Transaction B: Mechanical Engineering
Vol. 17, No. 3, pp. 194–204

A Continuous Vibration Theory for Beams with a Vertical Edge Crack

M. Behzad[1,*], A. Ebrahimi[1] and A. Meghdari[1]

Abstract. *In this paper, a continuous model for flexural vibration of beams with an edge crack perpendicular to the neutral plane has been developed. The model assumes that the displacement field is a superposition of the classical Euler-Bernoulli beam's displacement and of a displacement due to the crack. The additional displacement is assumed to be a product between a function of time and an exponential function of space. The unknown functions and parameters are determined based on the zero stress conditions at the crack faces and the concept of J-integral from fracture mechanics. The governing equation of motion for the beam has been obtained using the Hamilton principle and solved using a modified Galerkin method. The results have been compared with finite element results and an excellent agreement is observed.*

Keywords: *Vibration; Cracked beam; Vertical crack; J-integral.*

INTRODUCTION

Fatigue and crack initiation and propagation in structures and machinery subjected to dynamic loading are one of the main concerns of designers and users. An uncontrolled crack can lead to a catastrophic failure under certain conditions. The importance of early detection of cracks makes researchers study various aspects of the behavior of structures defected by cracks. One of these aspects is the vibration of cracked structures. Crack creation and development in a system changes the dynamic and vibration behavior of that system. With measurement and analysis of these vibrations, the cracks can be identified well in advance and appropriate actions can be taken to prevent more damage to the system.

The vibration behavior of cracked structures has been investigated by many researchers. Dimaragonas presented a review on the topic of the vibration of cracked structures [1]. His review contains vibration of cracked rotors, bars, beams, plates, pipes, blades and shells. Two more literature reviews are also available on the dynamic behavior of cracked rotors by Wauer and Gasch [2,3].

Beams are important elements in structures and machinery, so the vibration behavior of cracked beams has been studied by researchers widely. One may define a crack with its edge parallel or perpendicular to the neutral axis as horizontal or vertical cracks, respectively, as shown in Figure 1.

There exist three methods for the vibration modeling of beams with horizontal transverse cracks:

1. Discrete models with a local flexibility model for cracks.
2. Continuous models with a local flexibility model for cracks.
3. Continuous models with a continuous model for the crack.

The local flexibility model for the crack has been suggested by Dimaragonas for the first time [4]. He replaced the cracked beam with two undamaged half beams connected by a rotational spring. The stiffness of this spring is obtained from the concept of the J-integral in fracture mechanics. Papadopoulos presented a complete literature review on the method of using the J-integral for finding the local flexibility of

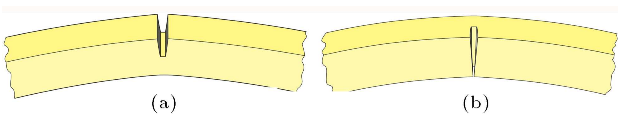

Figure 1. A beam with (a) horizontal edge crack and (b) vertical edge crack under bending.

1. *School of Mechanical Engineering, Sharif University of Technology, Tehran, P.O. Box 11155-9567, Iran.*
*. *Corresponding author. E-mail: m_behzad@sharif.edu.*

Received 18 May 2009; received in revised form 2 February 2010; accepted 6 April 2010

cracks [5]. The local flexibility idea has been followed by several researchers till now. Some researchers modeled two undamaged half beams discretely and added the flexibility of the rotational spring to the flexibility matrix of the system [6,7]. Others modeled two undamaged half beams continuously and used appropriate boundary conditions for each part to link them through the rotational spring [8,9]. Some other researchers have tried to modify and improve the local flexibility model of the crack by adding one or two linear springs besides the rotational one [10]. These methods have also been extended for beams with more than one crack [11-13]. The local flexibility model for the crack is a simple approach and has a relatively good result in finding the fundamental natural frequency of a cracked beam. However, this method cannot be implemented for finding stress at the crack area under dynamic loads, mode shapes in free vibrations and operational deformed shapes in forced vibrations.

Another approach to the vibration analysis of cracked beams is continuous modeling of the crack. Christides and Barr developed a continuous theory for the vibration of a uniform Euler-Bernoulli beam containing one or more pairs of symmetric cracks [14]. They suggested some modifications on the familiar stress field of an undamaged Euler-Bernoulli beam in order to consider the crack effect. The differential equation of motion and corresponding boundary conditions are given as the results. However, in their model, two different and incompatible assumptions have been made for displacement and strain fields. Although the accuracy of the results in finding the natural frequencies is acceptable for some applications, their model is still not reliable for more accurate analyses such as stress analysis near the crack tip under dynamic loading and mode shape analysis. In addition, the obtained partial differential equation is complicated and dependent on some constants that are unknown and must be calculated by correlating the analytically obtained results with those calculated by finite element in each case. Several researchers followed the Christides and Barr approach by modifying their method and gained some improvements [15-19]. However, there still exists the inconsistency between strain and displacement fields, which causes inaccuracy in the results, especially in mode shapes and stress analysis.

Behzad et al. presented a new continuous theory for the bending analysis of a cracked beam [20]. A bilinear displacement field has been suggested for the beam strain and stress calculations and the bending differential equation has been obtained using equilibrium equations. The model can predict the load-deflection relation of the beam near or far from the crack tip accurately and can be also used for stress-strain analysis in a cracked beam. This model is also used for the vibration analysis of a cracked beam and showed an excellent performance in dynamic loading too [21,22]. They used this method also for the force vibration analysis of beams with a horizontal edge crack [23].

In all the above approaches, the crack is assumed to be horizontal. This type of crack is more probable to be created and other forms of crack tend to grow horizontally in bending. However, by using a pre-cracked element with specific orientation in a structure, the crack may become horizontal. The other application of this research is in the area of rotor dynamics, where a cracked beam rotates. Figure 1 shows a beam with horizontal and vertical cracks.

In this paper, a continuous approach for the flexural vibration analysis of a beam with a vertical edge crack has been presented for the first time. The crack is assumed to be an open edge notch and the crack edge is perpendicular to the neutral plane. A quasi-linear displacement field has been suggested for the cracked beam and the strain and stress fields have been calculated. The differential equation of motion of the cracked beam has been obtained using the Hamilton principle. This partial differential equation has been solved with a special numerical algorithm based on the Galerkin projection method. The constants needed in this model can be obtained using fracture mechanics. The results of this study are compared with the finite element results for verification.

CRACK BEHAVIOR ANALYSIS IN BENDING AND HYPOTHESES

The basic assumption in the Euler-Bernoulli vibration theory for beams is that the plane sections of a beam which are perpendicular to the neutral axis remain plane and perpendicular to the neutral axis after deformation. In the presence of an edge crack, this assumption, especially near the vicinity of the crack, is no longer correct.

Behzad et al. suggested that for a horizontal crack, the crack faces have an additional displacement due to the absence of the normal stress [20,21]. They discussed that this additional displacement is inherited by the adjacent area with lesser magnitude, and dissipates away with an exponential regime along the beam length. Consequently, for planes far from the crack tip, this warping will be negligible and the displacement field can be assumed linear.

