Transaction B: Mechanical Engineering
Vol. 16, No. 5, pp. 413–424

SCIENTIA IRANICA

A Harmonic Balance Approach for the Analysis of Flexible Rotor Bearing Systems on Non-Linear Support

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Abstract. *The purpose of this article is to describe the theoretical background to the Harmonic Balance approach; adopted and further developed for the analysis of general multi degree of freedom rotor bearing systems with nonlinear supports. System equations of motion are prepared for dynamic systems with any number of degrees of freedom. Nonlinear behaviour can be associated with any number of these freedoms. A computer program which uses the harmonic balance method to solve the system equations of motion is also written. These equations are partitioned into linear and nonlinear parts. The nonlinear sets of equations need to be solved prior to solving the linear sets of equations. Verification of the proposed method of solution is justified through two examples. The frequency response of a well known rotor bearing, the so called Jefcott rotor, is examined and tested against data reported by some other researchers. Also, the versatility of this method is tested by comparing the harmonic balance approach with the transient solution and some experimental measurements involving the nonlinear squeeze film bearing supports, which have already been reported by this author. It is shown that by utilizing harmonic balance with appropriate condensation, it is possible to considerably reduce the number of simultaneous nonlinear equations inherent to such systems. The stability (linear) of the equilibrium solutions may be conveniently evaluated using the Floquet theory.*

Keywords: *Non-linear dynamics; Flexible rotor bearing systems; Squeeze film bearings; Harmonic balance approach; Floquet theory.*

INTRODUCTION

The trend of increasing power-to-weight ratios in high speed rotating machinery results in more flexible rotors and higher operating speeds. In many applications of high speed rotating machinery, rolling element bearings are preferred to hydrodynamic bearings, due to instability problems in the latter and their rapid failure in case of malfunction. However, rolling element bearings provide very little damping. Significant vibration isolation can be achieved by mounting these bearings in appropriately designed damped flexible supports. As shown by Glienicke and Stanski [1], Squeeze Film Dampers (SFD's) have proved extremely useful for this purpose. An SFD is a journal bearing, wherein the journal is mechanically prevented from rotating. In its simplest form, an SFD can be designed such that the rotor locates itself within the clearance space, or the rotor is preloaded within the clearance space by retainer springs.

Squeeze film damper bearings are highly nonlinear elements with vibrational amplitude-dependent stiffness and damping coefficients. Regardless of the damper design, satisfactory linearization of a rotor bearing system incorporating such dampers has not been found possible owing to the strongly nonlinear fluid film forces that are motion dependent.

Many rotor bearing systems are inherently nonlinear because of the existence of nonnegotiable nonlinear sources such as bearings, dampers, seals, etc. They cannot be approximately analyzed by a linear model. Many numerical-analytical techniques have been proposed to study nonlinear rotor bearing problems [2,3].

The influence of squeeze film dampers on the dynamic behaviour of rigid and flexible rotors has been the subject of many theoretical and experimental investigations [4,5]. Properly designed, such dampers can

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Received 24 February 2008; received in revised form 1 December 2008; accepted 9 March 2009

significantly reduce vibration amplitudes and bearing transmitted forces due to rotor unbalance, and permit safe passage through critical speeds. In application, the dampers are usually either centrally preloaded using centralizing springs or such support springs are dispensed with altogether so that the damper journal depends on unbalance excitation for lift-off from the damper bearing surface. Squeeze film dampers that operate without a centralizing spring have the obvious advantage of simplicity with a resulting reduction in both production cost and complexity of assembly. However, their analysis is more complex [6].

Circular orbit type equilibrium solutions and the stability of centrally preloaded dampers have been well documented [4,7]. For dampers which are not centrally preloaded, the steady state journal centre orbit need not be circular and its determination generally necessitates transient solutions [8,9]. It is often computationally prohibitive to carry out parametric design studies on the vibration behaviour of such rotor bearing systems, and various attempts have been made to quasi-linearize the damper forces [10]. Such solutions assume that the journal centre motions are synchronous with the excitation frequency and make no allowance for the possibility of sub and super-harmonic vibrations. More recently, trigonometric collocation and harmonic balance techniques have been successfully tried out over a limited range of relevant parameters [11] for rigid rotors. Extension to general rotor bearing systems necessitates a condensation of the potentially large number of nonlinear simultaneous equations to a manageable size, and the technique used for determining equilibrium orbits in this report is similar to that of [12]. As evidenced by the unexpected instabilities discovered in [4], the stability evaluation of equilibrium orbits is an essential requirement for the assumed equilibrium solution analysis. Since the perturbed orbits may now result in linear differential equations with periodic coefficients, even with rotating coordinates, the theory is developed with the damper forces expressed directly in terms of stationary coordinates, thereby, simplifying application of the Floquet theory [13] in evaluating system stability. The versatility of the technique is illustrated using systems with and without centralizing springs and of increasing complexity. Of particular interest is the applicability of this approach to unsupported systems with relatively large unidirectional loadings, i.e. at high orbit eccentricities as occurs when the damper has just lifted off as well as to confirmation of the instability results reported in [4].

BASIC THEORY

Empirical methods for the analysis of large rotor bearing systems must include many degrees of freedom. A general trend for the development of numerical-analytical methods is to avoid unnecessary complications. Hence, linear methods of analysis are preferable. The presence of nonlinear effects associated with any of the system degrees of freedom contradicts the simplicity of the methods. Transient methods of solution resorting to the direct integration of system equations of motion are a general remedy for such cases. However, transient methods are time consuming and run the risk of numerical instabilities. In order to reduce the computation time, many other techniques have also been developed. The accuracy and stability of numerical solutions combined with the total execution time of computer software are major concerns for analysts of rotating machinery.

The harmonic balance method of solution presented in this article is a numerical-analytical method for the prediction of the steady-state periodic response of large order nonlinear rotor dynamic systems. Using this method, the set of nonlinear differential equations governing the motion of rotor systems is transformed to a set of nonlinear algebraic equations. A condensation technique is proposed to reduce the nonlinear algebraic equations to only those related to the physical coordinates associated with nonlinear components.

The condensation technique can result in a substantial reduction for a large order system with a small number of nonlinear coordinates, compared to that of system degrees of freedom.

