Research Note

Optimizing a Joint Economic Lot Sizing Problem with Price-Sensitive Demand

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Abstract. *This paper considers the problem of a vendor-buyer integrated production-inventory model. The vendor manufactures the item at a finite rate and delivers the final goods at a lot-for-lot shipment policy to the buyer. We relax the assumption of uniform demand in the hitherto existing joint economic lot sizing models and analyze the problem where the end customer demand is price-sensitive. The relation between demand and price is considered to be linear. The model proposed, based on the integrated expected total relevant profits of both buyer and vendor, finds out the optimal values of order quantity and mark-up percentage, using an analytical approach. Some numerical examples are also used to analyze the effect of the price-sensitivity of demand on the improvements in joint total profit over individually derived policies.*

Keywords: *Joint economic lot sizing; Mark-up pricing policy; Price-sensitive demand.*

INTRODUCTION

In the cases where no coordination exists between supply chain members, the vendor and the buyer will act independently to maximize their own profit. This independent decision behavior usually cannot assure that the two parties, as a whole, reach the optimal state. In traditional inventory management, the optimal inventory and shipment policies for manufacturer and buyer in a two-echelon supply chain are managed independently. As a result, the optimal lot size for the purchaser may not result in an optimal policy for the vendor and vice versa. To overcome this difficulty, the integrated vendor-buyer model is developed, where the joint total relevant cost for the purchaser as well as the vendor is minimized. Consequently, determining the optimal policies, based on integrated total cost function rather than buyer or supplier individual cost function, results in a reduction of the total inventory cost of the system.

The integrated vendor-buyer problem is called the Joint Economic Lot Sizing (JELS) problem and can be considered as the building block for wider supply chain systems. The global supply chain can be very complex

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and link-by-link understanding of joint policies can be very useful.

Goyal [1] was the first who introduced the idea of a joint total cost for a single-vendor and a single-buyer scenario, under the assumption of having an infinite production rate for the vendor and a lot for lot policy for the shipments from the vendor to the buyer. Banerjee [2] relaxed the infinite production rate assumption. Then, Goyal [3] contributed to the efforts of generalizing the problem by relaxing the assumption of lot for lot. He assumed that the production lot is shipped in a number of equal-size shipments. Later, Goyal [4] developed a model where the shipment size increases by a factor equal to the ratio of production rate to demand rate. He formulated the problem and developed an optimal expression for the first shipment size as a function of the number of shipments. Hill [5] generalized the model of Goyal [4] by taking the geometric growth factor as a decision variable. He suggested a solution method based on an exhaustive search for both the growth factor and the number of shipments in certain ranges. Later, Hill [6] relaxed the assumptions of the shipment policy and developed an optimal solution of the problem. He showed that the structure of the optimal policy includes shipments increasing in size, according to a geometric series, followed by equal-sized shipments.

Joint economic lot sizing models have been extended in many different directions. It is beyond

the scope of this paper to discuss all works in detail here. Broadly speaking, the existing literature on JELS may be divided into different categories such as quality (e.g. [7]), controllable lead-time (e.g. [8]), setup and order cost reduction (e.g. [9]) and transportation (e.g. [10]). Readers are referred to [11] for a comprehensive review of the JELS problems.

Despite the large amount of research extending different dimensions of JELS problems, most of them are limited to deterministic conditions in which none of the parameters are dependent on each other. However, in practice, there may be some negative or positive coordination between different parameters (e.g. price-sensitive demand, time-dependent demand, stock-dependent demand, lot-size dependent lead-times, etc). In this paper, we develop an integrated production-inventory system consisting of a single vendor and single buyer, where the lot from the vendor is transferred to the buyer in a lot-for-lot shipment policy. Unlike previous work in the literature, the demand is considered to be price-sensitive. Here, we analyze how the coordination between two supply chain members will be affected when the end customer demand is price-sensitive.

The paper is organized as follows. In the following section, the notations and assumptions of the problem are introduced. Then, a discussion on independent policies for buyer and vendor as well as on the integrated model is given and the optimal value of decision variables is also obtained. Later, some numerical examples and sensitivity analyses are presented. Finally, the paper findings and directions for future research are summarized.

