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Research Note

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Convergence Analysis of Spline Solution of Certain Two-Point Boundary Value Problems

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Abstract. *The smooth approximate solution of second order boundary value problems are developed by using non-polynomial quintic spline function. We obtained the classes of numerical methods, which are second, fourth and six-order. For a specific choice of the parameters involved in a non-polynomial spline, truncation errors are given. A new approach convergence analysis of the presented methods are discussed. Three test examples are considered in our references. By considering the maximum absolute errors in the solution at grid points and tabulated in tables for different choices of step size, we conclude that our presented methods produce accurate results in comparison with those obtained by existing methods.*

Keywords: *Two-point boundary value problem; Non-polynomial Quintic spline; Convergence analysis.*

INTRODUCTION

We consider the two-point boundary value problem:

$$u'' + g(x)u = q(x),$$

$$u(a) = \alpha,$$

$$u(b) = \beta,$$

$$a \le x \le b, \tag{1}$$

where $g(x)$ and $q(x)$ are continuous functions on $[a, b]$ and a, b, α and β are arbitrary real finite constants. Such problems arise in the theory which describes the deflection of plates and a number of other scientific applications. In general, it is difficult to obtain the analytical solution of Equations 1 for arbitrary choices of $g(x)$ and $q(x)$. We usually resort to a numerical method for obtaining an approximate solution of the problem (Equations 1). A more commonly used finite difference method for solving Equations 1 numerically is discussed by many authors and we refer the reader, in particular, to Fox [1], Henrici [2], Aziz et al. [3], Bramble et al. [4], Fischer et al. [5] and Usmani [6]. The possibility of using spline functions for obtaining a smooth approximate solution of Equations 1 is briefly discussed by Ahlberg et al. [7]. Since then, Albasiny and Hoskins [8], Bickley [9], Fyfe [10] and Sakai and Usmani [11] have used the cubic spline for obtaining approximations. Bhatta et al. [12] have used the spline functions of degrees seven and eight and Usmani and Wasrt [13] used the Quintic spline. Also, Usmani and Sakai [14] used a cubic and a quartic spline. Khan [15] used a parametric cubic spline function to develop a numerical method for computing smooth approximations to the solution for second order boundary value problems. Recently, Ramadan et al. [16] developed a sixth-order method based on a quintic non-polynomial spline function for the solution of high order two point boundary value problems, but in application, they solved only second and fourth order boundary value problems. Their method suffers from boundary conditions and due to this, the order of accuracy of their method is reduced. Besides, in the convergence analysis, they assumed more restrictions on Equations 1 and an arising coefficient matrix.

In this paper, we have derived a uniformly convergent mesh difference scheme using a non-polynomial spline for the solution of Equations 1. Analysis of the methods shows a second, fourth and sixth-

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order convergent for arbitrary α, β, p, r and s. In this article, first, the consistency relation of our non-polynomial quintic spline in [17] is used for the solution of Equations 1. Then, the methods and the development of boundary conditions are described and classes of the methods are discussed. Following that a new approach for convergence analysis is presented. Here, we obtained the restriction on function g only. Finally, some numerical evidence is included to show the practical applicability and superiority of our methods.

DESCRIPTION OF THE METHODS AND DEVELOPMENT OF BOUNDARY CONDITIONS

Let us consider a mesh with nodal points x_i on $[a,b]$, such that:

$$\Delta : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

where $h = \frac{b-a}{n}$ for $i = 1(1)n$. We also denote the function value $u(x_i)$ by u_i.

