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Improvements to the Mathematical Model of Acoustic Wave Scattering from Transversely Isotropic Cylinders

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Abstract. *This paper considers the scattering of an infinite plane acoustic wave from a long immersed, solid, transversely isotropic cylinder. The mathematical model which has already been developed for this problem does not work in the case of a normally incident wave. Modifications to the mathematical model are proposed in order to make it applicable to all incidence angles, including $\alpha = 0$. Numerical results are used to demonstrate the correctness of the modified equations. Moreover, using a mathematical discussion, it is shown that at normal incidence, the whole displacement field is constrained within the isotropic plane of the cylinder (cylinder cross section) and only the two elastic constants characterizing this plane appear in the remaining equations. A perturbation study on the five elastic constants of the transversely isotropic cylinder confirms this result.*

Keywords: *Acoustic wave; Scattering; Transversely isotropic; Cylinder.*

INTRODUCTION

Circular components such as cylinders, rods, pipes, and tubes are widely used in oil, gas, petrochemical, transportation and power industries. The nondestructive evaluation of cylindrical components has received much attention in recent years. Among various techniques used for characterizing material properties and detecting defects, ultrasonic techniques are the most widely used. When the immersion ultrasonic technique is employed for material characterization, a theoretical model of acoustic scattering (or reflection) from the sample is needed for a quantitative evaluation [1]. Most previous studies are concerned with isotropic cylinders. However, many engineering components have anisotropically embedded reinforcements or unintended anisotropy produced during manufacturing processes. Examples are axially fiber-reinforced composite rods manufactured by the extrusion processes.

Resonance Acoustic Spectroscopy (RAS) is the study of resonance effects present in reflected acoustic echoes from an elastic target. These resonance effects are caused by the excitation of eigenvibrations of the target by an incident acoustic wave. RAS and other acoustic scattering techniques have been used for nondestructive evaluation of materials, material characterization and remote classification of submerged targets [2-4].

The interest in acoustic wave scattering from solid obstacles dates back to the time of Lord Rayleigh [5]. Early studies of wave scattering from solid elastic cylinders, conducted by Faran [6], dealt with normally incident compression waves incident on a submerged infinite homogeneous elastic isotropic rod. The more general problem of the scattering of an obliquely incident plane wave from an infinite elastic cylinder was studied by Flax et al. [7]. Similar problems for a cylindrical shell were studied by Leon et al. [8] and Veksler [9]. Until a decade ago, all mathematical models developed for acoustic wave scattering from elastic targets only dealt with isotropic materials. The first mathematical model for the scattering of plane acoustic waves from an anisotropic cylinder was developed by Honarvar and Sinclair in 1996 [10]. They used a normal mode expansion method for solving the acoustic wave scattering problem. The anisotropy symmetry considered was hexagonal (transverse isotropy). An alternative formulation, based on the same mathematical method for this problem, was later presented by Ahmad and Rahman [11]. These works were complemented by a

number of other papers discussing various aspects of this problem [12-14].

The mathematical models developed in [10,11] have been used by many researchers in studying different problems dealing with the scattering of acoustic waves from transversely isotropic cylinders. Following the formulation presented in [10], Qian et al. [15,16] studied the problem of the scattering of P-waves from 1-3 piezo-composite cylinders. They considered the scattering of a longitudinal plane elastic wave incident at arbitrary angles on a transversely isotropic piezo-electric cylinder surrounded by a polymeric matrix medium. Pan et al. [17] studied the acoustic field of a transversely isotropic cylinder generated by a laser line pulse in either the ablation or thermo-elastic regime. In another study, this group considered waves generated by a laser point source in an isotropic cylinder [18] and used it for measurement of the elastic constants of the cylinder [19]. They also modeled bulk and surface acoustic waves generated in a transversely isotropic cylinder by a laser point source [20] and used it for stiffness tensor measurements [21].

All the above mathematical models use analytical or semi analytical methods for solving acoustic wave scattering problems. There are also a number of numerical methods for modeling acoustic wave scattering problems including finite element [22-24], infinite element [22], boundary element, and coupled finite/boundary element methods [25].

The mathematical models presented for the scattering of plane acoustic waves from immersed transversely isotropic cylinders in [10,11], although correct, both suffer from a deficiency; they cannot be used for normally incident waves.

In this paper, we show how these mathematical models can be modified to overcome this deficiency. Moreover, using the derived equations, we show that in the case of a normally incident wave, the cylinder behaves exactly as an isotropic material and the wave only has displacement components in the isotropic plane (cross section) of the cylinder. The normal mode expansion method will be used throughout this paper for solving the acoustic wave scattering problem.

OVERVIEW OF CURRENT FORMULATION OF THE PROBLEM

To present the proposed modifications, we first need to review the mathematical model developed for the scattering of a plane acoustic wave from an immersed infinite transversely isotropic solid elastic cylinder following [10]. In formulating the problem, an infinite plane acoustic wave of circular frequency, ω, incident at an angle, α, on an infinite submerged transversely isotropic cylinder, is considered (Figure 1). A cylindrical coordinate system (r, θ, z) is chosen such that
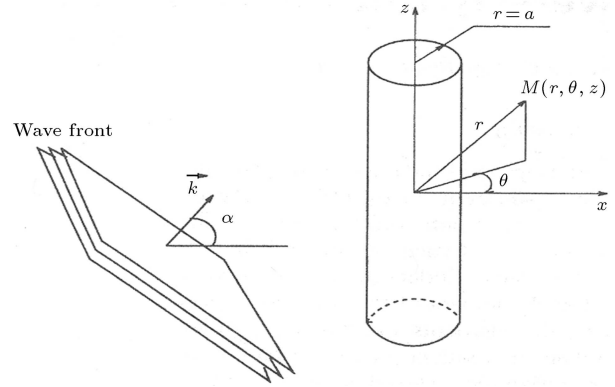

Figure 1. Infinite plane acoustic wave obliquely incident on a submerged infinite transversely isotropic cylinder.