This idea can be followed here for a vertical crack with some modifications. In order to have a better sense of the flexural vibrations in a cracked beam, a finite element model has been produced in this research, and the mid span vertical crack flexural behavior can be seen in Figure 2. This finite element model is made using ANSYS software [24]. The crack is modeled as a vertical U-shape notch at the mid-span as a crack, and a fine singular mesh is used. Note that the grid lines of Figure 2 are only some hypothetical lines that show the deformation field of the beam, and these lines are not referring to the finite element mesh.

Near the crack area, the plane sections will no longer remain plane. In fact, the crack faces have an additional rotation, in comparison with the remaining part of the section, due to the absence of normal stress. This additional rotation dissipates gradually, while the distance from the crack tip increases. With a good approximation, it can be supposed that each plane section turns into two straight planes after deformation. Each straight plane section turns into two planes with different slopes, one on the right side and the other on the left side of the crack edge. The slope difference between these two planes decreases with distance from the crack tip. These two straight planes connect to each other through a nonlinear part near the xz-plane. Figure 3 shows the coordinate system and the parameter definitions, graphically.

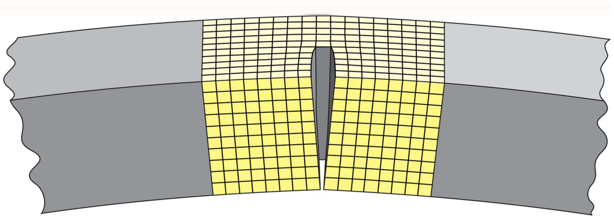

Figure 2. Displacement field illustration in a beam with a vertical edge crack subject to bending.

In order to find the stress, strain and deformation functions for a beam with a vertical crack in flexural vibration, a displacement field for the beam has been suggested in this research. In fact, it is assumed that each plane section turns into two straight planes and a nonlinear connector after deformation. In this research, the beam is assumed to be a slender prismatic beam and the crack is considered as an open edge U-shape notch. The cross section of the beam is assumed to be symmetric about the y-axis, so the y-axis can be assumed to be the neutral axis in pure bending. The displacements and stresses are supposed to be small and the crack does not grow. Finally, the material is assumed to be linear elastic.

DISPLACEMENT FIELD DEFINITION

With reference to Figures 2 and 3 and the above assumptions, the displacement field for a beam with a vertical edge crack can be defined. It is well known that the displacement and stress fields near the crack tip are 3D functions. In this paper, at first, a 3-dimensional displacement field has been introduced for the beam but, afterwards, the equations are integrated over the cross-section area of the beam and a 1-dimensional relation is obtained for the beam vibrations. This

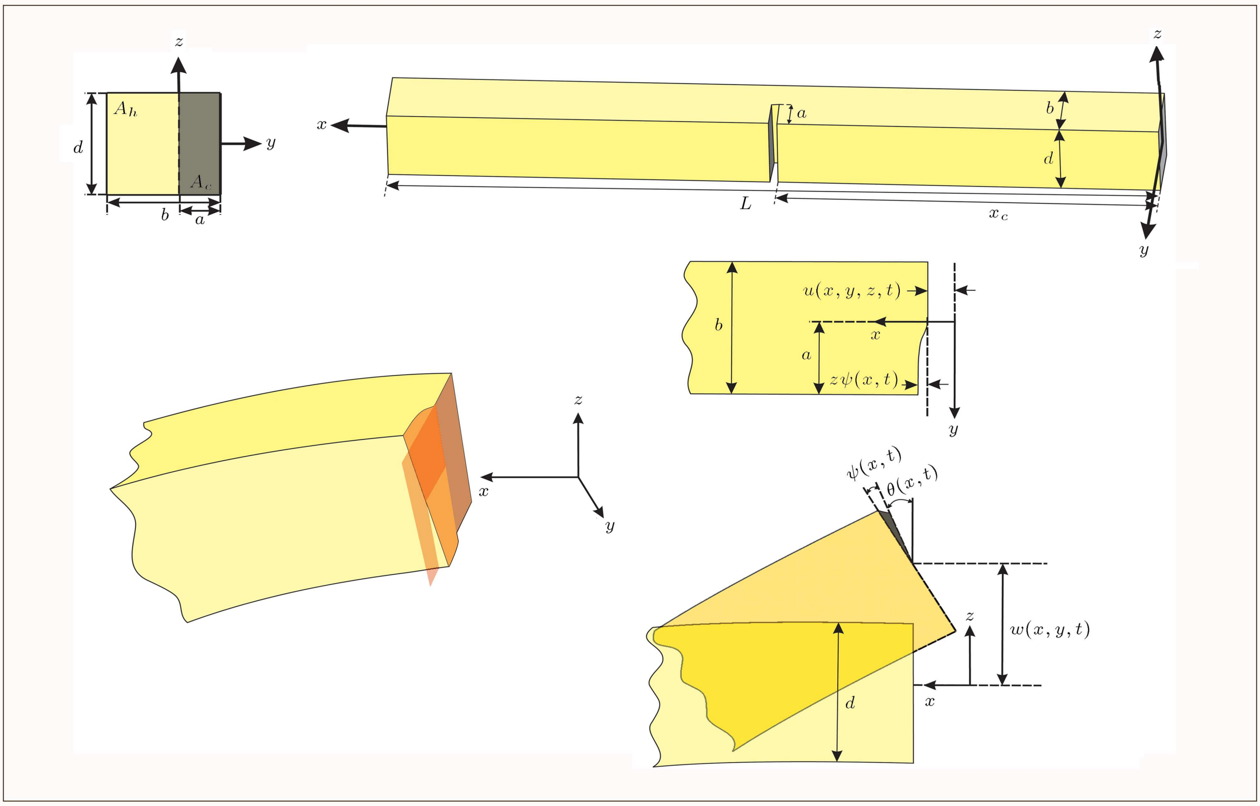

Figure 3. Coordinate system and parameters definition.

equation is not an exact equation, but the results of this research show the good engineering approximation of this relation.

The crack section consists of two parts: The crack face, which is denoted in Figure 2 by A_c, and the remaining part of the section, which is denoted by A_h in this research. Under pure bending, the healthy part of the cross section (A_h) rotates about its neutral axis, which is coincident with the y-axis in this research. This planar part remains plane after rotation and perpendicular to the neutral axis. The crack face rotates about the y-axis too, but more than the remaining part of the section and, consequently, does not remain perpendicular to the neutral axis due to the shear stress near the crack tip. The crack face can also be assumed to remain plane after deformation, except at a small area near the crack tip. The rotation difference between the crack face and the remaining part of the section inherits to the adjacent cross sections but, gradually, the magnitude of this difference decreases. As a side effect, deformation of the beam along the z-axis is a function of y. In fact, the parts of the beam sections which have more rotation cause more vertical displacement, too. The numerical simulations confirm this phenomenon.