System idealization and its equilibrium solutions are explained in the next sections. Squeeze film dampers are the source of nonlinearity for the system. The addition of unbalance masses to the rotor degrees of freedom is the source of external excitation to the system. This excitation can be in the form of force or couple unbalances. Nonlinear hydrodynamic bearing forces associated with damper degrees of freedom are calculated based on the short bearing approximation, as explained later in the text. System mass, stiffness and damping matrices and the amount of its unbalance excitation are input to the program. After partitioning the system equations of motion into their nonlinear and linear parts, the nonlinear sets of equations need to be solved simultaneously, prior to solving the linear sets of equations. A computer program based on the proposed methodology is developed.

System Idealization

Consider an r-degree of freedom rotor bearing system with nonlinear forces associated with q of these degrees of freedom running in one or more squeeze film damped flexible support. The multi-mass flexible rotor in Figure 1 is an example of such a system, wherein highly nonlinear damper forces exist at each damper location.

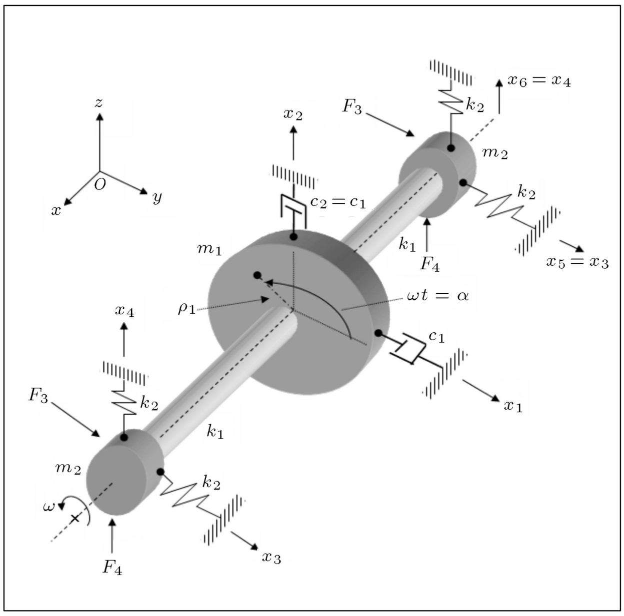

Figure 1. Flexible, unbalanced symmetric rotor supported on identical squeeze film dampers and retainer springs.

One can write the equations of motion as:

$$M\ddot{X} + C\dot{X} + KX = F, \tag{1}$$

where the first p equations do not involve non-linear motion dependant forces. The nonlinearity at the damper locations is reflected in the corresponding force (F) terms, i.e.:

$$p = r - q. \tag{2}$$

Equilibrium Solutions

If steady-state conditions have been reached with the system being subjected to a periodic excitation force of frequency ω, such as unbalance excitation, one can assume that the equilibrium or steady state solutions are of the form:

$$X_E = A_0 + \sum_{k=1}^{n}(A_k \cos\lambda_k t + B_k \sin\lambda_k t), \tag{3}$$

where:

$$\lambda_k = k\omega/N = k\Omega. \tag{4}$$

It is assumed that there are n harmonics of the fundamental frequency (Ω). Required are the $(2n+1)r$ coefficients, $A_0, A_1, \cdots, A_n$, $B_1, \cdots, B_n$, which are to be found by harmonic balance. Thus:

$$\dot{X}_E = -\sum_{k=1}^{n}\lambda_k(A_k \sin\lambda_k t - B_k \cos\lambda_k t), \tag{5}$$

and:

$$\ddot{X}_E = -\sum_{k=1}^{n}\lambda_k^2(A_k \cos\lambda_k t + B_k \sin\lambda_k t), \tag{6}$$

and Equation 1 becomes:

$$M\ddot{X}_E + C\dot{X}_E + KX_E = F_E, \tag{7}$$

where:

$$F_E = C_0 + \sum_{k=1}^{n}(C_k \cos\lambda_k t + S_k \sin\lambda_k t). \tag{8}$$

The Fourier coefficients of F_E, viz. C_0, C_k and S_k are functions of $X_E, \dot{X}_E$ and the external excitation.

F_E in Equation 8 generally represents the force component associated with the system degrees of freedom. This research has considered two sources for these external forces; the first one being the hydrodynamic bearing forces at damper degrees of freedom and the second one being unbalance excitation that can be applied to any of the rotor degrees of freedom.

On substituting Equations 3, 5, 6 and 8 into Equation 7, one can solve the set of equations represented by Equation 7 by equating the similar terms on the left and right sides of this equation. Therefore, equating coefficients for the constant terms, equating coefficients for the cosine terms and equating coefficients for the sine terms are the three steps required in that equation.

Starting with equating coefficients for the constant terms, one obtains:

$$KA_0 = C_0. \tag{9}$$

K represents the system stiffness matrix.

As the solution procedure includes solving for the nonlinear degrees of freedom before solving for the linear freedoms, one needs to partition the system equations of motion into linear and nonlinear parts.

In the case of Equation 9, the partitioned form is:

$$\begin{bmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{bmatrix} \begin{bmatrix} A_0^p \\ A_0^q \end{bmatrix} = \begin{bmatrix} C_0^p \\ C_0^q \end{bmatrix}, \tag{10}$$

where K_{pq} is a matrix of order $p \times q$ and C_0^p is a vector of order p.

At this stage, it is necessary to eliminate all coefficients corresponding to linear freedoms, A_0^p, and to keep all coefficients corresponding to nonlinear degrees of freedom, A_0^q.

By eliminating the A_0^p from Equation 10, one obtains:

$$[K_{qq} - K_{qp}K_{pp}^{-1}K_{pq}]A_0^q + K_{qp}K_{pp}^{-1}C_0^p = C_0^q. \tag{11}$$

Equation 11 is a set of q nonlinear simultaneous equations in the $(2n+1)q$ unknowns, namely

$A_0^q, A_1^q, \cdots, A_n^q, B_1^q, \cdots, B_n^q$, which determine the C_0^q. Note that in the absence of linear spring forces $K = 0$, and the left hand side of Equation 11 is then zero.