the z direction coincides with the axis of the cylinder. The incident wave pressure, p_i, at an arbitrary point, $M(r, \theta, z)$, is:

$$p_i = p_\circ \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(k_\perp r) \cos(n\theta) e^{i(k_z z - \omega t)}, \qquad (1)$$

where $k_z = k \sin \alpha$, $k_\perp = k \cos \alpha$, and $k = \omega/c$, c is the compression wave velocity in the liquid medium surrounding the cylinder, ε_n is the Neumann factor ($\varepsilon_0 = 1$ and $\varepsilon_n = 2$ for $n > 0$), $p_\circ$ is the incident pressure wave amplitude, and J_n are the Bessel functions of the first kind of order n. The scattered wave pressure, p_s, at an arbitrary point, M, is symmetrical about $\theta = 0$ and of the following form:

$$p_s = p_\circ \sum_{n=0}^{\infty} \varepsilon_n i^n A_n H_n^{(1)}(k_\perp r) \cos(n\theta) e^{i(k_z z - \omega t)}, \quad (2)$$

where $H_n^{(1)}$ are the Hankel functions of the first kind of order n, and A_n are unknown scattering coefficients.

A transversely isotropic material is characterized by five independent elastic constants, c_{11}, c_{12}, c_{13}, c_{33}, and c_{44}. The general Hooke's law for a transversely isotropic material is:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{z\theta} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{z\theta} \\ \varepsilon_{rz} \\ \varepsilon_{r\theta} \end{Bmatrix}, \qquad (3)$$

where σ_{ij} are stress components and ε_{ij} are strain components. In a cylindrical coordinate system, the

equations of motion of a continuum in the absence of body forces can be written as:

$$
c_{44}\left(\frac{\partial^2 U_z}{\partial z \partial r}+\frac{\partial^2 U_r}{\partial z^2}\right)+c_{11}\left(\frac{\partial^2 U_r}{\partial r^2}+\frac{1}{r}\frac{\partial U_r}{\partial r}\right.
$$

$$
-\frac{3}{2r^2}\frac{\partial U_\theta}{\partial \theta}+\frac{1}{2r^2}\frac{\partial^2 U_r}{\partial \theta^2}-\frac{U_r}{r^2}+\frac{1}{2r}\frac{\partial^2 U_\theta}{\partial r \partial \theta}\right)
$$

$$
+\frac{1}{2r^2}c_{12}\left[\frac{\partial}{\partial \theta}\left(\frac{\partial U_\theta}{\partial r}-\frac{1}{r}\frac{\partial U_r}{\partial \theta}+\frac{U_\theta}{r}\right)\right]
$$

$$
+c_{13}\frac{\partial^2 U_z}{\partial z \partial r}=\rho_c\frac{\partial^2 U_r}{\partial t^2}, \tag{4}
$$

$$
c_{44}\left[\frac{\partial}{\partial z}\left(\frac{\partial U_\theta}{\partial z}+\frac{1}{r}\frac{\partial U_z}{\partial \theta}\right)\right]+c_{12}\left(-\frac{1}{2r}\frac{\partial U_\theta}{\partial r}\right.
$$

$$
+\frac{1}{2}\frac{U_\theta}{r^2}-\frac{1}{2}\frac{\partial^2 U_\theta}{\partial r^2}-\frac{1}{2r^2}\frac{\partial U_r}{\partial \theta}+\frac{1}{2r}\frac{\partial^2 U_r}{\partial r \partial \theta}\right)
$$

$$
+c_{11}\left(\frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial \theta^2}+\frac{1}{2r}\frac{\partial U_\theta}{\partial r}+\frac{1}{2}\frac{\partial^2 U_\theta}{\partial r^2}+\frac{3}{2r^2}\frac{\partial U_r}{\partial \theta}\right.
$$

$$
\left.+\frac{1}{2r}\frac{\partial^2 U_r}{\partial r \partial \theta}-\frac{1}{2}\frac{U_\theta}{r^2}\right)+\frac{c_{13}}{r}\frac{\partial^2 U_z}{\partial z \partial \theta}=\rho_c\frac{\partial^2 U_\theta}{\partial t^2}, \tag{5}
$$

$$
c_{44}\left(\frac{1}{r^2}\frac{\partial^2 U_z}{\partial \theta^2}+\frac{\partial^2 U_z}{\partial r^2}+\frac{1}{r}\frac{\partial^2 U_\theta}{\partial z \partial \theta}+\frac{1}{r}\frac{\partial U_z}{\partial r}+\frac{1}{r}\frac{\partial U_r}{\partial z}\right.
$$

$$
\left.+\frac{\partial^2 U_r}{\partial z \partial r}\right)+c_{13}\left[\frac{\partial}{\partial z}\left(\frac{U_r}{r}+\frac{1}{r}\frac{\partial U_\theta}{\partial \theta}+\frac{\partial U_r}{\partial r}\right)\right]
$$

$$
+c_{33}\frac{\partial^2 U_z}{\partial z^2}=\rho_c\frac{\partial^2 U_z}{\partial t^2}, \tag{6}
$$

where ρ_c is the cylinder density and U_r, U_θ, U_z are displacement components in the r, θ and z directions, respectively.