ASSUMPTIONS

The assumptions of the model are summarized as follows:

1. The integrated system of single-vendor and single-buyer for a single product is considered.

2. The buyer faces a linear demand as a function of the selling price.

3. Selling price is set based on the unit purchasing price plus a constant percentage mark-up.

4. A finite production rate for the vendor is considered, which is greater than the demand rate.

5. The buyer orders a lot of size Q, when the on-hand inventory reaches the reorder point.

6. The inventory holding cost for the buyer is more than that for the vendor, i.e. $h_b > h_v$.

7. Shipments from the vendor to the buyer use a lot-for-lot policy.

8. Shortage is not allowed.

MATHEMATICAL MODELING

The optimal order quantity and profit margin of the integrated system is derived in this section. We first obtain the optimal policies if each supply chain member tries to maximize its benefit. Then, the policies and profits are compared with the case of an integrated system when they cooperate with each other. We assume that the buyer faces a linear demand, $D(\delta) = a - b\delta(a > b > 0)$, as a function of his/her unit retail price, which increases as the price decreases. Moreover, we employ a mark-up pricing policy where the selling price is set based on the unit purchasing prices, c, plus a constant percentage mark-up, i.e. $\delta = (1 + \alpha)c$.

Since $D(\alpha) = a - bc - \alpha bc > 0$, the maximum percentage mark-up is $a/bc - 1$. The buyer's yearly profit is equal to the gross revenue minus the sum of the purchasing cost, the order processing cost, and inventory holding cost. The buyer wishes to maximize his/her yearly profit function, TBP, through the optimal percentage mark-up, α, and order quantity Q, i.e.:

$$TBP(\alpha, Q) = \alpha c(a - bc - \alpha bc) - \frac{(a - bc - \alpha bc)A_b}{Q}$$

$$- \frac{h_b Q}{2}. \tag{1}$$

For the buyer's percentage mark-up, α, which in turn determines the annual demand, $D(\alpha)$, the buyer's optimal order size is $Q^* = \sqrt{2(a - bc - \alpha bc)A_b/h_b}$. Substituting the optimal order quantity into Equation 1 and simplifying, we obtain:

$$TBP(\alpha) = \alpha c(a - bc - \alpha bc) - \sqrt{2(a - bc - \alpha bc)A_b h_b}.$$

Using the approximation used by Qin et al. [12], the above expression can be rewritten as:

$$TBP(\alpha) = \alpha c(a - bc - \alpha bc) - \sqrt{2A_b h_b a}$$

$$\times [d_0(1 + \alpha)^2 c^2 + d_1(1 + \alpha)c + d_2],$$

where $d_0 = (-8 + 4\sqrt{2})(b/a)^2$, $d_1 = (12 - 7\sqrt{2})(b/a)$ and $d_2 = 3\sqrt{2} - 4$.

Substituting $c = \delta(1 + \alpha)^{-1}$ and taking the second derivation of TBP with respect to δ, we obtain:

$$\frac{\partial^2 TBP}{\partial \delta^2} = -2b - 2\sqrt{2A_b h_b a} d_0.$$

The above expression is negative if $a^3 > 11A_b h_b b^2$. In practice, a is usually very large (see [12]) and, thus, the buyer's profit function is concave in δ, and its optimal value is uniquely determined by equating the first derivation of TBP to zero.

$$\frac{\partial TBP}{\partial \delta} = a - 2b\delta + bc - \sqrt{2A_b h_b a} \times (2d_0\delta + d_1).$$

Solving $\partial TBP/\partial \delta = 0$ and substituting $\delta = c(1 + \alpha)$, the optimal percentage mark-up can be obtained as follows:

$$\alpha^* = \frac{a - bc - \sqrt{2A_b h_b a}[d_1 + 2cd_0]}{2bc + 2c\sqrt{2A_b h_b a}d_0}. \quad (2)$$

Thus, the optimal order quantity can be obtained as:

$$Q^* = \left[\frac{b(a - bc)A_b + \sqrt{2A_b h_b a}A_b(2ad_0 + bd_1)}{h_b(b + \sqrt{2A_b h_b a}d_0)}\right]^{\frac{1}{2}}. \quad (3)$$