It should be mentioned that, while partitioning system motion equations into linear and nonlinear parts, there need to be attempts to calculate the corresponding Fourier coefficients A_o, A_k and B_k accordingly. Since the total degrees of freedom for the general case of a rotor bearing system is defined as r, the total number of Fourier coefficients to be calculated is equal to $(2n+1)r$. In order to find the $(2n+1)q$ unknown coefficients, $A_0^q, A_1^q, \cdots, A_n^q$, $B_1^q, \cdots, B_n^q$, corresponding to all nonlinear degrees of freedom, one needs to configure $(2n+1)q$ simultaneous nonlinear equations. Likewise, the procedure to calculate the $(2n+1)p$ unknown coefficients corresponding to linear degrees of freedom, $A_0^p, A_1^p, \cdots, A_n^p$, $B_1^q, \cdots, B_n^p$, needs the configuring of $(2n+1)p$ simultaneous linear equations.

Again, on equating coefficients for the cosine terms in Equation 7 for each kth harmonic, one obtains:

$$[K - \lambda_k^2 M]A_k + \lambda_k C B_k = C_k, \tag{12}$$

or:

$$QA_k + RB_k = C_k. \tag{13}$$

Similarly, on equating coefficients for the sine terms for each kth harmonic, one obtains:

$$QB_k - RA_k = S_k. \tag{14}$$

Elimination of A_k from Equations 13 and 14 by pre-multiplying Equation 13 by Q^{-1} and substituting into Equation 14 gives:

$$[Q + RQ^{-1}R]B_k = S_k + RQ^{-1}C_k, \tag{15}$$

or:

$$TB_k = W. \tag{16}$$

In partitioned form:

$$\begin{bmatrix} T_{pp} & T_{pq} \\ T_{qp} & T_{qq} \end{bmatrix} \begin{Bmatrix} B_k^p \\ B_k^q \end{Bmatrix} = \begin{pmatrix} W^p \\ W^q \end{pmatrix}. \tag{17}$$

Note that W^p and W^q are functions of X_E and $\dot{X}_E$. Elimination of B_k^p from Equation 17 gives:

$$[T_{qq} - T_{qp}T_{pp}^{-1}T_{pq}]B_k^q + T_{qp}T_{pp}^{-1}W^p = W^q. \tag{18}$$

Equation 18 constitutes a further set of nq nonlinear simultaneous equations in the $(2n+1)q$ unknowns.

Again, elimination of B_k from Equations 13 and 14 by pre-multiplying Equation 14 by Q^{-1} and substituting into Equations 13 gives:

$$TA_k = C_k - RQ^{-1}S_k = V. \tag{19}$$

By partitioning, as previously done to obtain Equation 17 from Equation 16, one can solve for the A_k^q to obtain:

$$[T_{qq} - T_{qp}T_{pp}^{-1}T_{pq}]A_k^q + T_{qp}T_{pp}^{-1}V^p = V^q, \tag{20}$$

where again V^p and V^q are functions of X_E and $\dot{X}_E$. Equation 20 constitutes yet another set of nq nonlinear simultaneous equations in the $(2n+1)q$ unknowns. Hence, together with Equations 11 and 18, one has a set of $(2n+1)q$ nonlinear simultaneous equations in the $(2n+1)q$ unknowns, $A_0^q, A_1^q, \cdots, A_n^q$, $B_1^q, \cdots, B_n^q$.

These equations need to be solved by some iterative procedure, such as Newton-Raphson, which is the procedure adopted in this report. Convergence is assumed when changes in the successive values of the unknowns are less than 0.0001C. Significant values for amplitudes of the highest assumed harmonics indicate the need for including additional harmonics, and such further addition of harmonics continues until there is no significant change in the lower harmonic values. Once found, the remaining $(2n+1)p$ unknowns, $A_0^p, A_1^p, \cdots, A_n^p$, $B_1^q, \cdots, B_n^p$, can be found from the present simultaneous linear sets of equations obtained by eliminating A_0^q from Equation 10, B_k^q from Equation 17 and A_k^q from Equation 19.

Note that no matter how many degrees of freedom there are in the system, the number of nonlinear simultaneous equations to be solved is still only $(2n+1)q$. In general, each damper introduces nonlinear forces into four equations of motion, reducing to two equations when the damper connects to the ground. Thus, $q = 4$ or 2 for a system with one damper only, whereas there is no limit to the total degrees of freedom, r. Also, for physical systems with real values for system mass, stiffness and damping matrices, Q^{-1} and T_{pp}^{-1} always exist. Therefore, the only computational problem is generally associated with numerical iterative schemes, viz. convergence to all possible solutions. The other potential disadvantage of this approach is the initial choice of the fundamental frequency. Subharmonic solutions (solutions with frequency components lower than the lowest excitation frequency) are occasionally possible [14]. These are catered for, by assuming a fundamental frequency of $\Omega = \omega/N$ where N is assumed to be an integer. However, there is no sure way of knowing whether all possible values on N have been exhausted, as multi-equilibrium solutions of the same or of different fundamental frequencies to the excitation frequency are occasionally possible.

Stability of Equilibrium Solutions

The stability in the linear sense of the above equilibrium solutions still has to be addressed. Consider a small perturbation, ΔX, to the equilibrium solution, X_E, so that Equation 1 is relevant.

Then, subtraction of Equation 7 from Equation 1 gives:

$$M(\ddot{X}-\ddot{X}_E)+C(\dot{X}-\dot{X}_E)+K(\mathrm{X}\text{-}\mathrm{X}_E)=F-F_E, \tag{21}$$

or:

$$M\Delta\ddot{X}+C\Delta\dot{X}+K\Delta\mathrm{X}=\Delta\mathrm{F}=\begin{Bmatrix}0^{\mathrm{p}}\\ \Delta\mathrm{F}^{\mathrm{q}}\end{Bmatrix}. \tag{22}$$

Since:

$$F^q=F(X^q,\dot{X}^q), \tag{23}$$

$$\Delta F^q=\sum_{j=1}^{q}\left(\frac{\partial F_i^q}{\partial \mathrm{X}_\mathrm{j}^\mathrm{q}}\Delta\mathrm{X}_\mathrm{j}^\mathrm{q}+\frac{\partial F_i^q}{\partial \dot{\mathrm{X}}_\mathrm{j}^\mathrm{q}}\Delta\dot{X}_j^q\right)$$

$$+\text{ Higher order terms,}$$

$$\text{for } i=1,2,\cdots,q. \tag{24}$$

All partial derivatives are evaluated at the equilibrium state at any given time.