The potential function method is used for solving this problem and the displacement vector is written in terms of three scalar potential functions, ϕ, χ and ψ, as follows:

$$
\vec{u}=\nabla\phi+\nabla\times(\chi\hat{e}_z)+a\nabla\times\nabla\times(\psi\hat{e}_z), \tag{7}
$$

where a is the radius of the cylinder; a constant with dimensions of length. The substitution of Equation 7 into Equations 4-6 gives:

$$
\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\left\{c_{11}\nabla^2\phi+(c_{13}+2c_{44}-c_{11})\frac{\partial^2\phi}{\partial z^2}\right.
$$

$$
-\rho_c\frac{\partial^2\phi}{\partial t^2}+a\frac{\partial}{\partial z}\left[(c_{11}-c_{13}-c_{44})\nabla^2\psi\right.
$$

$$
\left.\left.+(c_{13}+2c_{44}-c_{11})\frac{\partial^2\psi}{\partial z^2}-\rho_c\frac{\partial^2\psi}{\partial t^2}\right]\right\}=0, \tag{8}
$$

$$
\frac{\partial}{\partial z}\left[(c_{13}+2c_{44})\nabla^2\phi+(c_{33}-c_{13}-2c_{44})\frac{\partial^2\phi}{\partial z^2}\right.
$$

$$
\left.-\rho_c\frac{\partial^2\phi}{\partial t^2}\right]+a\left(\frac{\partial^2}{\partial z^2}-\nabla^2\right)\left[c_{44}\nabla^2\psi\right.
$$

$$
\left.+(c_{33}-c_{13}-2c_{44})\frac{\partial^2\psi}{\partial z^2}-\rho_c\frac{\partial^2\psi}{\partial t^2}\right]=0, \tag{9}
$$

$$
\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\left[\frac{(c_{11}-c_{12})}{2}\nabla^2\chi\right.
$$

$$
\left.+\left(c_{44}-\frac{(c_{11}-c_{12})}{2}\right)\frac{\partial^2\chi}{\partial z^2}-\rho_c\frac{\partial^2\chi}{\partial t^2}\right]=0. \tag{10}
$$

Equations 8 and 9 represent the L (longitudinal) and SV (vertically polarized shear) waves. The longitudinal wave represented by ϕ, and the SV wave represented by ψ are coupled. According to Equation 10, the SH (horizontally polarized shear) wave, represented by χ, is decoupled from the other two wave types. To solve Equations 8-10 for ϕ, ψ and χ, the normal mode expansion method is used and solutions of the following forms are assumed:

$$
\phi=\sum_{n=0}^{\infty}B_n J_n(sr)\cos n\theta e^{i(k_z z-\omega t)}, \tag{11}
$$

$$
\psi=\sum_{n=0}^{\infty}C_n J_n(sr)\cos n\theta e^{i(k_z z-\omega t)}, \tag{12}
$$

$$
\chi=\sum_{n=0}^{\infty}D_n J_n(sr)\sin n\theta e^{i(k_z z-\omega t)}. \tag{13}
$$

Substituting Equations 11 and 12 into 8 and 9 gives:

$$
\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}B_n\\C_n\end{bmatrix}=0, \tag{14}
$$

where:

$$
a_{11}=-[-c_{11}s^2-(c_{13}+2c_{44})k_z^2+\rho_c\omega^2]s^2,
$$

$$
a_{12}=-aik_z[-(c_{11}-c_{13}-c_{44})s^2-c_{44}k_z^2+\rho_c\omega^2]s^2,
$$

$$
a_{21}=ik_z[-(c_{13}+2c_{44})s^2-c_{33}k_z^2+\rho_c\omega^2],
$$

$$
a_{22}=as^2[-(c_{44}s^2-(c_{33}-c_{13}-c_{44})k_z^2+\rho_c\omega^2]. \tag{15}
$$

For a nontrivial solution, the coefficient determinant of Equation 14 must vanish. This yields the following characteristic equation:

$$
c_{11}c_{44}s^4-\xi s^2+\zeta=0, \tag{16}
$$

where:

$$\xi = (c_{13} + c_{44})^2 k_z^2 + c_{11}(\rho_c \omega^2 - c_{33} k_z^2)$$

$$+ c_{44}(\rho_c \omega^2 - c_{44} k_z^2),$$

$$\zeta = (\rho_c \omega^2 - c_{44} k_z^2)(\rho_c \omega^2 - c_{33} k_z^2). \qquad (17)$$

Solving Equation 16 yields two roots, s_1 and s_2, as follows:

$$s_1^2 = \frac{\xi - \sqrt{\xi^2 - 4\zeta c_{11} c_{44}}}{2 c_{11} c_{44}}, \qquad (18)$$

$$s_1^2 = \frac{\xi + \sqrt{\xi^2 - 4\zeta c_{11} c_{44}}}{2 c_{11} c_{44}}. \qquad (19)$$

Therefore, the potential functions, ϕ and ψ, should be of the form:

$$\phi = \sum_{n=0}^{\infty} [B_n J_n(s_1 r) + q_2 C_n J_n(s_2 r)] \cos n\theta e^{i(k_z z - \omega t)}, \qquad (20)$$

$$\psi = \sum_{n=0}^{\infty} [q_1 B_n J_n(s_1 r) + C_n J_n(s_2 r)] \cos n\theta e^{i(k_z z - \omega t)}, \qquad (21)$$

where:

$$q_1 = -\frac{-c_{11} s_1^2 - (c_{13} + 2c_{44})k_z^2 + \rho_c \omega^2}{a i k_z [-(c_{11} - c_{13} - c_{44})s_1^2 - c_{44} k_z^2 + \rho_c \omega^2]}, \qquad (22)$$

$$q_2 = -\frac{a i k_z [-(c_{11} - c_{13} - c_{44})s_2^2 - c_{44} k_z^2 + \rho_c \omega^2]}{-c_{11} s_2^2 - (c_{13} + 2c_{44})k_z^2 + \rho_c \omega^2}. \qquad (23)$$

Moreover, for the scalar potential function, χ, corresponding to the SH wave:

$$s_3^2 = \frac{2(\rho_c \omega^2 - c_{44} k_z^2)}{c_{11} - c_{12}}, \qquad (24)$$

and therefore:

$$\chi = \sum_{n=0}^{\infty} D_n J_n(s_3 r) \sin n\theta e^{i(k_z z - \omega t)}. \qquad (25)$$

The boundary conditions at $r = a$ are:

$$-\frac{1}{\rho} \frac{\partial}{\partial r}(p_i + p_s) = \frac{\partial^2 U_r}{\partial t^2},$$

$$\sigma_{rr} = -(p_i + p_s), \qquad \sigma_{r\theta} = 0, \qquad \sigma_{rz} = 0, \qquad (26)$$

where ρ is the density of the fluid surrounding the cylinder. Inserting the potential functions from Equations 20, 21 and 25 in Equation 26, a system of four

linear algebraic equations is obtained for each value of n:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix}. \qquad (27)$$

Elements a_{ij} and b_{ij} of the matrices appearing in Equation 27 are given in the Appendix. Equation 27 can be solved for A_n for any desired frequency and position angle, θ. The usual approach is to solve the problem in the far field ($r \gg a$) at a specific angle, θ, for a range of frequencies. The resulting far-field amplitude spectrum, which is called the 'form function', is obtained from the following equation [26]:

$$|f_\infty| = \left(\frac{2r}{a}\right)^{1/2} \left(\frac{p_s}{p_o}\right) e^{-ikr}. \qquad (28)$$

The above formulation cannot be used if the wave incidence angle is zero. In the following section, we present the required modifications in order to avoid this deficiency.