For each segment $[x_i, x_{i+1}], i = 0, 1, 2, \cdots, n-1$, by using the non-polynomial quintic spline relation derived in our paper [17], we have:

$$pM_{i-2} + rM_{i-1} + sM_i + rM_{i+1} + pM_{i+2} = \frac{1}{h^2}[\alpha(u_{i+2} + u_{i-2}) + 2(\beta - \alpha)(u_{i+1} + u_{i-1}) + (2\alpha - 4\beta)u_i], \quad (2)$$

where:

$$p = \alpha_1 + \frac{\alpha}{6},$$

$$r = 2\left[\frac{1}{6}(2\alpha + \beta) - (\alpha_1 - \beta_1)\right],$$

$$s = 2\left[\frac{1}{6}(\alpha + 4\beta) + (\alpha_1 - 2\beta_1)\right],$$

$$\alpha = \left(\frac{1}{\theta^2}\right)(\theta \csc\theta - 1),$$

$$\beta = \left(\frac{1}{\theta^2}\right)(1 - \theta \cot\theta),$$

$$\alpha_1 = \frac{1}{\theta^2}\left(\frac{1}{6} - \alpha\right),$$

$$\beta_1 = \frac{1}{\theta^2}\left(\frac{1}{3} - \beta\right).$$

At mesh point x_i, the proposed differential Equations 1 may be discretized by:

$$M_i + g_i u_i = q_i, \quad (3)$$

where:

$$M_i = S_i''(x_i),$$

$$g_i = g(x_i),$$

$$q_i = q(x_i),$$

and:

$$u_i = u(x_i).$$

Substituting Equation 3 in spline Relation 2, we obtain:

$$(\alpha + ph^2 g_{i-2})u_{i-2} + (2(\beta - \alpha) + h^2 r g_{i-1})u_{i-1} + (2\alpha - 4\beta + sh^2 g_i)u_i + (2(\beta - \alpha) + h^2 r g_{i+1})u_{i+1} + (\alpha + ph^2 g_{i+2})u_{i+2} = h^2(p(q_{i-2}) + r(q_{i-1}) + s(q_i) + r(q_{i+1}) + p(q_{i+2})), \quad (4)$$

$$i = 2(1)n - 2.$$

To obtain a unique solution for this system (Equation 4), we need two more equations to be associated, so we use the following boundary conditions:

(a) Following [17], the second-order boundary formula is:

$$u_1 - 2u_2 + u_3 = \frac{h^2}{6}(u_1'' + 4u_2'' + u_3''),$$

$$i = 1,$$

$$u_{n-3} - 2u_{n-2} + u_{n-1} = \frac{h^2}{6}(u_{n-3}'' + 4u_{n-2}'' + u_{n-1}''),$$

$$i = n - 1, \quad (5)$$

using Equation 3, we have:

$$(1 + \frac{h^2}{6}g_1)u_1 + (-2 + \frac{4h^2}{6}g_2)u_2 + (1 + \frac{h^2}{6}g_3)u_3 = \frac{h^2}{6}[q_1 + 4q_2 + q_3],$$

$$(1 + \frac{h^2}{6}g_{n-3})u_{n-3} + (-2 + \frac{4h^2}{6}g_{n-2})u_{n-2} + (1 + \frac{h^2}{6}g_{n-1})u_{n-1} = \frac{h^2}{6}[q_{n-3} + 4q_{n-2} + q_{n-1}],$$

(b) Following [17], the fourth-order boundary formula is:

$$u_1-2u_2+u_3=\frac{h^2}{12}(u_1''+10u_2''+u_3''),$$

$$i=1,$$

$$u_{n-3}-2u_{n-2}+u_{n-1}=\frac{h^2}{12}(u_{n-3}''+10u_{n-2}''+u_{n-1}''),$$

$$i=n-1, \qquad (6)$$

using Equation 3, we have:

$$(1+\frac{h^2}{12}g_1)u_1+(-2+\frac{10h^2}{12}g_2)u_2+(1+\frac{h^2}{12}g_3)u_3=\frac{h^2}{12}[q_1+10q_2+q_3],$$

$$(1+\frac{h^2}{12}g_{n-3})u_{n-3}+(-2+\frac{10h^2}{12}g_{n-2})u_{n-2}+(1+\frac{h^2}{12}g_{n-1})u_{n-1}=\frac{h^2}{12}[q_{n-3}+10(q_{n-2})+q_{n-1}],$$