Based on the above explanations, the following displacement field is introduced for a beam with a vertical edge crack in flexural vibration:

$$\begin{cases} u = \begin{cases} -z\theta(x,t) & y<0 \\ -z(\theta(x,t)+\psi(x,t)) & y>0 \end{cases} \\ w = \begin{cases} w_0(x,t) & y<0 \\ w_0(x,t)+\Delta(x,t) & y>0 \end{cases} \\ v=0 \end{cases} \tag{1}$$

In which u, v and w are the displacement components along x, y and z axes. $\theta(x,t)$ is the rotation of that part of the section with $y<0$, as shown in Figure 3. $\psi(x,t)$ is the additional rotation of that part of the section with $y>0$ and $\Delta(x,t)$ is the additional vertical displacement of the beam for $y>0$. By assuming that the plane sections in $y<0$ remain perpendicular to the neutral axis, one has:

$$\theta(x,t) = \frac{\partial w_0(x,t)}{\partial x}. \tag{2}$$

The additional rotation $\psi(x,t)$ of that part of the plane sections with $y>0$ has its maximum value at the crack face and decreases gradually with distance from the crack tip. This additional rotation is a nonlinear and complicated variable with respect to x. Here, in this research, an exponential regime has been assumed for function $\psi(x)$ along the x-axis as follows:

$$\psi(x,t) = m(t)e^{-\alpha\frac{|x-x_c|}{b}}\operatorname{sgn}(x-x_c). \tag{3}$$

In Equation 3, $m(t)$ is the magnitude of additional rotation of the crack faces, α is a dimensionless exponential decay rate, which will be obtained later in this paper, x_c is the crack position, b is the depth of the beam and $\operatorname{sgn}(x-x_c)$ is the sign function, which is -1 for $x<x_c$ and $+1$ for $x>x_c$. The application of a sign function is due to the fact that the additional rotation function has a discontinuity at the crack position and the sign of its value changes when passing through the crack tip.

In order to find the value of $m(t)$, zero normal stress conditions at the crack faces can be used. The normal strain function can be found using Equation 1:

$$\varepsilon_x = u_{,x} = \begin{cases} -zw_{0,xx} & y<0 \\ -z\left(w_{0,xx} - m(t)\frac{\alpha}{b}e^{-\alpha\frac{|x-x_c|}{b}} + 2m(t)e^{-\alpha\frac{|x-x_c|}{b}}\delta(x-x_c)\right) & y>0 \end{cases} \tag{4}$$

in which $\delta(x-x_c)$ is the Dirac delta function and the subscript $,x$ denotes the partial derivative with respect to x. The normal stress at the crack faces where $y>0$ and $x=x_c^+$ or x_c^- should be zero, so one has:

$$m(t) = \frac{b}{\alpha}w_{0,xx}(x_c,t). \tag{5}$$

The additional displacement, $\Delta(x,t)$, which also decreases gradually with distance from the crack tip can be assumed to be a function similar to $\psi(x,t)$ as follows:

$$\Delta(x,t) = n(t)e^{-\alpha\frac{|x-x_c|}{b}}, \tag{6}$$

where $n(t)$ is the magnitude of additional displacement at the crack face, which can be found using a zero shear stress condition at the crack faces. The shear strain function, γ_{xz}, can be found using Equation 1:

$$\gamma_{xz} = \frac{1}{2}(u_{,z}+w_{,x}) = \begin{cases} 0 & y<0 \\ -\left(m(t)+n(t)\frac{\alpha}{b}\right)e^{-\alpha\frac{|x-x_c|}{b}}\operatorname{sgn}(x-x_c) & y>0 \end{cases} \tag{7}$$

The shear stress at the crack faces, where $y>0$ and $x=x_c^+$ or x_c^-, should be zero so one has:

$$n(t) = -\frac{b}{\alpha}m(t) = -\frac{b^2}{\alpha^2}w_{0,xx}(x_c,t). \tag{8}$$

To avoid discontinuity and considering the nonlinearity at the crack tip, it is assumed that the displacement field at $y>0$ transforms into the defined functions in Equation 1 with an exponential regime from the displacement field at $y < 0$. So, the displacement field is modified in this paper as follows:

$$u = \begin{cases} -zw_{0,x} & y < 0 \\ -z\left(w_{0,x} + \frac{b}{\alpha}\left(1 - e^{-\beta\frac{y}{d}}\right)\right. \\ \left.\qquad w_{0,xx}(x_c)e^{-\alpha\frac{|x-x_c|}{b}}\mathrm{sgn}(x-x_c)\right) & y > 0 \end{cases}$$

$$w = \begin{cases} w_0(x,t) & y < 0 \\ w_0(x,t) - \frac{b^2}{\alpha^2}\left(1 - e^{-\beta\frac{y}{d}}\right) \\ \qquad w_{0,xx}(x_c)e^{-\alpha\frac{|x-x_c|}{b}} & y > 0 \end{cases}$$

$$v = \begin{cases} 0 & y < 0 \\ \frac{b^2}{\alpha^2}\frac{\beta}{d}ze^{-\beta\frac{y}{d}}e^{-\alpha\frac{|x-x_c|}{b}} & y > 0 \end{cases} \quad (9)$$

In Equation 9, β is a dimensionless parameter and will be discussed later in this paper. The term $(1 - e^{-\beta\frac{y}{d}})$ prevents the discontinuity at the crack tip. The displacement component, v, is modified in order that γ_{yz} becomes zero at the crack faces.

EQUATION OF MOTION

Now, the strain field can be extracted from the displacement field. The normal strain component of the stress field can be written using Equation 9 as follows:

$$\varepsilon_x = \begin{cases} -zw_{0,xx} & y < 0 \\ -z\left(w_{0,xx} - \left(1 - e^{-\beta\frac{y}{d}}\right)\right. \\ \left(1 - 2\frac{b}{\alpha}\delta(x-x_c)\right) \\ \left.\quad w_{0,xx}(x_c)e^{-\alpha\frac{|x-x_c|}{b}}\right) & y > 0 \end{cases} \quad (10)$$

The normal stress energy of the beam can be obtained using the following relation:

$$V = \frac{1}{2}\int_V \sigma_x\varepsilon_x d\mathrm{V} = \frac{1}{2}\mathrm{E}\int_V \varepsilon_x^2 d\mathrm{V}. \quad (11)$$

In which V is the normal strain energy function, V is the volume of the beam and E is the modulus of elasticity.

In this research, the cracked beam is assumed to be slender. So, the Euler-Bernoulli assumption can be used and one can neglect the shear strain energy in comparison with the normal strain energy. The displacement field is defined in order to force average shearing strain components to be zero; similar to an undamaged Euler-Bernoulli beam.

The kinetic energy of the cracked beam can be also calculated as follows:

$$T = \frac{1}{2}\int_{\mathrm{V}} \rho w_{,t}^2 d\mathrm{V}. \quad (12)$$

In Equation 12, the rotational inertia has been neglected, similar to the undamaged Euler-Bernoulli beam theory.