PROPOSED MODIFICATIONS TO THE FORMULATION

Although the above mathematical formulation is derived for any arbitrary angle of incidence, it turns out to be singular when the wave is normally incident ($\alpha = 0$) on the cylinder. In this case, the axial wave vector, k_z, equals zero ($k_z = k \sin \alpha = 0$) and, consequently, Equation 22 tends to infinity.

We suggest the following modifications to Equations 22 and 23 in order to avoid this situation and make the formulation applicable to all values of α. For having a nontrivial solution to Equation 14, the determinant of the coefficient matrix should vanish, i.e:

$$a_{11}/a_{12} = a_{21}/a_{22}. \qquad (29)$$

From Equations 15, 22 and 23, we note that:

$$q_1 = -\frac{-c_{11} s_1^2 - (c_{13} + 2c_{44})k_z^2 + \rho_c \omega^2}{a i k_z [-(c_{11} - c_{13} - c_{44})s^2 - c_{44} k_z^2 + \rho_c \omega^2]}$$

$$= -\frac{a_{11}}{a_{12}}, \qquad (30)$$

$$q_1 = -\frac{a i k_z [-(c_{11} - c_{13} - c_{44})s_2^2 - c_{44} k_z^2 + \rho_c \omega^2]}{-c_{11} s_2^2 - (c_{13} + 2c_{44})k_z^2 + \rho_c \omega^2}$$

$$= -\frac{a_{12}}{a_{11}}. \qquad (31)$$

The terms on the right hand side of Equations 30 and 31 can be replaced with their equivalent values based on Equation 29. This replacement gives:

$$q_1^* = -\frac{a_{11}}{a_{12}} = -\frac{a_{21}}{a_{22}}$$

$$= -\frac{ik_z[-(c_{13} + 2c_{44})s_1^2 - c_{33}k_z^2 + \rho_c\omega^2]}{as_1^2[-c_{44}s_1^2 - (c_{33} - c_{13} - c_{44})k_z^2 + \rho_c\omega^2]}, \quad (32)$$

$$q_2^* = q_2$$

$$= -\frac{aik_z[-(c_{11} - c_{13} - c_{44})s_2^2 - c_{44}k_z^2 + \rho_c\omega^2]}{-c_{11}s_2^2 - (c_{13} + 2c_{44})k_z^2 + \rho_c\omega^2}. \quad (33)$$

The above transformation does not affect the final results and makes the equations applicable to the case of normally incident waves. The transformation has moved the k_z term from the denominator to the numerator of Equation 32. The substitution of Equations 32 and 33 into Equations 20 and 21 gives:

$$\phi = \sum_{n=0}^{\infty}[B_n J_n(s_1 r) + q_2^* C_n J_n(s_2 r)]\cos n\theta e^{i(k_z z - \omega t)}, \quad (34)$$

$$\psi = \sum_{n=0}^{\infty}[q_1^* B_n J_n(s_1 r) + C_n J_n(s_2 r)]\cos n\theta e^{i(k_z z - \omega t)}. \quad (35)$$

The same modification can be applied to the mathematical model presented in [11] and would lead to similar results.

THE CASE OF A NORMALLY INCIDENT WAVE

From the physics of the problem it is known that for a normally incident wave, the surface waves travel along the circumference of the cylinder and, therefore, do not depend on the axial properties of the cylinder. However, considering that the elastic constants, c_{13} and c_{44}, relate the elastic properties of the isotropic plane (cylinder cross section) to those in the axial direction, it cannot readily be recognized whether or not, for normally incident waves, the scattering field depends on these constants. In the following, we try to answer this question.

For the case of a normally incident wave $k_z = 0$ ($\alpha = 0$) and Equations 32 and 33 reduce to:

$$q_1^* = q_2^* = 0. \quad (36)$$

Consequently, the potential functions will be of the following forms:

$$\phi = \sum_{n=0}^{\infty} B_n J_n(s_1 r)\cos n\theta e^{-i\omega t}, \quad (37)$$

$$\psi = \sum_{n=0}^{\infty} C_n J_n(s_2 r)\cos n\theta e^{-i\omega t}, \quad (38)$$

$$\chi = \sum_{n=0}^{\infty} D_n J_n(s_3 r)\sin n\theta e^{-i\omega t}, \quad (39)$$

where:

$$s_1^2 = \rho_c\omega^2/c_{11} = \frac{\omega^2}{(c_{11}/\rho_c)}, \quad (40)$$

$$s_2^2 = \rho_c\omega^2/c_{44} = \frac{\omega^2}{(c_{44}/\rho_c)}, \quad (41)$$

$$s_3^2 = \frac{\rho_c\omega^2}{[(c_{11} - c_{12})/2]}. \quad (42)$$

Although Equations 37-39 are similar to the potential functions of an isotropic cylinder, as given in [6], in the new formulation, these potential functions are functions of three independent elastic constants, c_{11}, c_{12} and c_{44}, while, for an isotropic material, there are only two independent elastic constants.