If there is no cooperation between supply chain members, then the optimal values of order quantity and mark-up percentage are adopted just based on the buyer profit function. Thus, the orders are received by the vendor at known intervals, T (Figure 1). Therefore, the vendor's yearly profit function is as follows:

$$TVP = (a - bc - \alpha bc)\left[c - \frac{A_v}{Q} - \frac{h_v Q}{2P}\right].$$

Considering the case in which the buyer (purchaser) is free to choose his/her own pricing and ordering policies, (α, Q), then it is straightforward that the individually derived total system profit, $TIP(\alpha, Q)$, is equal to the summation of sub-optimal buyer and vendor profits, i.e. $TIP(\alpha, Q) = TBP + TVP$.

Suppose that both parties decide to cooperate and agree to follow the jointly optimal integrated policy. Therefore, the total system profit is going to be maximized, i.e.:

$$TSP(\alpha, Q) = (1 + \alpha)c(a - bc - \alpha bc)$$

$$- \frac{(a - bc - \alpha bc)(A_b + A_v)}{Q}$$

$$- \frac{Q}{2}[h_b + h_v(a - bc - \alpha bc)/P]. \quad (4)$$

It can easily be shown that the total system profit is concave in Q for the known values of the buyer

Inventory level

— Vendor inventory
- - - Buyer inventory
········· System inventory
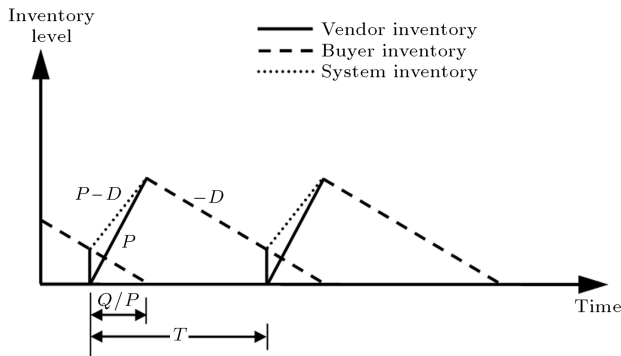
$P - D$ $-D$

P

Q/P

T

Time

Figure 1. Inventory level against time for buyer and vendor.

percentage mark-up, α. Therefore, the optimal order quantity can be obtained as:

$$Q^* = \left[\frac{2(a - bc - \alpha bc)(A_b + A_v)}{h_b + h_v(a - bc - \alpha bc)/P}\right]^{\frac{1}{2}}. \quad (5)$$

Substituting the optimal order size into Equation 4 and simplifying, we obtain:

$$TSP(\alpha) = (1 + \alpha)c(a - bc - \alpha bc)$$

$$- \sqrt{2(a - bc - \alpha bc)(A_b + A_v)[h_b + h_v(a - bc - \alpha bc)/P]}.$$

Using the same approach employed to optimize the buyer profit, the above expression can be rewritten as:

$$TSP(\alpha) = (1 + \alpha)c(a - bc - \alpha bc)$$

$$- \sqrt{2(A_b + A_v)[h_b + h_v(a - bc - \alpha bc)/P]a}$$

$$\times [d_0(1 + \alpha)^2 c^2 + d_1(1 + \alpha)c + d_2].$$

Substituting $c = \delta(1 + \alpha)^{-1}$ and taking the second derivation of TSP, with respect to δ, we obtain:

$$\frac{\partial^2 TSP}{\partial \delta^2} = -2b$$

$$- 2\sqrt{2(A_b + A_v)[h_b + h_v(a - bc - \alpha bc)/P]ad_0}.$$

The above expression is negative and, thus, the total system profit function is concave in δ and its optimal value is uniquely determined by equating the first derivation of TSP to zero.

$$\frac{\partial TSP}{\partial \delta} = a - 2b\delta + bc$$

$$- \sqrt{2(A_b + A_v)[h_b + h_v(a - bc - \alpha bc)/P]a}$$

$$\times (2d_0\delta + d_1).$$

By solving $\partial TSP/\partial \delta = 0$ and substituting $\delta = c(1 + \alpha)$, the optimal percentage mark-up under the vendor-buyer coordination can be obtained as follows:

$$\alpha^* =$$

$$\frac{a - bc - \sqrt{2(A_b + A_v)[h_b + h_v(a - bc - \alpha bc)/P]a}[d_1 + 2cd_0]}{2bc + 2c\sqrt{2(A_b + A_v)[h_b + h_v(a - bc - \alpha bc)/P]ad_0}}. \quad (6)$$