(c) In order to obtain the sixth-order boundary formula, we define the following identities:

$$\sum_{k=0}^{3}a_ku_k+h^2\sum_{k=0}^{5}b_ku_k''+t_1h^8u_0^{(8)},$$

$$i=1,$$

$$\sum_{k=0}^{3}a_ku_{n-k}+h^2\sum_{k=0}^{5}b_ku_{n-k}''+t_{n-1}h^8u_n^{(8)},$$

$$i=n-1, \qquad (7)$$

in order to obtain unknown coefficients a and b in Relations 7, by Taylor's expansion, we obtain:

$$(a_0,a_1,a_2,a_3)=(-10,19,-8,-1),$$

$$t_1=t_{n-1}=(\frac{2179}{60480}),$$

$$(b_0,\ b_1,\ b_2,\ b_3,\ b_4,\ b_5)=\left(\frac{179}{240},\frac{1057}{120},\frac{39}{40},\frac{41}{60},\frac{-61}{240},\frac{1}{24}\right).$$

CLASSES OF THE METHODS

By expanding Equation 4 in the Taylor's series about x_i, we obtain the following local truncation error:

$$t_i=\left[\frac{1}{6}(7\alpha+\beta)-(4p+r)\right]h^4u_i^{(4)}+\left[\frac{1}{180}(31\alpha+\beta)-\frac{1}{12}(16p+r)\right]h^6u_i^{(6)},+\left[\frac{1}{131040}(1611\alpha+31\beta)-\frac{1}{360}(4p+r)\right]h^8u_i^{(8)}+O(h^9),\qquad 2\le i\le n-2. \qquad (8)$$

By using the above truncation error to eliminate the coefficients of various powers, h, we can obtain classes of the methods. For any choice of α, β, p, r and s, whose $\alpha+\beta=\frac{1}{2}$ and with boundary formulas (Relations 5 to 7) we obtain the following methods.

Second-Order Method

For:

$$(\alpha,\beta)=\left(\frac{1}{4},\frac{1}{4}\right),$$

and:

$$p=0.040634839941134321703,$$

$$r=0.25412730690212937985,$$

$$s=0.41047570631347259688,$$

we obtain the second-order method, $t_i=O(h^4)$.

Fourth-Order Method

For:

$$(\alpha,\beta)=(\frac{1}{6},\frac{1}{3}),$$

and:

$$p=\frac{1}{120},$$

$$r=\frac{26}{120},$$

$$s=\frac{66}{120},$$

we obtain the fourth-order method, $t_i=O(h^6)$.

Sixth-Order Method

For

$$(\alpha,\beta)=(\frac{1}{12},\frac{5}{12}),$$

and:

$$p=\frac{1}{360},$$

$$r=\frac{56}{360},$$

$$s=\frac{246}{360},$$

we obtain the sixth-order method, $t_i=O(h^8)$.

CONVERGENCE ANALYSIS

In this section, we investigate the new approach convergence analysis of the sixth-order associated method, a boundary formulas (Relations 7). The given system can be considered in matrix form as:

$$\begin{aligned}
&\text{(i)} \quad A\overline{U}=C+T,\\
&\text{(ii)} \quad AU=C,\\
&\text{(iii)} \quad AE=T, \qquad (9)
\end{aligned}$$

where:

$$\overline{U}=(\overline{u}_i),\qquad U=(u_i),\qquad C=(c_i),$$

$$T=(t_i),\qquad E=(e_i),$$

are $(n-1)$-dimensional column vectors. Matrix A is defined by:

$$A=(A_0A_1+6A_0)+B,$$

where A is a monotone five band matrix of order $n-1$, and $A_0=(a_{ij})$ is a tri-diagonal matrix defined by:

$$a_{ij}=\begin{cases}2, & i=j=1,2,\cdots,n-1,\\ -1, & |i-j|=1,\\ 0, & \text{otherwise},\end{cases} \qquad (10)$$

and $A_1=(a^*_{ij})$, is a tri-diagonal matrix defined by:

$$a^*_{ij}=\begin{cases}4, & i=j=1,2,\cdots,n-1,\\ 1, & |i-j|=1,\\ 0, & \text{otherwise},\end{cases} \qquad (11)$$