Equations 40-42 indicate that, in the case of a normally incident wave, ϕ is a function of c_{11}, ψ is a function of c_{44}, and χ is a function of both c_{11} and c_{12}. Moreover, ϕ corresponds to the longitudinal wave, and ψ and χ correspond to shear waves. The substitution of $k_z = 0$ into the coefficients matrix of Equation 27 gives (see Appendix):

$$a_{13} = a_{23} = a_{33} = a_{42} = a_{44} = 0, \quad (43)$$

and therefore:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix}. \quad (44)$$

This requires that:

$$C_n = 0. \quad (45)$$

According to Equation 38, if C_n is equal to zero, then, $\psi = 0$, which means that when the wave is normally incident on the cylinder, only the longitudinal wave and one type of shear wave are present. Moreover, these two waves depend only on the elastic constants characterizing the cross-sectional plane (isotropic plane) of the cylinder, i.e. c_{11} and c_{12}. By taking $\psi = 0$, the

displacement decomposition introduced in Equation 7 will reduce to the familiar Helmholtz decomposition for potential functions, ϕ and χ, which is commonly used for solving problems dealing with isotropic materials.

We observe that in the case of a normally incident wave, s_1 and s_3 are functions of only two elastic constants (c_{11} and c_{12}). Therefore, it can be concluded that, in this case, the cylinder is indeed behaving as an isotropic material.

NUMERICAL RESULTS

To verify the modified mathematical model, form functions of both isotropic and transversely isotropic cylinders are calculated at different incidence angles.

First, the scattered field from an immersed isotropic aluminum cylinder is calculated at $\alpha = 0$, and $\alpha = 3$ degrees. The physical properties of aluminum are given in Table 1. The corresponding form functions are shown in Figures 2 and 3 for the frequency range of $0 < ka < 20$. These results are identical to earlier results, which are reported in [27] and shown in Figure 4. In Figures 5 and 6, the scattered field of the immersed isotropic aluminum cylinder calculated at higher incident angles of $\alpha = 20°$ and $50°$ is shown.
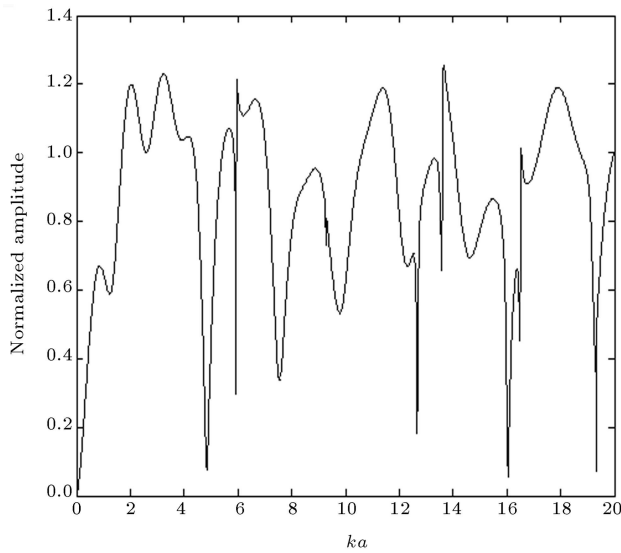
Next, the scattered pressure field from a cobalt

cylinder with transversely isotropic elastic properties is calculated at incidence angles of $\alpha = 0°$ and $\alpha = 5°$. The corresponding functions are shown in Figures 7 and 8, respectively. The physical properties of cobalt are given in Table 1. In these numerical calculations, and in those which follow, the number of normal modes, N, used in evaluating the series is $N = ka_{\max} + 5$ where $ka_{\max}$ is the maximum value of the
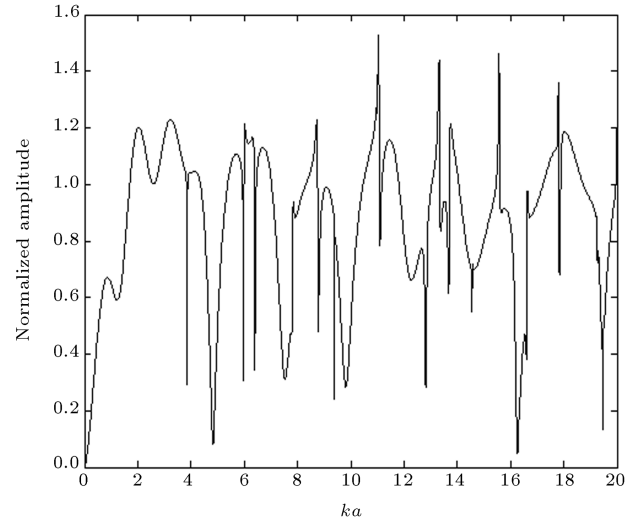

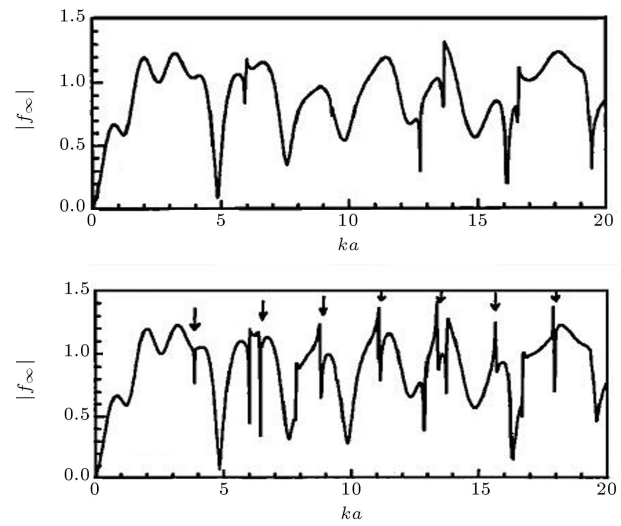**Figure 3.** Form function of an aluminum cylinder at $\alpha = 3°$.

Figure 4. Form function of an aluminum cylinder at $\alpha = 0°$ and $3°$ [27].

Figure 2. Form function of an aluminum cylinder at $\alpha = 0°$.

Table 1. Material properties.