Thus, the optimal order quantity can be obtained as:

$$Q^* = \sqrt{A_b + A_v}\left[\frac{b(a - bc) + \bar{\omega}(2ad_0 + bd_1)}{h_b(b + \bar{\omega}d_0)}\right]^{\frac{1}{2}}, \quad (7)$$

where:

$$\bar{\omega} = \sqrt{2(A_b + A_v)[h_b + h_v(a - bc - \alpha bc)/P]a}.$$

NUMERICAL EXAMPLES

We consider an example with the following data, $P = 3200$/year, $A_v = \$400$/setup, $A_b = \$25$/order, $h_v = \$4$/unit/year, $h_b = \$5$/unit/year, $a = 1500$, $b = 10$, and $c = \$60$/unit.

The optimal values of α and Q and the total system profit of the individually optimized model are 0.75, 66.9 and 44106.5, respectively. The corresponding values for jointly optimized model are 0.26, 326.3 and 54310.1. The improvement in joint total profit over individually derived policies is 23.13%, which should be shared in some equitable manner through the mechanism of a side payment to the buyer from the vendor, or a price discount scheme in order to entice the buyer to change his/her lot size and selling price.

In order to gain insight into the effect of some factors such as the price-sensitivity of demand, holding costs and ordering costs, different sets of parameter specifications have been considered:

1. Four levels for a : $a \in [1300, 1500, 1700, 1900]$.

2. Six levels for b : $b \in [1, 3, \cdots, 11]$.

3. Seven levels for buyer holding cost: $h_b \in [1, 5, 10, 20, 30, 40, 50]$.

4. Three levels for buyer ordering cost: $A_b \in [12.5, 25, 50]$.

5. Seven levels for proportion of vendor setup cost to buyer ordering cost: $A_v/A_b \in [1, 2, 4, 8, 12, 16, 20]$.

To represent improvements in joint total profit over individually derived policies, we define the percentage improvement, PI, as $(TSP - TIP)/TIP \times 100$. Looking at the results in Table 1, we see that the optimal mark-up percentage is higher in supply chains in which each party tries to maximize his own benefit (non-cooperative), compared with the case of a joint system when they cooperate. Since then, the selling prices to end customers will be higher in a non-cooperative situation. Therefore, based on the negative relation between selling price and demand, the incoming demand is less than that in a cooperative situation. As can be seen in Table 1, the demand is

Table 1. Cooperative optimization vs. non-cooperative optimization.

Parameter		Non-Cooperative Vendor-Buyer				Cooperative Vendor-Buyer			
a	b	α	Q	D	TIP	α	Q	D	TSP
1300	1	10.34	78.7	619.8	418017.1	9.85	308.1	649.2	420708.6
	3	3.11	74.8	559.5	134711.1	2.62	307.8	647.6	139043.1
	5	1.67	70.6	499.1	76745.6	1.18	307.5	646.1	82711.1
	7	1.05	66.2	438.7	50979.0	0.56	307.2	644.5	58569.4
	9	0.71	61.5	378.2	35952.5	0.22	306.9	642.9	45158.0
	11	0.49	56.4	317.6	25815.1	0.00	306.3	640.0	36623.7
1500	1	12.00	84.8	719.9	557735.0	11.51	327.5	749.2	560555.0
	3	3.67	81.2	659.5	181084.2	3.18	327.3	747.7	185556.1
	5	2.00	77.4	599.2	104439.1	1.51	327.0	746.2	110557.3
	7	1.29	73.4	538.8	70657.5	0.80	326.7	744.7	78415.6
	9	0.89	69.2	478.4	51169.0	0.40	326.4	743.2	60559.5
	11	0.64	64.6	417.9	38182.5	0.15	326.1	741.7	49197.0
1700	1	13.67	90.5	819.9	717470.7	13.18	345.1	849.3	720407.6
	3	4.22	87.2	759.6	234144.2	3.73	344.9	847.8	238742.0
	5	2.34	83.6	699.3	136155.5	1.85	344.6	846.4	142409.7
	7	1.53	79.9	638.9	93219.6	1.04	344.4	844.9	101125.1
	9	1.08	76.1	578.5	68638.6	0.59	344.1	843.5	78189.6
	11	0.79	72.0	518.1	52404.5	0.30	343.9	842.0	63594.7
1900	1	15.34	95.9	919.9	897221.1	14.85	361.1	949.3	900265.2
	3	4.78	92.7	859.6	293887.1	4.29	360.9	947.9	298599.5
	5	2.67	89.4	799.3	171890.2	2.18	360.7	946.5	178267.2
	7	1.76	86.0	739.0	118659.3	1.27	360.5	945.1	126696.8
	9	1.26	82.4	678.6	88353.6	0.77	360.3	943.7	98047.0
	11	0.94	78.6	618.3	68471.6	0.45	360.1	942.3	79815.6