$$\therefore \Delta F^q=F^x\Delta\mathrm{X}^q+F^{\dot{x}}\Delta\dot{X}^q. \tag{25}$$

Substitution of Equation 25 into Equation 22 and multiplying through by M^{-1} gives:

$$\Delta\ddot{X}+M^{-1}C^*\Delta\dot{X}+M^{-1}K^*\Delta\mathrm{X}{=}0, \tag{26}$$

where:

$$C^*=\begin{bmatrix}C_{pp} & C_{pq}\\ C_{qp} & C_{qq}-F^{\dot{x}}\end{bmatrix}, \tag{27}$$

and:

$$K^*=\begin{bmatrix}K_{pp} & K_{pq}\\ K_{qp} & K_{qq}-F^{x}\end{bmatrix}. \tag{28}$$

Equation 26 represents a set of r linear second-order differential equations with periodic coefficients of period $2\pi/\Omega$ (because the elements of F^x and $F^{\dot{x}}$ are periodic, with period $2\pi/\Omega$). Linear stability requires that ΔX approach zero with time. The Floquet theory [13] may be conveniently used to test for stability. Thus, Equation 26 may be written as the $2r$ first-order linear equations:

$$\begin{bmatrix}\Delta\dot{X}\\ \Delta\dot{Y}\end{bmatrix}-\begin{bmatrix}0 & \mathrm{I}\\ M^{-1}K^* & M^{-1}C^*\end{bmatrix}\begin{bmatrix}\Delta\mathrm{X}\\ \Delta\mathrm{Y}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}, \tag{29}$$

where:

$$Y=\dot{X}. \tag{30}$$

Let G be the $2r\times 2r$ matrix whose columns contain the $2r$ solutions at time $t=2\pi/\Omega$ of the above equations, having as initial conditions the corresponding columns of a $2r\times 2r$ identity matrix. Then, the system is stable if all the eigenvalues of G have magnitudes less than unity. The $2r$ solutions of Equation 29 may be carried out by a variety of techniques; a 4th order Runge-Kutta with variable step size is used in this report. Though various alternative schemes have been suggested to reduce the computational effort, it is doubtful whether the alleged time savings warrant the increased complexity involved, particularly since stability predictions can be very sensitive to numerical inaccuracies. Computation of the eigenvalues of G to a sufficient degree of accuracy for large systems, may itself be problematic.

APPLICATION TO SQUEEZE FILM DAMPERS

The above theory is developed quite generally and may be applied to any system with nonlinear motion dependent forces. However, the illustrative examples in this report involve nonlinear forces arising from end feed squeeze film dampers. Therefore, it is necessary to introduce a brief discussion about such dampers and the corresponding nonlinear hydrodynamic forces.

Damper Forces

A section view of such a damper is shown in Figure 2. It is assumed that the fluid is Newtonian with constant properties at some mean temperature, the flow is

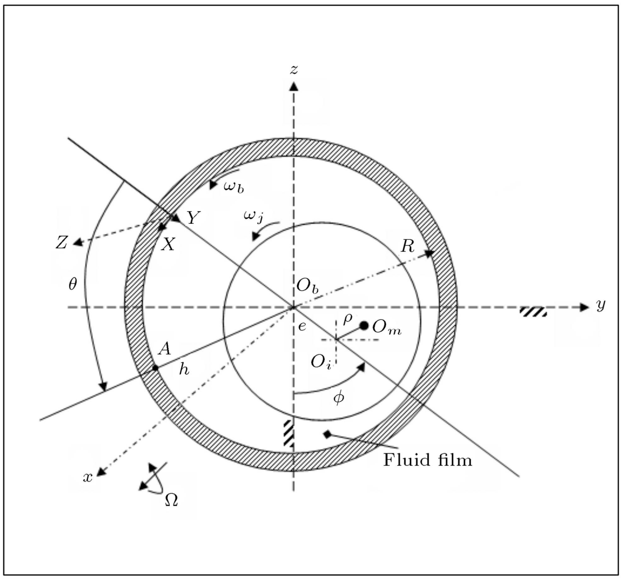

Figure 2. Schematic view of a squeeze film damper.

laminar, the fluid inertia forces are negligible, there is no slip at the bearing surfaces, h/L is of order 10^{-3}, the short bearing approximation is applicable (valid provided $L/D < 0.25$ [15]), and there is no variation in film thickness in the axial direction. The momentum and continuity equations for the damper fluid then result in the simplified Reynolds equation [16], viz:

$$\left(h^3 \frac{\partial^2 p}{\partial Z^2}\right) = 12\mu \frac{dh}{dt}, \tag{31}$$

where:

$$h = C - z\sin\psi - y\cos\psi. \tag{32}$$

The solution of Equation 31 with pressure boundary conditions pertaining to the end feed/drain oil supply, i.e.:

$$p = p_i \qquad \text{at} \qquad Z = -L/2,$$

$$p = p_0 \qquad \text{at} \qquad Z = +L/2, \tag{33}$$

yields:

$$p = p_i + \frac{p_0 - p_i}{L}\left(Z + \frac{L}{2}\right) + \frac{6\mu}{h^3}\left(Z^2 + \frac{L^2}{4}\right)\frac{dh}{dt}. \tag{34}$$

Assuming that $p_0 = p_i = 0$ gives:

$$p = \frac{6\mu}{h^3}\left(\frac{L^2}{4} - Z^2\right)(\dot{z}\sin\psi + \dot{y}\cos\varphi). \tag{35}$$

In general, this pressure distribution will not be continuous, owing to the emergence of dissolved gas bubbles at sub-supply pressures, fluid vaporization at the cavitation pressure and the possibility of sucking in air at the ends of the damper. Assuming further that once the pressure is less than the cavitation pressure, it will equal the cavitation pressure, the pressure is positive in the regions where:

$$\varphi^* \le \varphi \le \psi^* + \pi, \tag{36}$$

where:

$$\tan\psi^* = -\dot{y}/\dot{z}, \tag{37}$$

and:

$$y\sin\psi + z\cos\psi > 0. \tag{38}$$

This simplified cavitation condition does not satisfy continuity at the cavitation boundary and the more realistic Reynolds cavitation condition could have been used at the cost of introducing an additional iteration into the computation. Since the purpose of this report is to illustrate the utility of the harmonic balance approach to systems with a nonlinear component, such as a damper, the extra complication of more realistic cavitation conditions with only a minimal change to the damper forces was not felt to be warranted.