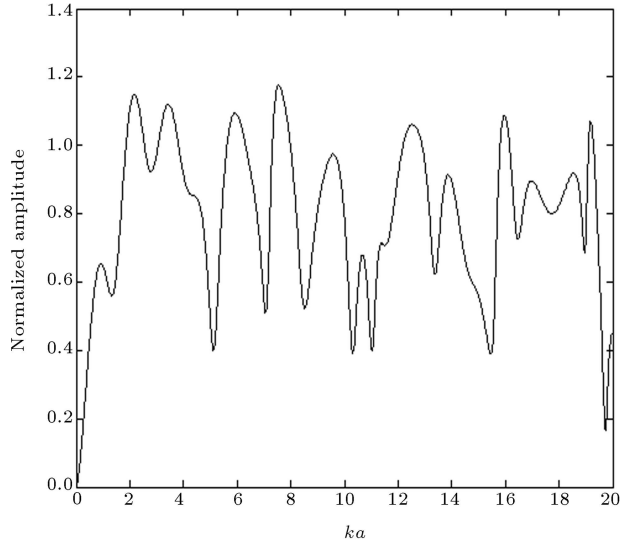| Material	Type	Stiffness $\times 10^{11}$ (N/m²)					Density (kg/m³)
		c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	
Aluminum	Isotropic	1.1087	0.6115	0.6115	1.1087	0.2486	2694
Cobalt	Trans. iso.	2.95	1.59	1.11	3.35	0.71	8900

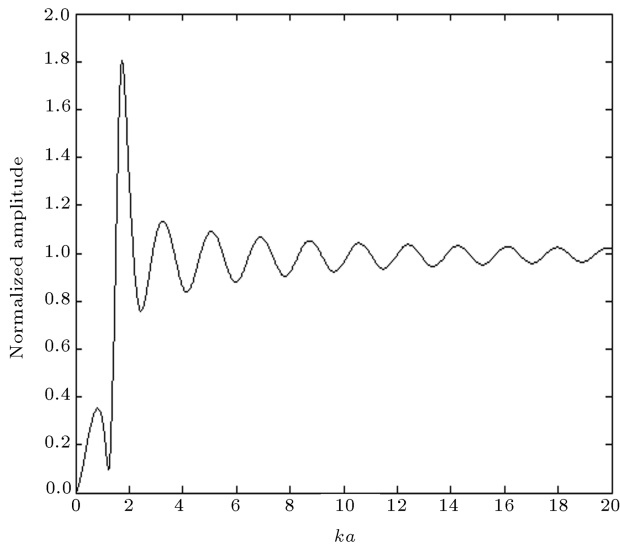**Figure 5.** Form function of an aluminum cylinder at $\alpha = 20°$.

Figure 6. Form function of an aluminum cylinder at $\alpha = 50°$.

normalized frequency, ka, on the graphs. This number of normal modes guarantees the correctness of the form function for the considered range of frequencies ([9], p. 245). Also, the runtime of the computations for the modified model is less than one second, which is similar to that of the original model discussed in [10].

As mentioned earlier, the mathematical models given in [10,11] cannot be used in the case of normally incident waves. If using these models, one would have to choose a very small incidence angle that would approximate normal incidence. However, no matter how small this angle is, it would produce extra resonances in the cylinder. To show the effect of small incidence angles, the three components of the

interior displacement field of the isotropic aluminum cylinder are plotted in Figure 9. In this figure, the displacements shown on the top are calculated by the modified model at $\alpha = 0°$ and those on the bottom are calculated based on the formulation of Ref. [10], at $\alpha = 0.001°$. It is observed that although the displacement fields in the r and θ directions are almost identical, the two displacement fields in the z direction are completely different. For the case of the normally incident wave, the displacement component in the z direction are zero over the entire cross section of the cylinder, while, at $\alpha = 0.001$ degrees, the displacement field along the cylinder axis is not zero. The nullity of the displacement field in the axial direction confirms our earlier discussion regarding the independency of the
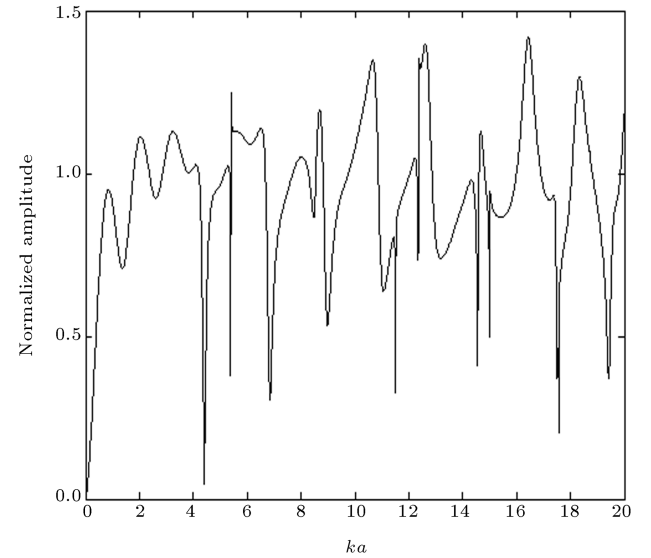

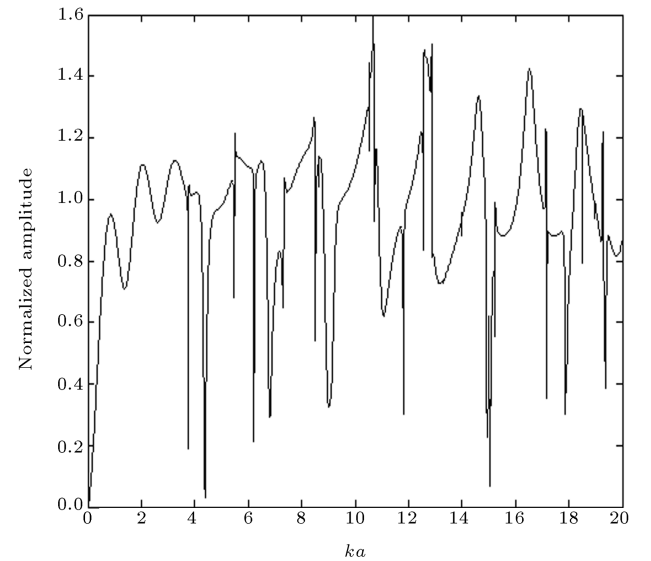**Figure 7.** Form function of a cobalt cylinder at $\alpha = 0°$.

