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Research Note

Dynamic Green Function Solution of Beams Under a Moving Load with Different Boundary Conditions

B. Mehri[1,*], A. Davar[2] and O. Rahmani[2]

Abstract. *This paper presents the linear dynamic response of uniform beams with different boundary conditions excited by a moving load, based on the Euler-Bernouli beam theory. Using a dynamic green function, effects of different boundary conditions, velocity of load and other parameters are assessed and some of the numerical results are compared with those given in the references.*

Keywords: *Beams; Green function; Euler-Bernoulli; Buckling problem.*

INTRODUCTION

In recent years, all branches of transport have experienced great advances, characterized by the increasingly high speed and weight of vehicles and other moving bodies. As a result, corresponding structures have been subjected to vibration and dynamic stress far longer than ever before. The moving load problem has been the subject of numerous research efforts in the last century. The importance of this problem is manifested in numerous applications in the field of transportation. Bridges, guideways, overhead cranes, cableways, rails, roadways, runways, tunnels, launchers and pipelines are examples of structural elements designed to support moving loads.

The literature concerning the forced vibration analysis of structures with moving bodies is sparse. The most used method for determining these vibrations is the expansion of applied loads and dynamic responses in terms of the eigenfunctions of the undamped beams. This method leads to solutions presented as infinite series, which will be truncated after a number of terms and approximate solutions are then obtained. Fryba [1] used the Fourier sine (finite) integral transformation and the Laplace-Carson integral transformation to determine the dynamic response of beams due to moving loads and obtained a response in the form of series solutions. Ting et al. [2] formulated and solved the problem using influence coefficients (static Green function). The distributed inertial effects of the beam were considered as applied external forces. Correspondingly, at each position of the mass, numerical integration had to be performed over the length of the beam.

Hamada [3] solved the response problem of a simply supported and damped Euler-Bernoulli uniform beam of finite length traversed by a constant force moving at a uniform speed, by applying the double Laplace transformation with respect to both time and the length coordinate along the beam. He obtained, in closed form, an exact solution for the dynamic deflection of the considered beam. Yoshimura et al. [4] presented the analysis of dynamic deflections of a beam, including the effects of geometric nonlinearity, subjected to moving vehicle loads. With the loads moving on the beam from one end to the other, the dynamic deflections of the beam and loads were computed using the Galerkin method. Lee [5] presented a numerical solution, based on integration programs, using the Runge-Kutta method for integrating the response of a clamped-clamped beam acted upon by a moving mass.

Esmailzadeh et al. [6] have studied the forced vibration of a Timoshenko beam with a moving mass. Gbadeyan and Oni [7] presented a technique based on modified generalized finite integral transforms and the modified Struble method, to analyze the dynamic re-

1. *Department of Mathematics, Sharif University of Technology, Tehran, P.O. Box 11155-3406, Iran.*
2. *Department of Mechanical Engineering, Khaje-Nasir Toosi University, Tehran, Iran.*
*. *Corresponding author. E-mail: mehri@math.sharif.ac.ir*

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sponse of finite elastic structures (Rayleigh beams and plates), having arbitrary end supports and under an arbitrary number of moving masses. Foda and Abduljabbar [8] used a Green function approach to determine the dynamic deflection of an undamped simply supported Euler-Bernoulli beam of finite length, subjected to a moving mass at constant speed. Visweswara Rao [9] studied the dynamic response of an Euler-Bernoulli beam under moving loads by mode superposition. The time-dependent equations of motion in modal space were solved by the method of multiple scales and the instability regions of the parametric resonance were identified.

Wu et al. [10] presented a technique using combined finite element and analytical methods for determining the dynamic responses of structures to moving bodies and applied this technique to a clamped-clamped beam subjected to a single mass moving along the beam. Sun [11] obtained Green's function of the beam on an elastic foundation by means of the Fourier transform. The theory of a linear partial differential equation was used to represent the displacement of the beam in terms of the convolution of Green's function. Next, he employed the theory of a complex function to seek the poles of the integrand of the generalized integral. The theorem of residue was then utilized to represent the generalized integral using a contour integral in the complex plane. Abu-Hilal [12] used a Green function method for determining the dynamic response of Euler-Bernoulli beams subjected to distributed and concentrated loads. He used this method to solve single and multi-span beams, single and multi-loaded beams and statically determinate and indeterminate beams.