$B=h^2QG$ with $G=\mathrm{diag}(g_i)$, $i=1,2,\cdots,n-1$, and:

$$(A_0A_1+6A_0)=$$

$$\begin{bmatrix}
19 & -8 & -1 & & & & & \\
-8 & 18 & -8 & -1 & & & & \\
-1 & -8 & 18 & -8 & -1 & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & & & & -1 & -8 & 18 & -8 & -1\\
 & & & & & -1 & -8 & 18 & -8\\
 & & & & & & -1 & -8 & 19
\end{bmatrix}, \qquad (12)$$

$$Q=$$

$$\left[\begin{matrix}
\frac{1057}{120} & \frac{39}{40} & \frac{41}{60} & \frac{-61}{240}\\
-\frac{672}{360} & -\frac{2952}{360} & -\frac{672}{360} & -\frac{12}{360}\\
-\frac{12}{360} & -\frac{672}{360} & -\frac{2952}{360} & -\frac{672}{360}\\
 & \ddots & \ddots & \ddots\\
 & & \ddots & \ddots\\
 & & & \ddots
\end{matrix}\right.$$

$$\left.\begin{matrix}
\frac{1}{24} & & & & \\
-\frac{12}{360} & & & & \\
\ddots & \ddots & & & \\
\ddots & \ddots & \ddots & & \\
\ddots & \ddots & \ddots & \ddots & \\
-\frac{12}{360} & -\frac{672}{360} & -\frac{2952}{360} & -\frac{672}{360} & -\frac{12}{360}\\
 & -\frac{12}{360} & -\frac{672}{360} & -\frac{2952}{360} & -\frac{672}{360}\\
\frac{1}{24} & \frac{-61}{240} & \frac{41}{60} & \frac{39}{40} & \frac{1057}{120}
\end{matrix}\right]. \qquad (13)$$

Vector **C** is given by:

$$c_i = \begin{cases} 10\alpha - h^2\left[\frac{179}{240}(q_0 - g_0\alpha) + \frac{1057}{120}q_1 + \frac{39}{40}q_2 + \frac{41}{60}q_3 - \frac{61}{240}q_4 + \frac{1}{24}q_5\right], \\ i = 1, \\ \alpha - \frac{12h^2}{360}[(q_0 - g_0\alpha) + 56q_1 + 246q_2 + 56q_3 + q_4], \\ i = 2, \\ -\frac{12h^2}{360}[q_{i-2} + 56q_{i-1} + 246q_i + 56q_{i+1} + q_{i+2}], \\ i = 3, \cdots, n-3, \\ \beta - \frac{12h^2}{360}[(q_n - g_n\beta) + 56q_{n-1} + 246q_{n-2} + 56q_{n-3} + q_{n-4}], \\ i = n-2, \\ 10\beta - h^2[\frac{179}{240}(q_n - g_n\beta) + \frac{1057}{120}q_{n-1} + \frac{39}{40}q_{n-2} + \frac{41}{60}q_{n-3} - \frac{61}{240}q_{n-4} + \frac{1}{24}q_{n-5}], \\ i = n-1. \end{cases} \tag{14}$$

Vector T is the local truncation error vector and is defined as:

$$t_i = \begin{cases} \frac{2179}{60480}h^8u^{(8)}(\xi_1) + O(h^9), & a < \xi_1 < x_5 \quad i = 1, \\ \frac{173}{262080}h^8u^{(8)}(\xi_i) + O(h^9), & x_{i-2} < \xi_i < x_{i+2} \quad i = 2, \cdots, n-2, \\ \frac{2179}{60480}h^8u^{(8)}(\xi_{n-1}) + O(h^9), & x_{n-5} < \xi_{n-1} < b \quad i = n-1. \end{cases} \tag{15}$$

Lemma 1

If M is a square matrix of order n and $\|\mathrm{M}\| < 1$, then $(I + \mathrm{M})^{-1}$ exists and $\|(I + \mathrm{M})^{-1}\| < \frac{1}{(1-\|\mathrm{M}\|)}$.