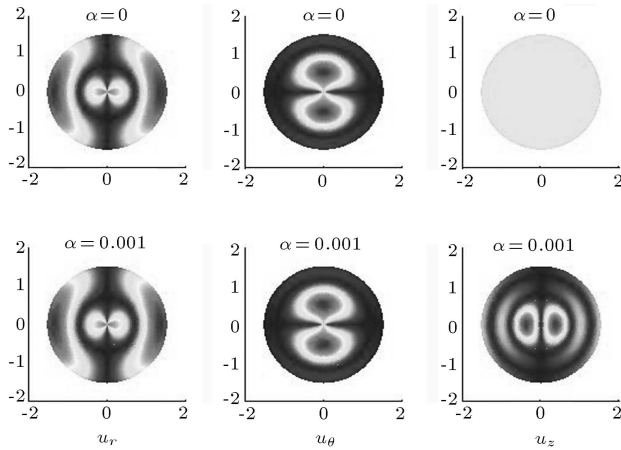**Figure 8.** Form function of a cobalt cylinder at $\alpha = 5°$.

Figure 9. Displacement fields of an aluminum cylinder in r, θ and z directions. Top: Calculated based on the modified formulation suggested in this paper at $\alpha = 0°$. Bottom: Calculated based on the formulation of [10] at $\alpha = 0.001°$.

surface waves from the axial properties of the cylinder when the wave angle is normal to the cylinder axis.

To demonstrate that the elastic constants, c_{13} and c_{44}, do not affect the scattered field in the case of a normally incident wave, all five elastic constants of

a transversely isotropic cobalt cylinder are perturbed, and the effect of this perturbation is studied on the form function. Figures 10a to 10e show the effect of perturbing each of the five elastic constants of a cobalt cylinder on its form function at $\alpha = 0$ for the frequency range $0 < ka < 10$. These figures show that only c_{11} and c_{12} affect the resonance frequencies at normal incidence and the other three elastic constants have no effect. This complies with our previous discussion, where it was shown that c_{13} and c_{44} have no effect on the scattered field when $\alpha = 0$.

CONCLUSIONS

The existing mathematical model for the scattering of a plane acoustic wave from an immersed, infinite solid, transversely isotropic cylinder fails in the case of normally incident waves. In this paper, modifications were suggested to make it work in the case of normally incident waves. It was also shown that in the case of a normally incident wave, the corresponding equations reduce to those of the isotropic cylinder and only elastic constants characterizing the isotropic plane of the material affect the resonance frequencies. The modified model was numerically verified for both isotropic
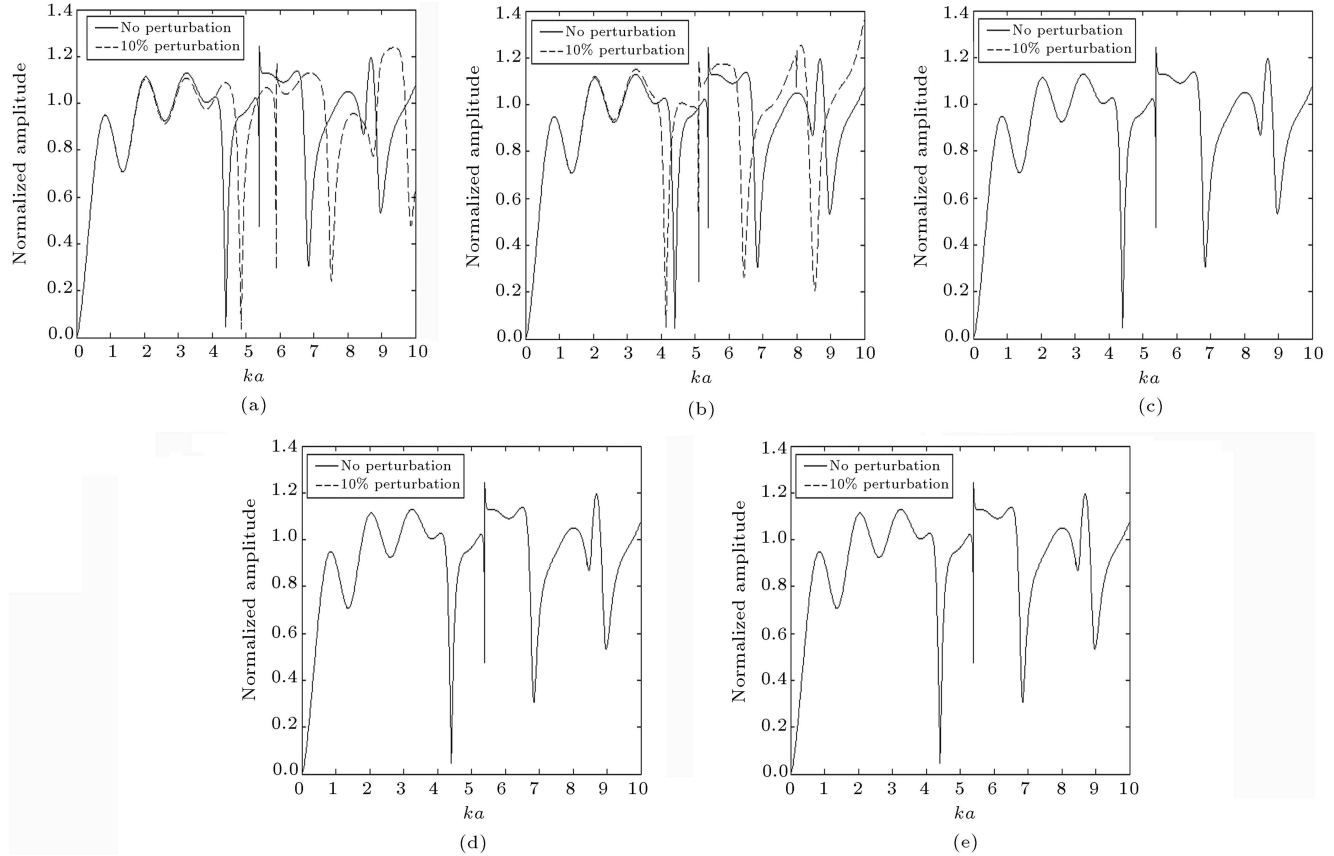