Using the dynamic Green function yields exact solutions in closed forms and the deflection expression for the beam to be written in a simple form; the computation, therefore, becoming more efficient. This is particularly essential for calculating dynamic stresses and determining the dynamic response of beams other than simply supported ones. Also, by use of the Green function method, the boundary conditions are embedded in the Green functions of the corresponding beams. Furthermore, by using this method, it is not necessary to solve the free vibration problem in order to obtain the eigenvalues and the corresponding eigenfunctions, which are required while using series solutions.

An exact and direct modeling technique is presented in this paper for modeling beam structures with various boundary conditions, subjected to a constant load moving at constant speed. This technique is based on the dynamic Green function. In order to demonstrate the procedure and to show the simplicity and efficiency of the method presented, quantitative examples are given. In addition, the influence of variation of the system speed parameters on the dynamic response is studied.

GREEN FUNCTION SOLUTION

The governing equation of a flexible beam subject to a concentrated force (shown in Figure 1) can be given by:

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\mu\frac{\partial^2 y(x,t)}{\partial t^2}=F(x,t), \tag{1}$$

where $y(x,t)$ represents the deflection of the beam, x represents the traveling direction of the moving load and t represents time. Also, EI is the rigidity of the beam, E is Young's modulus of elasticity, l is the cross sectional moment of inertia of the beam and μ is the mass per unit length of the beam. The beam length is l, traveling load velocity is v. The boundary conditions and the initial conditions for the general beam (Figure 1) are:

$$\frac{\partial^3 y(x,t)}{\partial x^3}=k_l y(x,t),$$

$$\frac{\partial^2 y(x,t)}{\partial x^2}=k_t\frac{\partial y(x,t)}{\partial x}, \qquad \text{for } x=0 \text{ and } l,$$

$$y(x,t)=\frac{\partial y(x,t)}{\partial t}=0, \tag{2}$$

where k_l and k_t are linear and twisting spring constants, preventing vertical motion and, in the $x-y$ plane, rotation of the beam ends, respectively. $F(x,t)$ is the external load and, for a moving concentrated load case, can be given by:

$$F(x,t)=P\delta(x-u), \tag{3}$$

where P is the amplitude of the applied load and δ is the Dirac-delta function, which is defined by:

$$\int_{-\infty}^{\infty}\delta(x-x_0)f(x)dx=f(x_0). \tag{4}$$

Using the dynamic Green function, the solution of Equation 1 can be written as:

$$y(x,t)=G(x,u)P, \tag{5}$$

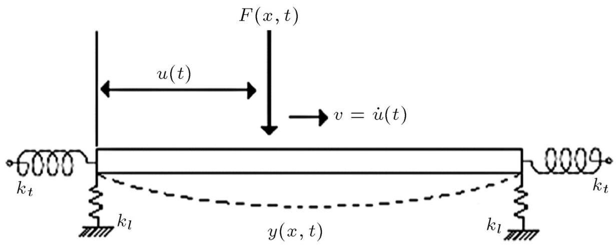

Figure 1. Moving mass on a beam with general boundary condition.

where $G(x,u)$ is the solution of the differential equation:

$$\frac{d^4y(x)}{dx^4} - \psi^4 y(x) = \delta(x-u), \tag{6}$$

in which ψ is the frequency parameter (separation constant) and is given by:

$$\psi^4 = \omega^2 \bar{m}/EI, \tag{7}$$

where ω is the circular frequency that expresses the motion of the mass and is equal to $\pi\nu/l$.