Thus, at any instant of time, the pressure is positive over half the circumferential extent, with the precise location of this π film-dependent on the instantaneous motion of the journal centre. The fluid film force components are then given by:

$$\begin{Bmatrix} F_y \\ F_z \end{Bmatrix} = -\mu R L^3 \int_{\psi^*}^{\psi^*+\pi} \frac{(y\cos\psi + z\sin\psi)}{(C - z\sin\psi - y\cos\psi)^3} \begin{Bmatrix} \cos\psi \\ \sin\psi \end{Bmatrix} d\psi. \tag{39}$$

Note that with more complicated damper geometries, e.g. oil feed holes, greater aspect ratios ($L/D > 0.25$) and cavitation pressure, $p_c \neq p_0$ or p_i, the damper force expressions will not be expressible as simply as in Equation 39, but will still be functions of $y, z, \dot{y}$ and $\dot{z}$, i.e. of X_E and $\dot{X}_E$ so that this approach is still applicable.

COMPUTER PROGRAM

Based on the basic theory presented in this report, a computer program for the harmonic balance analysis (HBA) of general squeeze film damped multi-degree of freedom rotor bearing systems is developed.

This computer program consists of a main program that opens input/output files, initializes data, sets system equations of motion according to input data, initializes solver subroutines and prints out the results. The main program is also accompanied by many subprograms to perform operations such as matrix algebra, matrix partitioning and nonlinear equation solvers, etc. A subprogram builds the idealized system equations of motion based on input data. System mass, damping and stiffness matrices are input to this program. It then reads in some constant terms in forcing functions and the amount of rotor unbalance. This program then calls in a second subprogram to reduce degrees of freedom to those corresponding to the number of non-linear equations.

Another subprogram reads in squeeze film damper parameters, the amount of preload on damper freedoms and the initial guessed values. Damper parameters include oil viscosity, clearance, radius and the length and number of dampers involved. It, then, calculates the bearing parameter. A general routine sets up the system of nonlinear equations. From displacements specified by the initial guessed values, a subroutine is called to evaluate the hydrodynamic bearing forces in the vertical and horizontal directions and returns them in the form of their Fourier components.

Program Input/Output

Elements of mass, damping and stiffness matrices are initialized with null in the main program. As a result, the user needs only to fill in the nonzero elements of the corresponding matrices. Squeeze film damper parameters, including oil viscosity, radial clearance, radius, length and the number of squeeze film dampers, should also be given as input data. The preload assumed to be effective on damper vertical freedoms, is also affected by gravity. Initial assumptions are essential for harmonic balance solutions. This program can find its initial guessed values from two sources: from data available in the input file, or from rewinding the temporary output file and reading the last stage data as initial data for the next stage.

Output from this program includes information about the system total degrees of freedom, the number of linear and nonlinear freedoms, the level of sub-harmonics included, the number of Fourier terms used and the rotation frequency for the rotor. Mass, damping and stiffness matrices plus damper specifications together with the amount of preloads on damper freedoms are also included. Displacement of damper inner and outer freedoms, corresponding absolute eccentricities, eccentricity of the inner ring relative to the outer ring and stability analysis data in the form of eigenvalues, are also part of the output data.

ILLUSTRATIVE EXAMPLES

Verification of the harmonic balance approach in predicting the dynamic response of the multi-degree of freedom rotor bearing systems with some non-linearity is examined through the following two examples.

Flexible Rotor-Centralized Damper

Figure 1 presents a flexible symmetric unbalanced rotor, the so-called Jeffcott rotor, supported on identical squeeze film dampers and centralizing springs of constant radial stiffness. The lumped mass at the bearing ends is m_2, the centralizing spring has stiffness k_2 and the rotor stiffness between the central and either end node is k_1. All unbalance is assumed to be at the disk, resulting in a disk mass eccentricity, ρ_1. Viscous damping at the disk is c_1. Damping at the disk is negligible compared with that provided by the damper; hence, may be neglected. Since the rotor is symmetric about the disk, it suffices to consider one half of the system only. Thus, for cylindrical whirl, the motion of the system will be described by the plane motion in the damper of a journal of mass, $m = m_1/2 + m_2$, with unbalance eccentricity, $\rho = \rho_1 m_1/(m_1 + 2m_2)$. Working frequency extends beyond the pin pin critical speed of the rotor, ω_c, centralizing springs are retained and the rotor is centrally preloaded. Such a system results in synchronous circular orbit type solutions, and has been analysed previously in the literature for both equilibrium solutions and their stability in the linear sense [4]. By virtue of the synchronous nature of the orbits, such stability analyses were possible without the need to resort to the Floquet theory, by writing the perturbed equations of motion with respect to a rotating reference frame; thereby, obtaining linear differential equations with constant coefficients. Once super and/or sub-harmonics of the excitation frequency are also present, as they will be in general, such a procedure no longer removes the periodicity of the coefficients in the perturbed equations of motion. Solutions for this flexible rotor model, therefore, proved particularly useful in evaluating the Floquet theory based stability analysis, since there are circular orbit solutions which are alleged to be unstable and, indeed, unexpectedly so [4]. Referring to Figure 1, the equations of motion are given by:

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1(x_1 - x_3) = \rho_1 m_1 \omega^2 \cos\alpha,$$

$$m_1\ddot{x}_2 + c_1\dot{x}_2 + k_1(x_2 - x_4) = \rho_1 m_1 \omega^2 \sin\alpha,$$

$$m_2\ddot{x}_3 + k_1(x_3 - x_1) + k_2 x_3 = F_y,$$

$$m_2\ddot{x}_4 + k_1(x_4 - x_2) + k_2 X_4 = F_z,$$

where:

$$y = x_3, \qquad z = x_4.$$

The equations are in the form of Equation 1 with $r = 4$ and $q = 2$. The following values of non-dimensional system parameters were used to allow comparison with [4]:

$$M_1 = 0.75,$$

$$M_2 = 0.25,$$

$$K_1 = 0.75/(\omega/\omega_c)^2,$$

$$K_2 = 0.25/(\omega/\omega_c)^2,$$

$$C_1 = 0.0075/(\omega/\omega_c),$$

$$U = 0.3,$$

$$\omega_r/\omega_c = 0.5.$$

The equilibrium solutions for various values of the bearing parameter, ω_b/ω_c, are reported in [4]. Using the generalized theory that is presented in this article, the same frequency response curve was obtained for $\omega_b/\omega_c = 0.3$ as indicated in Figure 3.