To explain the existence of A^{-1}, since $A = (A_0A_1 + 6A_0) + B$, we have to show $(A_0A_1 + 6A_0)$ is nonsingular. By using Lemma 1 and Henrici [2], we shall first require bounds for the element of $(A_0)^{-1}$.

If $A_0^{-1} = (a^*_{ij})$, then:

$$a^*_{ij} = \begin{cases} \frac{j(n-i)}{n}, & i \geq j, \\ \frac{i(n-j)}{n}, & i \leq j, \end{cases} \tag{16}$$

and we get:

$$\sum_{j=1}^{n} a^*_{ij} = \sum_{j=1}^{i} \frac{j(n-i)}{n} + \sum_{j=i+1}^{n} \frac{i(n-j)}{n} \leq \frac{(n)^2}{8}, \tag{17}$$

where the equality holds only if n is odd. Inequality can be written as:

$$\|A_0^{-1}\| \leq \frac{(b-a)^2}{8h^2}. \tag{18}$$

Also, by using [13] we have:

$$\|A_1^{-1}\| \leq \frac{1}{2},$$

$$(A_0A_1 + 6A_0)^{-1} = (I + \frac{1}{6}A_1)^{-1}(6A_0)^{-1}. \tag{19}$$

By using Lemma 1 $(I + \frac{1}{6}A_1)^{-1}$ exists and we get bounds for $\|(A_0A_1 + 6A_0)^{-1}\|$,

$$\|(A_0A_1 + 6A_0)^{-1}\| \leq \frac{\|(6A_0)^{-1}\|}{1 - \|\frac{1}{6}A_1\|} = \frac{(b-a)^2}{44h^2}, \tag{20}$$

where $\|.\|$ represents the ∞-norm in the matrix vector.

Lemma 2

Matrix $A = (A_0A_1 + 6A_0) + B$ is nonsingular, if:

$$\|g\| < \frac{11}{3(b-a)^2},$$

where $\|G\| \leq \|g\| = \max_{a \leq x \leq b}|g(x)|$.

Proof

Since

$$A^{-1} = [(A_0A_1 + 6A_0) + B]^{-1}$$

$$= [I + (A_0A_1 + 6A_0)^{-1}B]^{-1}(A_0A_1 + 6A_0)^{-1},$$

it is sufficient to show that $[I + (A_0A_1 + 6A_0)^{-1}B]$ is nonsingular. Moreover, we know that in the case of a sixth-order method we can obtain:

$$\|Q\| \leq 12.$$

Also, using Lemma 1, if $\|(A_0A_1 + 6A_0)^{-1}B\| < 1$, then $(I + (A_0A_1 + 6A_0)^{-1}B)^{-1}$ exists. Also, we get:

$$\|(I + (A_0A_1 + 6A_0)^{-1}B)^{-1}\| < \frac{1}{1 - \|(A_0A_1 + 6A_0)^{-1}B\|},$$

where:

$$\|(A_0A_1 + 6A_0)^{-1}B\| \leq \|(A_0A_1 + 6A_0)^{-1}\|\|B\| \leq \frac{(b-a)^2}{44h^2}(h^2\|Q\|\|g\|) < 1,$$

and then, we have:

$$\|g\| < \frac{11}{3(b-a)^2}. \square$$

Theorem 1

Let $u(x)$ be the exact solution of the boundary value problem (Equations 1) and assume that u_i, $i = 1, 2, \cdots, n-1$ be the numerical solution obtained by solving the system (Equation 9 (iii)), then we have:

$$\|E\| \equiv O(h^6),$$

$$\left(\text{provided } |g(x)| < \frac{11}{3(b-a)^2}, \quad \alpha = \frac{1}{12}, \quad \beta = \frac{5}{12}, \quad p = \frac{1}{360}, \quad r = \frac{56}{360}, \quad s = \frac{246}{360}\right).$$