22.69% lower in non-cooperative vendor-buyer supply chains, on average. Additionally, there is not a significant demand variation in joint optimization as price sensitivity change. However, the demand for individual/non-cooperative optimization shows a 40% reduction, as moving from $b = 1$ to $b = 11$, on average.

Some other facts can also be discerned from Table 1. The optimal order quantity is considerably higher in joint optimization versus individual optimization. The main reason is that since the setup cost of the vendor is higher than the ordering cost of the buyer, the model enlarges the order quantity to lessen the number of ordering and setups. Based on the results shown in Table 1, the order quantity of the cooperative model is almost 4.3 times bigger than that in the non-cooperative model.

Moreover, Table 1 shows that as the price-sensitivity of demand, b, increases, mark-up percentages decrease in both joint and individual optimizations. That is because the models try to condense the effect of selling prices and control the demand reductions.

Figure 2 shows the percentage improvement in joint total profit over individually derived policies for a range of a and b. As can be seen in Figure 2, PI increases for cases where demand is more price-sensitive. This result confirms that it will be more beneficial for the buyer and vendor to cooperate with each other in competitive environments where end customers have a number of purchasing choices and can easily shift to other less expensive supply chains.

Furthermore, the increase in the total system profit shows an exponential behavior where the change in PI is 1.84%, moving from $b = 1$ to $b = 3$, while this value is equal to 9.95%, moving from $b = 9$ to $b = 11$, on average.

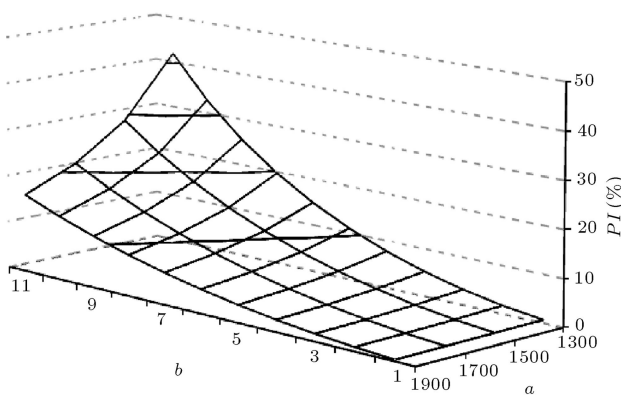We also endeavor to examine whether the buyer holding cost has any effect on the benefits of vendor-buyer coordination. As can be seen in Figure 3, the improvement percentage, PI, increases by buyer holding costs. However, this increase in the improvement percentage is of a diminishing kind, with most of the saving obtained from an initial increase. The reason is that as buyer holding costs increase, the optimal order quantity in the non-cooperative model decreases as well, much faster than that in the joint model. Thus, the total profit of non-cooperative models reduces and, consequently, cooperation becomes more attractive for supply chain members.