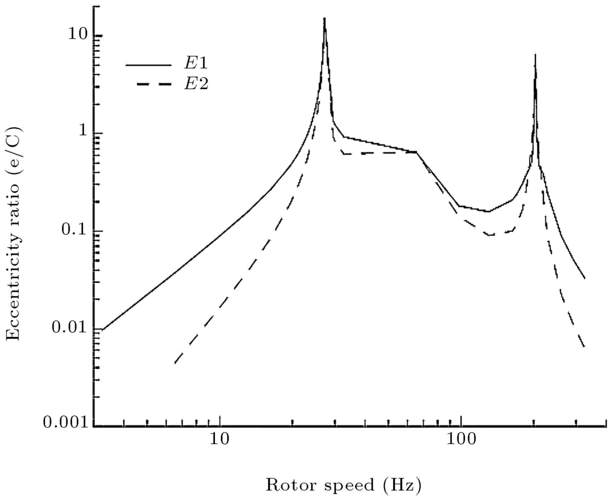

Figure 3. Predicted frequency response of circular orbit eccentricities for the Jeffcot rotor, a comparison between results of this work (marked as E1-solid line) and the equilibrium solutions reported in [4] (marked as E2-dashed lines); $U = 0.3$, $M_1 = 0.75$, $M_2 = 0.25$, $\omega_r/\omega_c = 0.5$, $\omega_b/\omega_c = 0.3$.

Flexible Rotor-Modified Damper-Under Gyroscopic Effect

Figure 4, reproduced from [6], is a schematic representation of a rotor-bearing rig, specifically designed to evaluate the ability of squeeze film dampers to attenuate critical speeds due to gyroscopic effects. This experimental test rig was also used to evaluate a transient method of solution for predicting the frequency response of general multi-mass rotor bearing systems supported by squeeze film dampers.

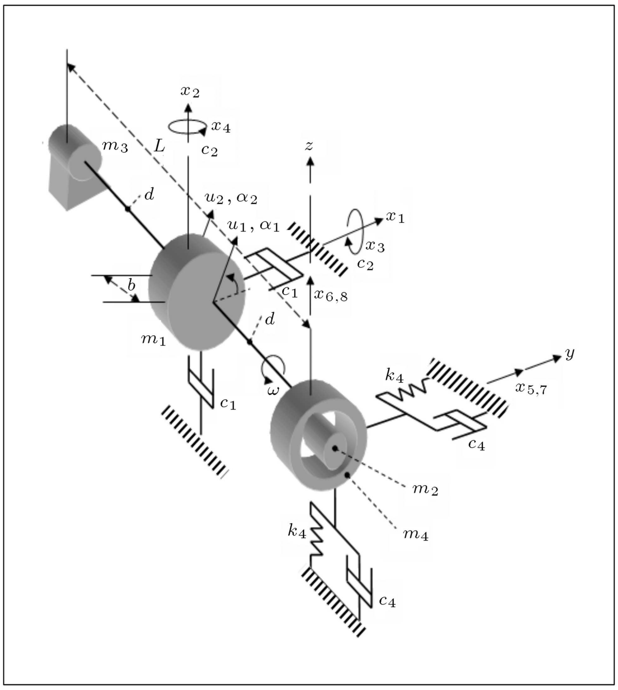

Figure 4. Dynamic model of test rig, reproduced from [6].

The left hand end of the rotor with lumped mass m_3 is supported by self-aligning ball bearings, the disc of mass m_1 is centrally located and the right-hand end of the rotor with lumped masses m_2 and m_4 is flexibly supported by a circumferentially grooved modified squeeze film damper together with centralizing springs of constant radial stiffness. All unbalance is assumed to be at the disk resulting in disk mass eccentricities, ρ_1 and ρ_2. Viscous damping corresponding to the disk, c_1 and c_2, are assumed to be equal in the y and z directions.

Working frequency extends beyond the pin-pin critical speed of the rotor, ω_c, centralizing springs are retained and the rotor is centrally preloaded. Such a system results in synchronous circular orbit type solutions. The equations of motion for this system, in the form of Equation 1, are given by:

$$
\begin{aligned}
&m_1\ddot{x}_1 + c_1\dot{x}_1 + k_{11}x_1 + k_{15}x_5 = m_1\rho_1\omega^2\cos\alpha_1 \\
&\quad + m_1\rho_2\omega^2\cos\alpha_2, \\
&m_1\ddot{x}_2 + c_1\dot{x}_2 + k_{22}x_2 + k_{26}x_6 = -m_1 g \\
&\quad + m_1\rho_1\omega^2\sin\alpha_1 + m_1\rho_2\omega^2\sin\alpha_2, \\
&I_d\ddot{x}_3 + c_2\dot{x}_3 + I_p\omega\dot{x}_4 + k_{33}x_3 + k_{36}x_6 = \\
&\quad - bm_1\rho_1\omega^2\sin\alpha_1 + bm_1\rho_2\omega^2\sin\alpha_2, \\
&I_d\ddot{x}_4 + c_2\dot{x}_4 - I_p\omega\dot{x}_3 + k_{44}x_4 + k_{45}x_5 \\
&\quad = bm_1\rho_1\omega^2\cos\alpha_1 - bm_1\rho_2\omega^2\cos\alpha_2, \\
&m_2\ddot{x}_5 + c_3\dot{x}_5 + k_{51}x_1 + k_{54}x_4 = 2F_y, \\
&m_2\ddot{x}_6 + c_3\dot{x}_6 + k_{62}x_2 + k_{63}x_3 = 2F_z - m_2 g + pl_1, \\
&m_4\ddot{x}_7 + c_4\dot{x}_7 + k_4x_7 = -2F_y, \\
&m_4\ddot{x}_8 + c_4\dot{x}_8 + k_4x_8 = -2F_z - m_4 g + pl_2,
\end{aligned}
\tag{40}
$$