Using the Hamilton principle, one has:

$$\delta\int_{t_0}^{t_1}(T - V)dt = 0. \quad (13)$$

Now, using Equations 10 to 13 and performing appropriate calculations, the following equations can be obtained:

$$E\frac{\partial^2}{\partial x^2}\left(Iw_{0,xx} - kw_{0,xx}(x_c,t)e^{-\alpha\frac{|x-x_c|}{b}}\right) + \rho Aw_{,tt} = 0,$$

$$k = \int_{A_c} z^2\left(1 - e^{-\beta\frac{y}{d}}\right)dA = I_c - \int_{A_c} z^2 e^{-\beta\frac{y}{d}}dA. \quad (14)$$

Equation 14 is the governing equation of motion of a beam with a vertical edge crack. In this equation, k is a geometrical factor which can be found for every given cross-section and crack. I and I_c are the moment inertia of the cross-section and the crack face, respectively. In an undamaged beam, the geometrical parameter, k, is zero and, hence, Equation 14 turns into a familiar form of the Euler-Bernoulli vibration equation for slender beams. The dimensionless exponential decay rates (α, β) are the only factors that have not been discussed yet. In the next section, the parameters, α and β, are calculated and then, the solution for the partial differential Equation 14 is presented.

EXPONENTIAL DECAY RATES α AND β CALCULATION

The exponential decay rates presented in this research can be obtained using the concepts of additional remote point rotation and the J-integral, which are two familiar concepts in fracture mechanics. Several researchers used a similar method to evaluate the crack properties [5,20].

When a pair of static bending moments, M, are applied to the cracked beam, an additional relative rotation, θ^*, will exist between two ends of the beam, due to the crack, as shown in Figure 4.

For an Euler-Bernoulli simply supported beam, the slope function of the neutral axis is as follows:

$$\theta_h = \frac{dw}{dx} = \frac{M}{EI}\left(x - \frac{l}{2}\right). \quad (15)$$

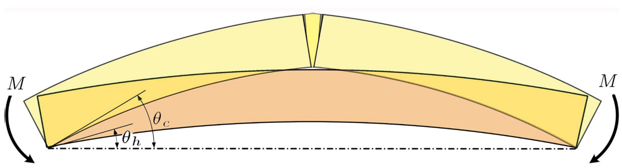

Figure 4. Additional rotation of a beam with a vertical crack under bending.

For a beam with a vertical crack under pure static bending, the time derivatives vanish from Equation 14. Integrating two sides of the obtained equation and using appropriate boundary conditions for pure bending, the following equation will be obtained:

$$\frac{d^2 w_0}{dx^2} = \frac{M}{EI}\left(1 + \frac{k}{I-k} e^{-\alpha \frac{|x-x_c|}{b}}\right). \tag{16}$$

Solving Equation 16 will result in the load-deflection relation of a beam with vertical crack under static pure bending. The results are as follows [20]:

$$w_0 = \begin{cases} \frac{M}{EI}\left(\frac{x^2}{2} + c_1 x + c_2 + \frac{b^2}{\alpha^2}\frac{k}{I-k} e^{\alpha \frac{x-x_c}{b}}\right) & x \le x_c \\ \frac{M}{EI}\left(\frac{x^2}{2} + c_3 x + c_4 + \frac{b^2}{\alpha^2}\frac{k}{I-k} e^{-\alpha \frac{x-x_c}{b}}\right) & x > x_c \end{cases} \tag{17}$$

in which constants c_1, c_2, c_3 and c_4 will be as follows [20]:

$$\begin{cases} c_1 = c_3 - 2\frac{b}{\alpha}\frac{k}{I-k} \\ c_2 = -\frac{b^2}{\alpha^2}\frac{k}{I-k} e^{-\alpha \frac{x_c}{b}} \\ c_3 = -\frac{l}{2} - \frac{c_4}{l} - \frac{1}{l}\frac{b^2}{\alpha^2}\frac{k}{I-k} e^{-\alpha \frac{l-x_c}{b}} \\ c_4 = c_2 - 2x_c \frac{b}{\alpha}\frac{k}{I-k} \end{cases} \tag{18}$$

Using Equations 15 and 17, one can obtain the additional remote point rotation, θ^*, as follows:

$$\theta^* = (\theta_c(0) - \theta_h(0)) - (\theta_c(l) - \theta_h(l)) = \frac{M}{EI}\frac{b}{\alpha}\frac{k}{I-k}\left(2 - e^{-\alpha \frac{x_c}{b}} - e^{-\alpha \frac{l-x_c}{b}}\right), \tag{19}$$

where θ_c and θ_h are the rotation of a cracked beam and an undamaged or healthy beam under static bending, respectively. In Equation 19, parameter k is a function of β. However, the finite element results, in comparison with those obtained by this model, show that parameter β is a large enough parameter and, accordingly, it can be assumed that parameter β tends to infinity. It must be noticed that, despite the fact that the exponential decay rate, β, is obtained here by finite element analysis and correlating analytical and finite element results, this value for β is a general value and, in other cases, can be used without separate calculations.

On the other hand, additional rotation θ^* means that the cracked beam accumulates more strain energy compared with an undamaged beam. This extra strain energy which is called U_T here is stored at the vicinity of the crack. The additional rotation of a beam subjected to a pair of bending moments at two ends, as shown in Figure 4, can be obtained using Castigliano's theorem as follows:

$$\theta^* = \frac{\partial U_T}{\partial M}. \tag{20}$$

This additional strain energy is due to the crack and can also be written in the following form [1,5]:

$$U_T = \int_{A_c} J_s(a) dA. \tag{21}$$

In Equation 21, A_c is the crack face area. Equation 21 is called the Paris equation, and J_s in this equation is the strain energy release rate. There are several experimental, analytical and numerical formulas to calculate the value of the J-integral based on the geometry, loading and type of crack [25,26]. In this case, the value of the J-integral could be obtained from the following equation [1,5]:

$$J_s = \frac{1}{E'}\left[\left(\sum_{i=1}^{6} K_{\mathrm{I}_i}\right)^2 + \left(\sum_{i=1}^{6} K_{\mathrm{II}_i}\right)^2 + m\left(\sum_{i=1}^{6} K_{\mathrm{III}_i}\right)^2\right], \tag{22}$$

where K_{I_i}, K_{II_i} and K_{III_i} are the Stress Intensity Factors (SIF), corresponding to three modes of fracture, which result for every individual loading mode, i. In pure bending, SIF is nonzero only for mode I. In Equation 22, if the plane stress assumption is used, then, $E' = E$ and, if the plane strain assumption is used, then $E' = E/(1-\nu^2)$. In this article the plane strain assumption is used.