Figure 10. Effect of the perturbation of the five elastic constants of a cobalt cylinder on resonance frequencies when $\alpha = 0°$. a) 10% increase in c_{11}, b) 10% increase in c_{12}, c) 10% increase in c_{33}, d) 10% increase in c_{13}, e) 10% increase in c_{44}.

and transversely isotropic cylinders. A perturbation study showed that, in the case of a normally incident wave, the only elastic constants affecting the resonance frequencies are those which characterize the isotropic plane of the transversely isotropic cylinder. This modified model can be employed in an examination of transversely isotropic cylinders by Resonance Acoustic Spectroscopy (RAS).

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APPENDIX

Elements of the matrices given in Equation 29 are as follows:

$$a_{11} = -\frac{p_0 i^n \varepsilon_n \left[n H_n^{(1)}(k_\perp a) - (k_\perp a) H_{n+1}^{(1)}(k_\perp a) \right]}{\rho_\omega \omega^2},$$
$$(A1)$$

$$a_{12} = (1 + iq_1 ak_z)[nJ_n(s_1 a) - s_1 a J_{n+1}(s_1 a)], \quad (A2)$$

$$a_{13} = (q_2 + iak_z)[nJ_n(s_2 a) - s_2 a J_{n+1}(s_2 a)], \quad (A3)$$

$$a_{14} = nJ_n(s_3 a), \quad (A4)$$

$$a_{21} = p_0 i^n \varepsilon_n a^2 H_n^{(1)}(k_\perp a), \quad (A5)$$

$$a_{22} = [c_{11} + i(c_{11} - c_{13})q_1 ak_z]$$
$$\lfloor (n^2 - n - s_1^2 a^2) J_n(s_1 a) + s_1 a J_{n+1}(s_1 a) \rfloor$$
$$+ [c_{12} + (c_{12} - c_{13})iq_1 ak_z]$$
$$[nJ_n(s_1 a) - s_1 a J_{n+1}(s_1 a)]$$
$$+ [-c_{13} a^2 k_z^2 - c_{12} n^2 + (c_{13}$$
$$- c_{12})in^2 q_1 ak_z] J_n(s_1 a), \quad (A6)$$

$$a_{23} = [c_{11} q_2 + i(c_{11} - c_{13})ak_z]$$
$$\lfloor (n^2 - n - s_2^2 a^2) J_n(s_2 a) + s_2 a J_{n+1}(s_2 a) \rfloor$$
$$+ [c_{12} q_2 + (c_{12} - c_{13})iak_z]$$
$$[nJ_n(s_2 a) - s_2 a J_{n+1}(s_2 a)]$$
$$[(-c_{13} a^2 k_z^2 - c_{12} n^2)q_2$$
$$+ (c_{13} - c_{12})in^2 ak_z] J_n(s_2 a), \quad (A7)$$

$$a_{24} = (c_{11} - c_{12})n[(n-1)J_n(s_3 a) - s_3 a J_{n+1}(s_3 a)], \quad (A8)$$

$$a_{31} = 0, \quad (A9)$$

$$a_{32} = 2n(1 + iq_1 ak_z)[(1-n)J_n(s_1 a) + s_1 a J_{n+1}(s_1 a)], \quad (A10)$$

$$a_{33} = 2n(q_2 + iak_z)[(1-n)J_n(s_2 a) + s_2 a J_{n+1}(s_2 a)], \quad (A11)$$

$$a_{34} = \lfloor s_3^2 a^2 - 2n(n-1) \rfloor J_n(s_3 a) - 2s_3 a J_{n+1}(s_3 a), \quad (A12)$$

$$a_{41} = 0, \quad (A13)$$

$$a_{42} = [q_1(s_1^2 a^2 - a^2 k_z^2) + 2iak_z]$$
$$[nJ_n(s_1 a) - s_1 a J_{n+1}(s_1 a)], \quad (A14)$$

$$a_{43} = (s_2^2 a^2 - a^2 k_z^2 + 2iak_z q_2)$$
$$[nJ_n(s_2 a) - s_2 a J_{n+1}(s_2 a)], \quad (A15)$$

$$a_{44} = inak_z J_n(s_3 a), \quad (A16)$$

$$b_1 = \frac{p_0 i^n \varepsilon_n [nJ_n(k_\perp a) - (k_\perp a)J_{n+1}(k_\perp a)]}{\rho_\omega \omega^2}, \quad (A17)$$

$$b_2 = -p_0 i^n \varepsilon_n a^2 J_n(k_\perp a). \quad (A18)$$

BIOGRAPHIES

Farhang Honarvar is an associate professor of Mechanical Engineering at K. N. Toosi University of Technology (Tehran, Iran) and an adjunct associate professor of Mechanical Engineering at the University of Toronto (Toronto, Canada). He received his PhD from University of Toronto (1997), MASc from University of Waterloo (1993), and BSc from University of Tehran (1989), all in mechanical engineering. Dr. Honarvar is the director of the Nondestructive Evaluation Laboratory (NDE Lab) at K. N. Toosi University of Technology. His scientific interests include advanced nondestructive testing techniques, theoretical and experimental studies of acoustic and elastic waves, signal processing of ultrasonic waves, and modeling of measurement systems. Dr. Honarvar has several inventions and patents in the area of ultrasonic testing of materials. He has published more than one hundred and fifty papers in national and international journals and conferences.

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