where k_{ij} is the spring force in the ith degree of freedom direction at the node associated with that degree of freedom upon unit displacement in the jth degree of freedom direction of the node associated with the jth degree of freedom; all other displacements being considered zero. The equations are in the form of Equation 1 with $r = 8$ and $q = 4$. From the rig measurements [6] the following data apply:

$$m_1 = 9.059 \text{ kg},$$

$m_2 = 2.106$ kg,

$m_4 = 0.500$ kg,

$I_d = 27.43 \times 10^{-3}$ kgm^2,

$I_p = 15.43 \times 10^{-3}$ kgm^2,

$c_1 = 4.46$ Ns/m,

$c_2 = 0.0854$ Ns/m,

$c_3 = 0.973$ Ns/m,

$c_4 = 0.973$ Ns/m,

$k_{11} = k_{22} = 380700$ N/m,

$k_{33} = k_{44} = -19273$ Nm/rad,

$k_{55} = k_{66} = 277050$ N/m,

$k_{51} = k_{62} = -190350$ N/m,

$k_{54} = -k_{63} = -42829$ N/rad,

$k_4 = 277050 \times 20$ N/m,

$R = 0.0327$ m,

$C = 0.2 \times 10^{-3}$ m,

$L = 0.01$ m,

$\omega = 0$ to 320 Hz,

$m_1\rho_1 = 0.498825 \times 10^{-3}$ kgm,

$m_1\rho_2 = 0.337875 \times 10^{-3}$ kgm,

$\alpha_1 = \alpha_2$,

$\mu = 15.717 \times 10^{-3}$ Ns/m^2 at 25°C (Tellus 15).

Note that m_4, c_4 and k_4 correspond to a modified squeeze film damper when the damper outer ring is also flexibly supported.

Using the above set of data, the frequency response for damper inner and outer rings calculated by using the harmonic balance method is summarized in Figures 5a and 5b.

The accuracy of the results presented in Figure 5 can be examined by comparing them with the results presented in [6]. Figure 6 from [6] presents the frequency response of the same test rig. This figure presents the frequency response predicted by a

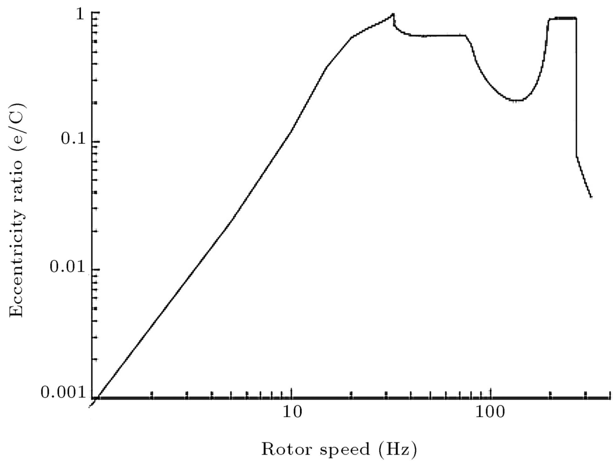

Figure 5a. Predicted frequency response of circular orbit eccentricities for the centrally preloaded experimental rotor, orbit of mass m_2 relative to mass m_4 (ε).

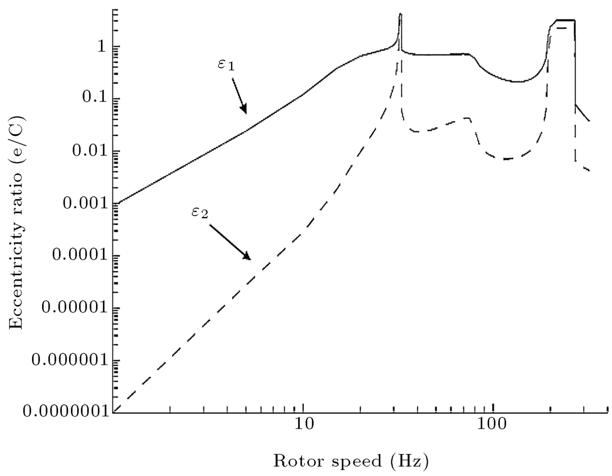

Figure 5b. Predicted frequency response of circular orbit eccentricities for the centrally preloaded experimental rotor, absolute eccentricities of masses m_2 and m_4 (ε_1 and ε_2, respectively).

transient method of solution in comparison with the experimentally measured frequency response. The points marked with squares present theoretical predictions while the points with an asterisk present measured frequency responses. The first critical speed of the test rig is equal to 32.6 Hz and the second critical speed is equal to 195.6 Hz [6]. The data points in Figure 6 are normalised for the 1st critical speed of the system. Bearing in mind the normalization factor of 32.6 Hz and comparing Figures 5a and 6, it becomes clear that the eccentricity values predicted by the harmonic balance method are comparable to those presented in Figure 6.

As the same data were used to obtain the results presented in Figures 5a and 6, it can clearly be noted that the damper eccentricity ratio and the critical speeds predicted by the harmonic balance approach are tightly related to the earlier predictions reported

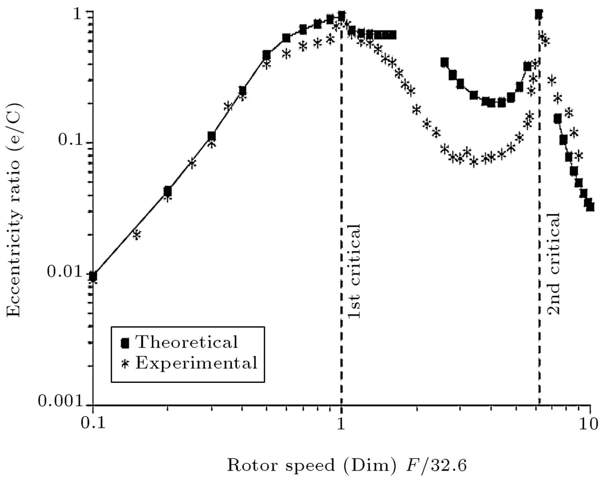

Figure 6. Frequency response of the test rig with centrally supported damper [6]. A comparison between predictions by a transient method of solution and experimental measurements.

in [6]. As the rotor speed increases, the trends of the damper movements predicted and measured by the three different methods are comparable; a credit to the predictions of the harmonic balance approach and to the accuracy of the strategy selected for the development of this method.