The beam with a vertical edge crack can be assumed to consist of a set of thin plates along a z-axis, and each plane to contain an edge crack and be subjected to axial tension or compression. This Stress Intensity Factor (SIF) for such plates is [26]:

$$K_I = \sigma_0 \sqrt{\pi a} F\left(\frac{a}{b}\right),$$

$$\sigma_0 = \frac{Mz}{I},$$

$$F\left(\frac{a}{b}\right) = \left(1 + 0.122\cos^4\frac{\pi a}{2b}\right)\sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}}. \tag{23}$$

Equation 23 has an accuracy of 0.5% for any a/b. From Equations 22 and 23, the energy release rate is:

$$J_s = \frac{K_I^2}{E'} = \frac{1-\nu^2}{E}\left(\frac{Mz}{I}\right)^2 \pi a F^2\left(\frac{a}{b}\right). \tag{24}$$

Now, substituting Equation 24 into 21 and, then, using Equation 20, the additional rotation of the cracked beam, θ^*, can be obtained in terms of bending moment and geometrical parameters. For a rectangular cross section, this additional rotation is:

$$\theta^* = \frac{1-\nu^2}{E}\frac{2M}{I^2}\pi\varphi,$$

$$\varphi = \int_{-\frac{d}{2}}^{\frac{d}{2}}\int_0^a z^2 sF^2\left(\frac{s}{b}\right)dsdz. \tag{25}$$

The additional rotation can be evaluated using the obtained relation in Equation 19 too. Comparing the two sides of Equations 25 and 19, one has:

$$\frac{b}{\alpha}\frac{k}{I-k}\left(2-e^{-\alpha\frac{x_c}{b}}-e^{-\alpha\frac{l-x_c}{b}}\right)=(1-\nu^2)\frac{2}{I}\pi\varphi. \tag{26}$$

The numerical solution of Equation 26 will lead to finding the value of exponential decay rate α for any values of geometrical parameters and simply supported ends. From Equation 26, it can be shown that exponential decay rate α is a function of a/d, l/d and x_c/l. However, Behzad et al. discussed that, for slender beams ($l/d \geq 10$), slenderness factor (l/d) and crack position ratio x_c/l have a minor effect on α for horizontal cracks [20]. It can be shown that exponential decay rate α is only a function of crack depth ratio (a/b) for slender beams with vertical cracks too. Figure 5 shows α versus a/b for slender beams ($l/d \geq 10$).

EIGEN SOLUTION FOR SIMPLY SUPPORTED BEAM WITH VERTICAL CRACK

In order to find the natural frequencies and mode shapes of a beam with vertical crack, the equation of motion presented in Equation 14 must be solved. However, this equation cannot be solved analytically, and a numerical method must be used. The especial form of Equation 14 in which the solution at the crack position is appeared in the governing equation prevents one from using the ordinary Galerkin projection method. Behzad et al. presented a modified Galerkin projection algorithm for solving this type of equation [21]. In this paper, a similar approach has been used. The beam is assumed to be simply supported in this section. However, for every desired boundary condition, the presented solution can be used.

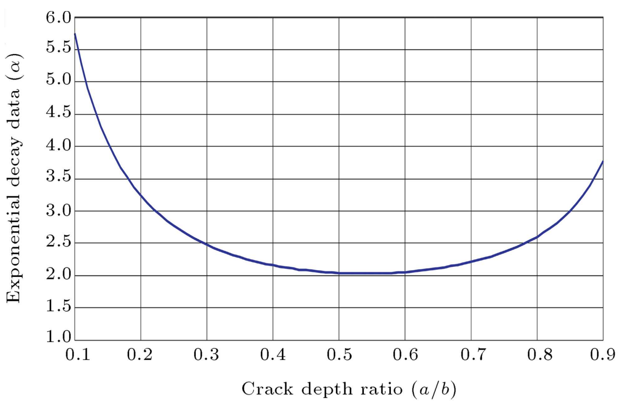

Figure 5. Exponential decay rate (α) versus crack depth ratio (a/b) for a slender simply supported beam with a vertical edge crack.

It can be assumed that the solution is a harmonic function, so one has:

$$w(x,t) = X(x)e^{i\omega t}, \tag{27}$$

in which ω is the natural frequency of the beam. Substituting Equation 27 into Equation 14 and assuming EI to be constant along the beam, the following eigenvalue problem will be obtained:

$$\begin{cases} \frac{d^2}{dx^2}\left(X'' - \frac{k}{I}X''(x_c)e^{-\alpha\frac{|x-x_c|}{b}}\right) - \frac{\rho A}{EI}\omega^2 X = 0 \\ X(0) = X(l) = 0 \\ X''(0) = X''(l) = 0 \end{cases} \tag{28}$$

In Equation 28, simply supported boundary conditions have been used. In a normal Strum-Louiville problem, one can easily consider function X to be in the form of $\sum c_i S_i(x)$ in which $S_i(x)$ are shape functions that satisfy the physical boundary conditions. However, in this research, the results show that such an approach will lead to a divergence of the results. Since the function $e^{-\alpha\frac{|x-x_c|}{d}}$ in Equation 28 is not a smooth function, it seems that the solution, especially for larger crack depth ratios, tends to have large derivatives near the crack tip. Accordingly, extracting the value of $X''(x_c)$ from X by derivation can lead to large fluctuations in the results and divergence. In order to avoid the divergence problem, function X'' and the value of $X''(x_c)$ are not extracted from X by direct derivation. Instead, X'' is discretized independently from X and, then, a constraint equation is provided to link X'' to X.

Considering the above discussion, the following relations can be written:

$$\begin{cases} X'' - \frac{\kappa}{I}X''(x_c)e^{-\alpha\frac{|x-x_c|}{d}} = \sum_{i=1}^{N} c_i S_i(x) \\ X = \sum_{i=1}^{N} c'_i S_i(x) \end{cases} \tag{29}$$

in which c_i and c'_i are two independent sets of constants, functions $S_i(x)$ are shape functions, which must satisfy physical boundary conditions, and N is the number of shape functions. Substituting Equation 29 into Equation 28, multiplying two sides of the equation by

$S_j(x)$, then, integrating along the length of the beam, one has:

$$\sum_{i=1}^{N} c_i \int_0^l S_i''(x) S_j(x) dx - \frac{\rho A}{EI}\omega^2 \sum_{i=1}^{N} c_i' \int_0^l S_i(x) S_j(x) dx = 0,$$

$$j = 1, 2, \cdots, N. \tag{30}$$

Or in the matrix form:

$$\mathbf{Kc} - \frac{\rho A}{EI_\eta}\omega^2 \mathbf{Pc}' = 0,$$

$$K_{ij} = \int_0^l S_i''(x) S_j(x) dx,$$

$$P_{ij} = \int_0^l S_i(x) S_j(x) dx. \tag{31}$$

On the other hand, if one substitutes the second equation of Equation 29 into the first one, the following relation will be obtained:

$$\sum_{i=1}^{N} c_i' \left(S_i''(x) - \frac{\kappa}{I} S_i''(x_c) e^{-\alpha \frac{|x-x_c|}{b}} \right) = \sum_{i=1}^{N} c_i S_i(x). \tag{32}$$