Proof

The main purpose is to drive a band on $\|E\|$. Using Equation 9 (ii) and Lemma 2, we have:

$$E = A^{-1}T = [I + (A_0A_1 + 6A_0)^{-1}B]^{-1}(A_0A_1 + 6A_0)^{-1}T,$$

$$\|E\| \leq \|[I + (A_0A_1 + 6A_0)^{-1}B]^{-1}\| \, \|(A_0A_1 + 6A_0)^{-1}\|\|T\|. \tag{21}$$

By using Lemma 1, we get:

$$\|E\| \leq \frac{\|(A_0A_1 + 6A_0)^{-1}\|\|T\|}{1 - \|[(A_0A_1 + 6A_0)^{-1}B]\|}, \tag{22}$$

provided $\|[(A_0A_1 + 6A_0)^{-1}B]\| < 1$, also we have:

$$\|T\| \leq \frac{2179h^8M_8}{60480}, \tag{23}$$

where $M_8 = \max_{a\leq\xi\leq b}|u^{(8)}(\xi)|$.

Using Equations 20, 22, 23 and Lemma 2, we obtain:

$$\|E\| \leq \frac{2179(b-a)^2h^6M_8}{60480(44 - 12(b-a)^2\|g\|)} \equiv O(h^6), \tag{24}$$

provided that:

$$\|g\| < \frac{11}{3(b-a)^2}. \tag{25}$$

NUMERICAL ILLUSTRATIONS

In order to test the viability of the proposed methods, based on a non-polynomial spline, and to demonstrate its convergence computationally, we consider the following four test boundary value problems.

Example 1

We consider the following boundary-value problem:

$$u'' = u + x^2 - 2, \qquad u(0) = 0, u(1) = 1,$$

with the exact solution, $u(x) = \frac{2sinh(x)}{sinh(1)} - x^2$.

This problem has been solved using our methods with different values of $n = 8, 16, 32, 64$ and the maximum absolute errors in solutions are tabulated in Table 1.

Table 1. Observed maximum absolute errors for Example 1.

n	Second-Order $\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	Fourth-Order $\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	Sixth-Order $\alpha = \frac{1}{12}, \beta = \frac{5}{12}$
8	1.09×10^{-4}	5.22×10^{-8}	8.75×10^{-11}
16	3.06×10^{-5}	2.31×10^{-9}	5.74×10^{-13}
32	8.11×10^{-6}	1.34×10^{-10}	2.30×10^{-14}
64	2.09×10^{-6}	8.42×10^{-12}	3.68×10^{-14}

Table 2. Observed maximum absolute errors for Example 2.

n	**Second-Order** $\alpha=\frac{1}{4}$, $\beta=\frac{1}{4}$	**Fourth-Order** $\alpha=\frac{1}{6}$, $\beta=\frac{1}{3}$	**Fourth-Order [13]**
8	2.42×10^{-3}	2.64×10^{-6}	1.10×10^{-6}
16	6.80×10^{-4}	1.26×10^{-7}	1.09×10^{-7}
32	1.80×10^{-4}	7.41×10^{-9}	7.51×10^{-9}
64	4.65×10^{-5}	4.65×10^{-10}	4.81×10^{-10}
128	1.18×10^{-5}	2.95×10^{-11}	3.03×10^{-11}
256	2.98×10^{-6}	1.78×10^{-12}	1.85×10^{-12}
512	7.47×10^{-7}	4.35×10^{-13}	6.48×10^{-13}
1024	1.87×10^{-7}	4.60×10^{-13}	7.64×10^{-13}

n	**Sixth-Order** $\alpha=\frac{1}{12}$, $\beta=\frac{5}{12}$	**Seventh-Order [12]**
8	8.02×10^{-9}	1.44×10^{-7}
16	5.01×10^{-11}	1.41×10^{-9}
32	1.16×10^{-12}	1.23×10^{-11}
64	2.55×10^{-15}	1.01×10^{-13}

Example 2

We consider the following boundary-value problem:

$$u''=u-4xe^{x},\qquad u(0)=u(1)=0,$$

with the exact solution, $u(x)=x(1-x)e^{x}$.