CONCLUSION

The harmonic balance approach for finding equilibrium solutions for general multi-degree of freedom rotor bearing systems with squeeze film dampers can result in a considerable reduction in the number of simultaneous non-linear equations required to be solved iteratively.

Weaknesses of the harmonic balance approach include dependence on an effective convergent iterative scheme for solving greatly reduced, yet high number of simultaneous non-linear equations, and the difficulty of ensuring that all possible equilibrium solutions have been exhausted. Also, the method implicitly assumes knowledge of the response fundamental frequency.

The converged equilibrium solution orbits are not necessarily stable in the linear sense. Perturbation of the equilibrium solutions results in as many second order linear differential equations with periodic coefficients as there are degrees of freedom. The Floquet theory may be conveniently applied to determine stability, particularly if the damper or bearing force components are expressed directly in terms of fixed reference axes directions.

Accuracy of the harmonic balance orbit predictions was successfully tested for uncentralized rigid rotors operating at high eccentricities for centralized dampers supporting the Jeffcott rotor and for a centralized and uncentralized flexible rotor with gyroscopic and bending criticals in the operating range.

NOMENCLATURE

A_k	$r \times 1$ vector of Fourier coefficients defined by Equation 3; $k = 0, \cdots, n$
B_k	$r \times 1$ vector of Fourier coefficients defined by Equation 3; $k = 0, \cdots, n$
B	angular velocity for non-dimensionalization $= \mu RL^3/[(m_1 + 2m_2)\omega C^3]$ in Figures 1 and 4
$c_1, \cdots, c_n$	the damping coefficients associated with degrees of freedom $x_1, \cdots, x_n$ in Figures 1 and 4
C	radial clearance of damper
C_1	$c_1/[(m_1/2 + m_2)\omega]$ non-dimensional damping in Figure 3
$\mathbf{C}$	$r \times r$ damping and gyroscopic matrix
C^*	$r \times r$ matrix defined by Equation 27
C_k	$r \times 1$ vector of Fourier coefficients defined by Equation 8; $k = 0, \cdots, n$
d	diameter of rotor in Figure 4
D	bearing or journal diameter
e, ε	journal eccentricity; $\varepsilon = e/C$
E	equilibrium value
F	$r \times 1$ vector of forces as defined in Equation 1
$F^x, F^{\dot{x}}$	$q \times q$ matrices of partial derivatives defined by Equations 24 and 25
F_y, F_z	fluid film force components in the y and z directions
G	$2r \times 2r$ fundamental matrix of the set of $2r$ linear differential equations of the perturbed equilibrium solution
h	fluid film thickness
I	$r \times r$ identity matrix
k	order of Fourier series component; $k = 0, 1, \cdots, n$
k_1, k_2	stiffness of retainer springs and rotor segments in Figure 1
K	$r \times r$ stiffness matrix
K_1	$2k_1/[(m_1 + 2m_2)\omega^2]$, non-dimensional stiffness in Figure 3
K_2	$2k_2/[(m_1 + 2m_2)\omega^2]$, non-dimensional stiffness in Figure 3
K^*	$r \times r$ matrix defined by Equation 28
L	length of axial land of damper
m	$m_1/2 + m_2$

m_1, m_2, m_3, m_4	the lumped masses in Figures 1 and 4
M	$r \times r$ mass matrix
M_1	$m_1/[m_1 + 2m_2]$, non-dimensional mass in Figure 3
M_2	$2m_2/[m_1 + 2m_2]$, non-dimensional mass in Figure 3
n	highest harmonic of truncated Fourier series
N	integer, usually 1 or 2
$\mathbf{O}$	$r \times r$ null matrix
O	$r \times 1$ null vector
p	degrees of freedom without nonlinear forces; also gauge pressure relative to cavitation pressure
p_i, p_o	inlet and outlet pressure
P	preload for centralizing the damper
q	degrees of freedom involving nonlinear forces
Q, R	$r \times r$ matrices defined by Equations 12 and 13
r	degrees of freedom of rotor bearing system
R	bearing radius
S_k	$r \times 1$ vector of Fourier coefficients defined by Equation 8; $k = 1, \cdots, n$
t	time
T	$r \times r$ matrix defined by Equations 15 and 16
U	unbalance parameter, $\rho_1 m_1/[(m_1 + 2m_2)C] = \rho/C$
V, W	$r \times 1$ vectors defined by Equations 19, 20, 15 and 16, respectively
W	Static load parameter $= g/(C\omega_b^2)$
x, y, z	coordinate system with x in direction of shaft rotation and origin located along the line joining bearing centres
X	$r \times 1$ vector of the degrees of freedom
Y	$r \times 1$ vector defined by Equation 30
Z	axial coordinate measured from bearing centre O_b in x direction; $\overline{Z} = Z/L$
$\alpha, \alpha_1, \alpha_2$	location of unbalance eccentricities at time (t) (Figures 1 and 4)
γ	speed or frequency ratio; $\gamma = \omega/\omega_b$
λ	eigenvalue of G
λ_k	defined by Equation 4
μ	absolute viscosity of lubricant
ρ	$\rho_1 m_1/(m_1 + 2m_2)$
ρ_1, ρ_2	unbalance eccentricities at lumped rotor mass in Figures 1 and 4
φ	angular location of point A from y axis in Figure 2
Ω	fundamental frequency of steady state response
ω	angular velocity of rotor
ω_b	a bearing parameter $= \mu R L^3/[(m_1 + 2m_2)C^3]$ in Figure 3
ω_c	a characteristic system frequency $= \sqrt{(2k_1/m_1)}$ in Figure 3
ω_r	a characteristic system frequency $= \sqrt{(k_2/[(m_1/2 + m_2)]}$ in Figure 3

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