Multiplying two sides of Equation 32 by $S_j(x)$ and, then, integrating along the length of the beam, one has:

$$\sum_{i=1}^{N} c_i' \left(\int_0^l S_i''(x) S_j(x) dx - \frac{\kappa}{I} S_i''(x_c) \int_0^l S_j(x) e^{-\alpha \frac{|x-x_c|}{b}} dx \right) = \sum_{i=1}^{N} c_i \int_0^l S_i(x) S_j(x) dx, \qquad j=1, \cdots, N. \tag{33}$$

Rearranging Equation 33 into matrix form, the following equation can be written:

$$\mathbf{Qc}' = \mathbf{Rc},$$

$$Q_{ij} = \int_0^l S_i''(x) S_j(x) dx - \frac{\kappa}{I} S_i''(x_c) \int_0^l S_j(x) e^{-\alpha \frac{|x-x_c|}{b}} dx,$$

$$R_{ij} = \int_0^l S_i(x) S_j(x) dx. \tag{34}$$

Now, from Equation 34, coefficients c_i' can be related to c_i, as follows:

$$\mathbf{c}' = \mathbf{Q}^{-1}\mathbf{Rc}. \tag{35}$$

Substituting Equation 35 into 31, the following equation will be obtained:

$$\left(\mathbf{K} - \frac{\rho A}{EI}\omega^2 \mathbf{M} \right) \mathbf{c} = 0,$$

$$\mathbf{M} = \mathbf{PQ}^{-1}\mathbf{R}. \tag{36}$$

The natural frequencies and corresponding mode shapes for the cracked beam can be calculated solving the matrix eigenvalue problem of Equation 36. In the next section, the results are presented for a simply supported beam with a rectangular cross-section.

RESULTS FOR A SIMPLY SUPPORTED BEAM WITH RECTANGULAR CROSS-SECTION

In this section, the eigenvalue problem of Equation 36 has been solved for free vibration analysis of a simply supported slender prismatic beam with a vertical edge crack and rectangular cross-section. In such a beam, the exponential decay rate, β, can be assumed to be infinite and the exponential decay rate, α, can be calculated from Equation 26. The values of α versus crack depth ratio (a/b) have been shown in Figure 5.

In a simply supported cracked beam, shape functions $S_i(x)$ can be assumed to be in the form of $\sin(i\pi x/l)$, which satisfy physical boundary conditions. The natural frequency and mode shapes can be calculated using the eigenvalue problem of Equation 36. In this research, the number of shape functions, N, is set to be 100. In order to generalize the results, the natural frequencies of the cracked beam have been divided into the corresponding values for an undamaged beam (ω_0). Figures 6, 7 and 8 show the fundamental, second and third natural frequency ratios of the cracked beam, respectively. In Figures 6 to 8, the natural frequency ratios have been plotted versus the crack depth ratio (a/b) for several crack positions.

In Figures 6 to 8, the results of Finite Element (FE) analysis are also presented for verification. The finite element results have been obtained using ANSYS software. In order to have an accurate and reliable model, the PLANE183 singular element has been used in the cracked area [22]. This element is an 8-node quadratic solid singular element, especially designed for crack analysis. In this research, a fine mesh has been used at the vicinity of the crack, and the dependency of the results on mesh size has been checked. In all results,

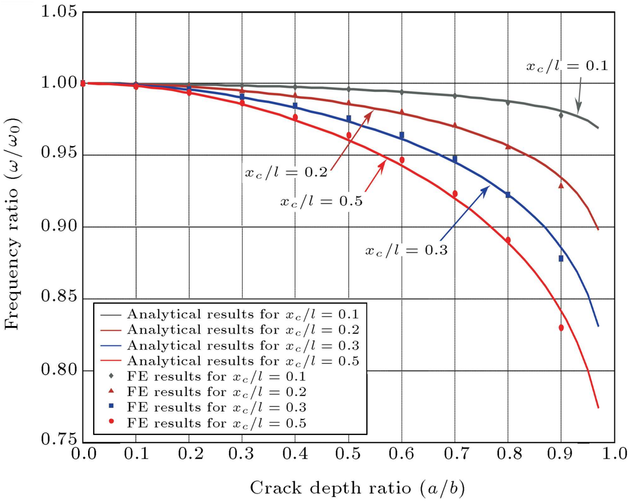

Figure 6. Fundamental natural frequency ratio for a beam with a vertical crack versus crack depth ratio.

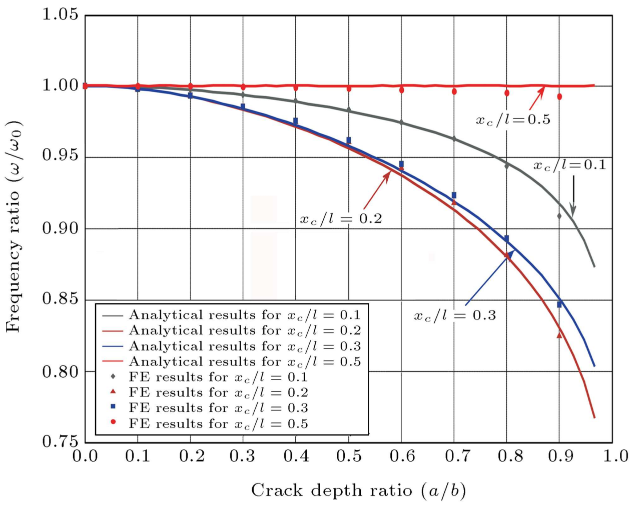

Figure 7. Second natural frequency ratio for a beam with a vertical crack versus crack depth ratio.

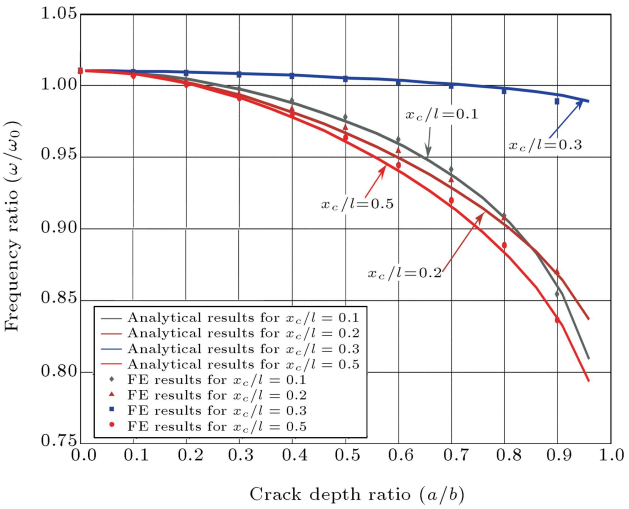

Figure 8. Third natural frequency ratio for a beam with a vertical crack versus crack depth ratio.

there is good agreement between analytical results and those obtained by FE analysis.