We applied our methods to solve this problem with $n=8,16,32,64,128,256,512,1024$ and the computed solutions are compared with the exact solution at grid points. The maximum absolute errors at the nodal points, $\max|u(x_i)-u_i|$, are given to compare with [12,13]. The observed maximum absolute errors are tabulated in Table 2.

Example 3

We consider the following example in [12,14,15]:

$$x^2u''=2u-x,\qquad u(2)=u(3)=0,$$

with the exact solution, $u(x)=\frac{(19x-5x^2-\frac{36}{x})}{38}$.

We applied our methods to solve this problem for $n=8,16,32,64$ and the computed solutions are compared with the exact solution at grid points. The observed maximum absolute errors are tabulated in Table 3. In this table we compared our results with the results given in [12,14,15]. This shows that our results are more accurate.

Example 4

We consider the following example in [16]:

$$u''=u+(4-2x^2)\sin x+4x\cos x,$$

$$u(0)=u(1)=0,$$

with the exact solution, $u(x)=(x^2-1)\sin x$.

We applied our sixth-order method to solve this problem for $n=8,16,32$ and 64. The computed solutions are compared with the exact solution at grid points. The observed maximum absolute errors are tabulated in Table 4. In this table, we compared our results with the results obtained by the methods in [16] and also with the results obtained by [18,19] which are reported in [16]. This shows that our results are more accurate.

CONCLUSION

The approximate solutions of second-order linear boundary-value problems using a non-polynomial spline, show that our methods are better in the sense of accuracy and applicability. These have been verified by the maximum absolute errors, $\max|e_i|$, given in the tables.

Table 3. Observed maximum absolute errors for Example 3.

n	Second-Order $\alpha=\frac{1}{4}$, $\beta=\frac{1}{4}$	Second-Order [14]	Fourth-Order $\alpha=\frac{1}{6}$, $\beta=\frac{1}{3}$	Fourth-Order [14]
8	3.30×10^{-5}	4.17×10^{-5}	9.21×10^{-8}	1.74×10^{-7}
16	9.25×10^{-6}	1.04×10^{-5}	4.11×10^{-9}	1.10×10^{-8}
32	2.45×10^{-6}	2.61×10^{-6}	2.29×10^{-10}	6.85×10^{-10}
n	**Fourth-Order [15]**	**Sixth-order $\alpha=\frac{1}{12}$, $\beta=\frac{5}{12}$**	**Seventh-order [12]**	
8	1.74×10^{-7}	1.78×10^{-9}	1.31×10^{-8}	
16	1.09×10^{-8}	1.62×10^{-11}	1.56×10^{-10}	
32	6.85×10^{-10}	7.15×10^{-13}	1.53×10^{-12}	
64	-	3.68×10^{-15}	1.33×10^{-14}	

Table 4. Observed maximum absolute errors for Example 4.

n	Sixth-Order	Ramadan [16]	Islam [18]	Al-Said [19]
8	7.35×10^{-9}	7.24×10^{-9}	2.37×10^{-5}	6.49×10^{-4}
16	1.68×10^{-11}	1.16×10^{-10}	1.60×10^{-6}	1.70×10^{-4}
32	5.06×10^{-13}	1.82×10^{-12}	1.03×10^{-7}	4.15×10^{-5}
64	4.12×10^{-15}	6.51×10^{-14}	6.60×10^{-9}	1.82×10^{-5}

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