae with in-plane free boundary.



Figure 7. Displacements of simply supported plate for different fiber arrangement, $\bar{q} = 200$.

in-plane conditions. The DR iterative algorithm is successfully applied and the stability of the algorithm is controlled by the correct and automatic selection of the fictitious densities and damping factors. A verification of the present numerical results is carried out by modeling the same plate in the Finite Element Method (FEM) code. The correlation of the present results and the FEM results is very satisfactory. Unpredictable displacement results were obtained, which justify presentation of the new results in the present study. The unusual effect of boundary conditions on the displacements for this type of plate configuration is appreciable.



Figure 8. Displacements of clamped plate for different fiber arrangement, $\bar{q} = 400$.

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NOMENCLATURE

 $\begin{array}{ll} A_{ij}(i,j=1,2,6) \text{ extensional stiffness} \\ A_{ij}(i,j=4,5) & \text{transverse shear stiffness} \\ B_{ij}(i,j=1,2,6) \text{ coupling stiffness} \\ D_{ij}(i,j=1,2,6) \text{ flexural stiffness} \\ C_u, C_v, C_w & \text{in and out-of-plane fictitious damping} \\ C_{\phi_w}, C_{\phi_{\theta_w}} & \text{rotational fictitious damping} \end{array}$

$K_i, K_j(i, j=4, 5)$)shear correction factor
k	layer number
$M_r, M_{\theta}, M_{r\theta}$	radial, circumferential and twisting stress couples
$N_r, N_{\theta}, N_{r\theta}$	radial, circumferential and shear stress resultants
q	uniform pressure
$Q_{ij}(i,j=1\cdots 6)$)reduced stiffness matrix
Q_r, Q_{θ}	transverse shear stress resultants
r_i, r_0	inner and outer radius of annular sector plate
r, heta	polar coordinate system
u, v, w	radial and circumferential in-plane displacements and deflection
\dot{u},\dot{v},\dot{w}	radial, circumferential and transverse velocities
z	distance from neutral axis
$\varepsilon_r^\circ, \varepsilon_\theta^\circ$	radial and circumferential middle-plane strains $% \left(\frac{1}{2} \right) = 0$
$\gamma^{\circ}_{r\theta}, \gamma^{\circ}_{rz}, \gamma^{\circ}_{\theta z}$	shear strains
δt	time increment
$\kappa_r^\circ,\kappa_ heta^\circ,\kappa_{r heta}^\circ$	middle-plane curvatures
$ ho_u, ho_v, ho_w$	in and out-of-plane fictitious densities
$ ho_{\phi_r}, ho_{\phi_ heta}$	rotational fictitious densities
σ	stress
Θ	sector angle
Θ_m	fiber angle
Θ_g	angle related to sector angle
$\phi_r, \phi_ heta$	rotations
$\dot{\phi}_r, \dot{\phi}_ heta$	rotational velocities

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APPENDIX

Using the transformation matrices, \mathbf{T}_{σ} and \mathbf{T}_{ε} , for stresses and strains, the compliance matrix in cylindrical coordinate $\mathbf{\bar{S}}$, at angle $\boldsymbol{\Theta}_{\mathbf{g}}$, in which the analysis is performed, in terms of stiffness in the material coordinate, \mathbf{S} , and the angle of fiber orientation in each ply, $\boldsymbol{\Theta}_{\mathbf{m}}$, is stated as:

$$\bar{S} = T_{\varepsilon}(\Theta_g) T_{\varepsilon}^{-1}(\Theta_m) S T_{\sigma}(\Theta_m) T_{\sigma}^{-1}(\Theta_g).$$
(A1)

BIOGRAPHIES

Manouchehr Salehi completed his BS (Hons) in production engineering in 1984 and his MPhil. in mechanical engineering in 1987 at Leeds Metropolitan University. He then moved to Lancaster University in the UK where he undertook a comprehensive research program on numerical and experimental analysis of stiffened and unstiffened sector plates towards his PhD degree, which he completed in 1990. He then returned to his homeland. Iran, to take up the position of Assistant Professor in solid mechanics at the Mechanical Engineering Department of Amirkabir University of Technology in Tehran, Iran, in 1990. Dr Salehi is now an Associate Professor in solid mechanics at the same university. He has presented fifty papers at International and National conferences in over ten countries. He has more than twenty referred journal publications and has authored a book on the 'Optimization of Composite Structures using Genetic Algorithm'.

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