As can be seen in Figure 6, the reduction rate of the fundamental natural frequency has a direct relation with the position of the crack. This rate reduces for cracks that have more distance from the mid span of the beam. For the cracks at $x_c/l = 0.1$, the fundamental natural frequency drops less than 1 percent when the crack reaches half of the beam depth, while for the cracks at the mid span, this value is about 4 percent.

The dependency of the reduction of the natural frequency on the crack position is also seen in the first few natural frequencies. For cracks at the mid span, the second natural frequency remains nearly constant with the crack depth, because this point coincides with the node of the second vibration mode of the beam.

Figures 9 and 10 compare the natural frequency

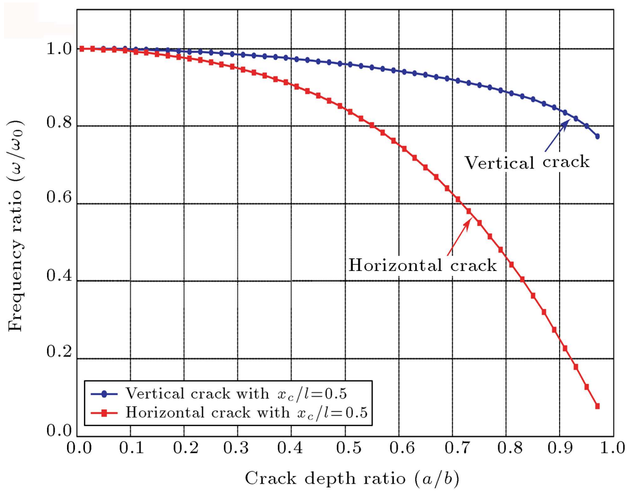

Figure 9. Fundamental natural frequency ratio versus crack depth ratio for vertical crack and horizontal crack [21] at $x_c/l = 0.5$.

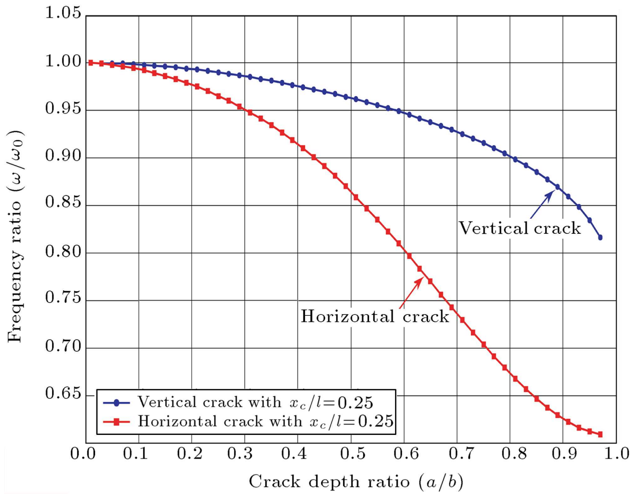

Figure 10. Second natural frequency ratio versus crack depth ratio for vertical crack and horizontal crack [21] at $x_c/l = 0.25$.

drop for horizontal and vertical cracks. In Figure 9, the fundamental frequency ratio for the mid span vertical crack has been compared with a horizontal one in the same position. It can be seen that a horizontal crack has much more effect than a vertical one on the natural frequency. This result was predictable, because a horizontal crack reduces bending stiffness more than a vertical crack. Figure 10 shows a similar comparison for the second natural frequency at $x_c/l = 0.25$. It can be seen that a horizontal crack has more effect on the second natural frequency, too.

Figures 11 to 13 show the first three normalized mode shapes for a cracked beam with $a/d = 0.5$ and $x_c/l = 0.1$, 0.3 and 0.5. Comparison of the analytic and finite element results in this set of figures shows the efficiency of the model presented in this research.

CONCLUSIONS

A continuous model for flexural vibration analysis of a beam with a vertical edge crack has been developed in this paper. It is assumed that the crack face rotates more than other parts of the section, as well as its adjacent area. The additional rotation decays with an exponential regime along the beam length. On the base of this assumption, a displacement field for the beam has been suggested and modified for compatibility, continuity and consistency. The strain and stress fields are calculated by direct derivation of the displacement field and by using the linear elastic material model. Then, the partial differential equation of motion has been obtained using the Hamilton principle. This equation has been evaluated for static conditions and the exponential decay rate has been obtained with the aid of the J-integral concept in fracture mechanics.

The obtained governing equation of motion for a simply supported beam with a rectangular cross-section and vertical edge crack has been solved with a modified Galerkin projection method. The obtained results have been compared with finite element results for a few first natural frequencies and mode shapes, and an excellent agreement has been observed.

The obtained results have also been used for studying the effect of crack parameters on natural frequencies and mode shapes. The calculated natural frequencies for a beam with vertical crack have been compared with those obtained for horizontal crack, and it is observed that the natural frequencies are more sensitive to horizontal cracks. Finally, it must be noticed that the developed theory in this research is only correct for open edge cracks without extension.

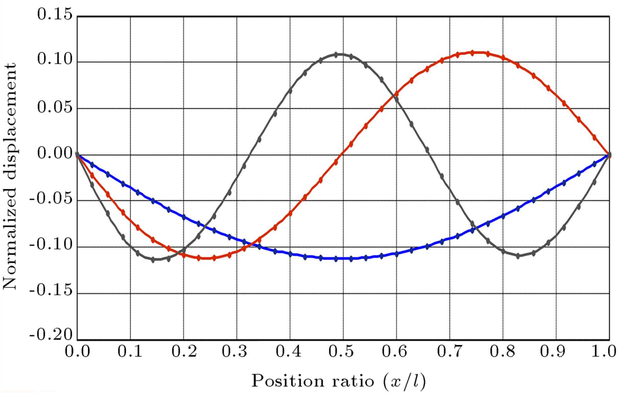

Figure 11. First three normalized mode shapes of a cracked beam with $x_c/l = 0.1$ and $a/b = 0.5$. (—): Analytical results; (• • ••): Finite element results.

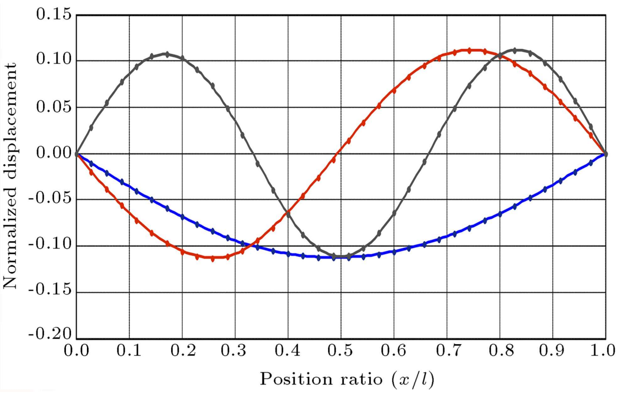

Figure 12. First three normalized mode shapes of a cracked beam with $x_c/l = 0.3$ and $a/b = 0.5$. (—): Analytical results; (• • ••): Finite element results.