Figure 4 illustrates the effect of the proportion of vendor to buyer ordering cost on PI. As illustrated, the improvement percentage increases by A_v/A_b. In other words, it will be more beneficial for supply chains to cooperate with each other, as their ordering and setup costs are far from each other. However, the improvement in PI is negatively affected by buyer ordering cost decreases. For example, PI improvement is 7.8%, moving from $A_v/A_b = 1$ to $A_v/A_b = 20$ (from 19.30% to 27.10) when $A_b = 50$, but it is 3.65% when $A_b = 12.5$.
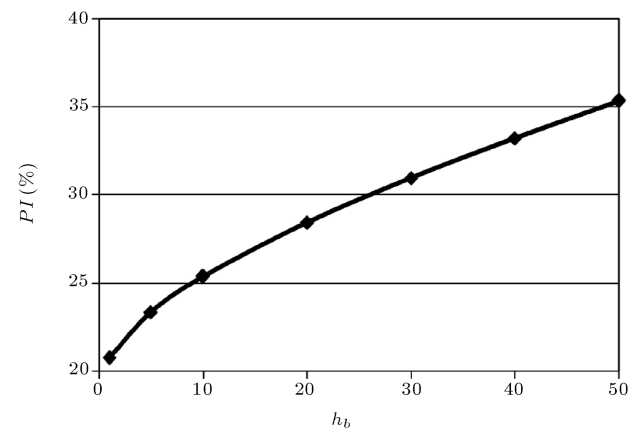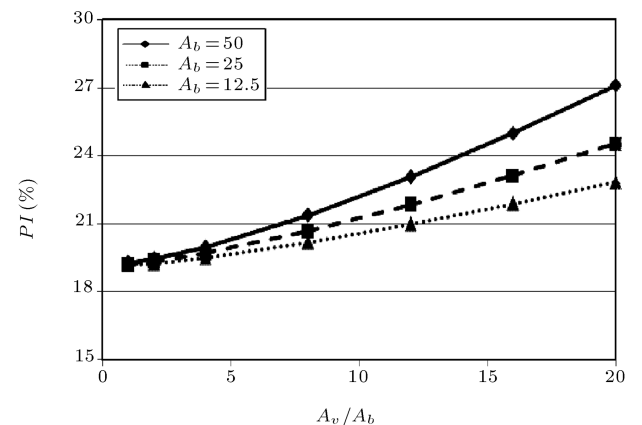

Figure 3. Effect of buyer's holding cost on PI.

Figure 2. Effect of price-sensitivity on the benefits of vendor-buyer coordination.

Figure 4. Effect of setup cost to ordering cost proportion on the benefits of vendor-buyer coordination.

CONCLUSION AND DIRECTIONS FOR FUTURE RESEARCH

We developed an integrated production-inventory model in which the objective is to maximize the joint total profit of the buyer and the vendor by optimizing the ordering and pricing policies. The developed model is a JELS model, where the lot from the vendor is transferred to the buyer in a lot-for-lot shipment policy. Unlike previous works in the literature, the demand is considered to be price-sensitive. Here, we used an analytical approach to find the optimal order quantity and mark-up percentage. Moreover, we analyzed how the coordination between two supply chain members will be affected when the end customer demand is price-sensitive.

We also compared the case of a cooperative system, when supply chain members cooperate with each other, over individually derived policies. The numerical example results showed that when demand is more price-sensitive, the system profit achieved by joint optimization is significantly larger than that obtained by individual optimization. Therefore, when demand is more price-sensitive, the supplier should employ some coordination mechanisms such as volume discount, to achieve higher channel coordination. This result confirms that it will be more beneficial for the buyer and vendor to cooperate with each other in competitive environments, where end customers have a number of purchasing choices and can easily shift to other less expensive supply chains, i.e. demand is more price-sensitive.

One of the future research directions is to extend this study for other shipment policies such as non-delayed, equal-sized shipment policies. Developing the model to the multi-supplier case is also proposed for future research. Moreover, in this paper, we considered a linear price-demand relation, however, it might be useful to analyze other demand functions (e.g. log-linear) as well.

NOMENCLATURE

D	demand rate as a function of unit selling price
P	production rate of the vendor
Q	buyer's order quantity
c	the buyer unit purchasing price
δ	the buyer unit selling price
α	mark-up percentage
A_v	vendor's setup cost
A_b	buyer's ordering cost
h_v	inventory holding cost for the vendor per unit time
h_b	inventory holding cost for the buyer per unit time

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