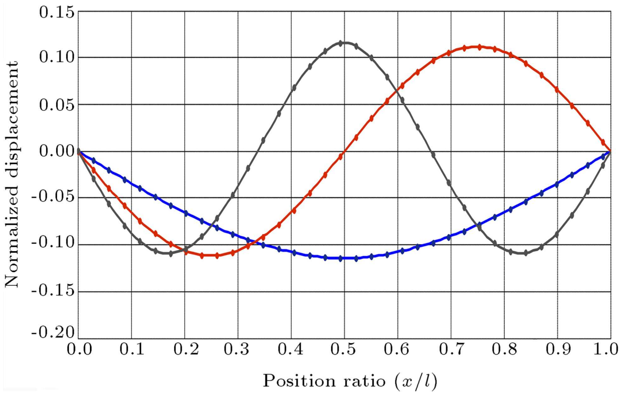

Figure 13. First three normalized mode shapes of a cracked beam with $x_c/l = 0.5$ and $a/b = 0.5$. (—): Analytical results; (• • ••): Finite element results.

ACKNOWLEDGMENT

The authors would like to acknowledge the financial assistance of the “Iranian Gas Transmission Co.” throughout this research.

REFERENCES

1. Dimarogonas, A.D. “Vibration of cracked structures-A state of the art review”, *Eng. Fract. Mech.*, **5**, pp.

831-857 (1996).

2. Wauer, J. "On the dynamics of cracked rotors: A literature survey", *Appl. Mech. Rev.*, **43**(1), pp. 13-17 (1990).
3. Gasch, R. "A survey of the dynamic behavior of a simple rotating shaft with a transverse crack", *J. Sound. Vib.*, **160**(2), pp. 313-332 (1993).
4. Dimarogonas, A.D. and Paipetis, S.A., *Analytical Methods in Rotor Dynamics*, London, Applied science publisher (1983).
5. Papadopoulos, C.A. "The strain energy release approach for modeling cracks in rotors: A state of the art review", *Mech. Syst. Signal Pr.*, **22**, pp. 763-789 (2008).
6. Zheng, D.Y. and Fan, S.C. "Vibration and stability of cracked hollow-sectional beams", *J. Sound. Vib.*, **267**, pp. 933-954 (2003).
7. Yang, J., Chen, Y., Xiang, Y. and Jia, X.L. "Free and forced vibration of cracked inhomogeneous beams under an axial force and a moving load", *J. Sound. Vib.*, **312**, pp. 166-181 (2008).
8. Lin, H.P. "Direct and inverse methods on free vibration analysis of simply supported beams with a crack", *Eng. Struct.*, **26**(4), pp. 427-436 (2004).
9. Zheng, D.Y. and Fan, S.C. "Vibration and stability of cracked hollow-sectional beams", *J. Sound. Vib.*, **267**, pp. 933-954 (2003).
10. Loya, J.A., Rubio, L. and Fernandez-Saez, J. "Natural frequencies for bending vibrations of Timoshenko cracked beams", *J. Sound. Vib.*, **290**, pp. 640-653 (2006).
11. Orhan, S. "Analysis of free and forced vibration of a cracked cantilever beam", *NDT&E Int.*, **40**, pp. 443-450 (2007).
12. Yang, X.F., Swamidas, A.S.J. and Seshadri, R. "Crack identification in vibrating beams using the energy method", *J. Sound. Vib.*, **244**(2), pp. 339-357 (2001).
13. Wang, J. and Qiao, P. "Vibration of beams with arbitrary discontinuities and boundary conditions", *J. Sound. Vib.*, **308**, pp. 12-27 (2007).
14. Christides, S. and Barr, A.D.S. "One-dimensional theory of cracked Bernoulli-Euler beams", *J. of Mech. Sci.*, **26**(11/12), pp. 639-648 (1984).
15. Shen, M.H.H. and Pierre, C. "Natural modes of Bernoulli-Euler beams with symmetric cracks", *J. Sound. Vib.*, **138**(1), pp. 115-134 (1990).
16. Shen, M.H.H. and Pierre, C. "Free vibrations of beams with a single-edge crack", *J. Sound. Vib.*, **170**(2), pp. 237-259 (1994).
17. Carneiro, S.H.S. and Inman, D.J. "Comments on the free vibration of beams with a single-edge crack", *J. Sound. Vib.*, **244**(4), pp. 729-737 (2001).
18. Chondros, T.G., Dimarogonas, A.D. and Yao, J. "A continuous cracked beam vibration theory", *J. Sound. Vib.*, **215**(1), pp. 17-34 (1998).
19. Chondros, T.G., Dimarogonas, A.D. and Yao, J. "Vibration of a beam with breathing crack", *J. Sound. Vib.*, **239**(1), pp. 57-67 (2001).
20. Behzad, M., Meghdari, A. and Ebrahimi, A. "A linear theory for bending stress-strain analysis of a beam with an edge crack", *Eng. Fract. Mech.*, **75**(16), pp. 4695-4705 (2008).
21. Behzad, M., Meghdari, A. and Ebrahimi, A. "A new continuous model for flexural vibration analysis of a cracked beam", *Pol. Mar. Res.*, **15**(2), pp. 32-39 (2008).
22. Behzad, M., Meghdari, A. and Ebrahimi, A. "A new approach for vibration analysis of a cracked beam", *Int. J. of Eng.*, **18**(4), pp. 319-330 (2005).
23. Behzad, M., Meghdari, A. and Ebrahimi, A. "A continuous model for forced vibration analysis of a cracked beam", *ASME Int. Mech. Eng. Cong. and Exp.*, Orlando, Florida USA (2005).
24. *ANSYS User's Manual for Rev.*, **8**, ANSYS Inc. (2004).
25. Barani, A. and Rahimi, G.H. "Approximate method for evaluation of the J-integral for circumferentially semi-elliptical-cracked pipes subjected to combined bending and tension", *Scientia Iranica*, **14**(5), pp. 435-441 (2007).
26. Tada, H., Paris, P.C. and Irvin, G.R., *The Stress Analysis of Cracks Handbook*, Hellertown, Pennsylvania, Del Research Corp. (1973).

BIOGRAPHIES

Mehdi Behzad has a PhD in Mechanical Engineering from the University of New South Wales, Sydney, Australia and is a faculty member of the mechanical engineering department of Sharif University of Technology in Tehran, Iran. Professor Behzad is also chairman of the Iran Maintenance Association.

Alireza Ebrahimi has a PhD in Mechanical Engineering from Sharif University of Technology in Tehran, Iran and is also a researcher of the condition monitoring center at that university.

Ali Meghdari has a PhD in Mechanical Engineering from the University of New Mexico, Albuquerque, U.S.A. and is a faculty member of the Mechanical Engineering Department of Sharif University of Technology in Tehran, Iran.

He is also a Distinguished Professor of Mechanical Engineering (MSRT), Vice-President of Academic Affairs and a Fellow of the American Society of Mechanical Engineers (ASME).