The solution of Equation 6 is assumed in the following form [13]:

$$G(x,u) = \begin{cases} C_1\cos(\psi x) + C_2\sin(\psi x) \\ \quad + C_3\cosh(\psi x) + C_4\sinh(\psi x), \\ \quad 0 \le x \le u \\ \\ C_5\cos(\psi x) + C_6\sin(\psi x) \\ \quad + C_7\cosh(\psi x) + C_8\sinh(\psi x), \\ \quad x \le u \le l \end{cases} \tag{8}$$

The constants $C_1, \cdots, C_8$ are evaluated such that the Green function, $G(x,u)$, satisfies the following conditions [14]:

(a) Two boundary conditions at each end of the beam, depending on the type of end support:

$$G''(0,u) = k_t G'(0,u),$$

$$G''(l,u) = k_t G'(l,u),$$

$$G'''(0,u) = k_l G(0,u),$$

$$G'''(l,u) = k_l G(l,u), \tag{9}$$

where the prime indicates a derivative with respect to x;

(b) Continuity conditions of displacement, slope and moment at $x = u$:

$$G(u^+,u) = G(u^-,u),$$

$$G'(u^+,u) = G'(u^-,u),$$

$$G''(u^+,u) = G''(u^-,u). \tag{10}$$

(c) Shear force discontinuity of magnitude one at $x = u$:

$$EI[G'''(u^+,u) - G'''(u^-,u)] = 1. \tag{11}$$

The Green function obtained by the above mentioned procedure has a general form. By leading linear and twisting spring constants (k_l and k_t) to extreme values (infinity and/or zero), one can obtain the appropriate Green function for the desired combinations of end boundary conditions. Denoting simply supported, clamped and free end boundary conditions by SS, C and F, a notation such as C-SS is employed to show the boundary conditions on the two ends of a clamped-simply supported beam.

For example, the Green function for a clamped-simply supported boundary condition (C-SS) is given by:

$$G(x,u) = \frac{1}{2EI\psi^3\Theta} \begin{cases} C_1\cos(\psi x) + C_2\sin(\psi x) \\ \quad +C_3\cosh(\psi x)+C_4\sinh(\psi x), \\ \quad 0 \le x \le u \\ \\ C_5\cos(\psi x) + C_6\sin(\psi x) \\ \quad +C_7\cosh(\psi x)+C_8\sinh(\psi x), \\ \quad x \le u \le l \end{cases} \tag{12}$$

where:

$$C_1 = a_1\sinh(\psi L) - a_2\sin(\psi L),$$

$$C_2 = a_2\cos(\psi L) - a_1\cosh(\psi L),$$

$$C_3 = a_2\sin(\psi L) - a_1\sinh(\psi L),$$

$$C_4 = a_1\cosh(\psi L) - a_2\cos(\psi L),$$

$$C_5 = -\sin(\psi L)^*(a_3\sinh(\psi L) + a_4\cosh(\psi L)),$$

$$C_6 = \cos(\psi L)^*(a_3\sinh(\psi L) + a_4\cosh(\psi L)),$$

$$C_7 = \sinh(\psi L)^*(a_3\sin(\psi L) + a_4\cos(\psi L)),$$

$$C_8 = -\cosh(\psi L)^*(a_3\sin(\psi L) + a_4\cos(\psi L)), \tag{13}$$

and:

$$a_1 = \sin[\psi(u-L)],$$

$$a_2 = \sinh[\psi(u-L)],$$

$$a_3 = \cos(\psi u) - \cosh(\psi u),$$

$$a_4 = \sinh(\psi u) - \sin(\psi u),$$

$$\Theta = \sin(\psi L)\cosh(\psi L) - \cos(\psi L)\sinh(\psi L). \tag{14}$$

NUMERICAL RESULTS

Before discussing the numerical results, the formulation developed herein is validated against available analytical solutions for a beam with different boundary conditions and acted upon by a moving load. First, a comparison of the present results with the example of a clamped-clamped beam, with the following (Equation 15) system parameters reported by Lee [5], is demonstrated in Figure 2.

$$l = 6\ \bar{m},$$

$$\nu = 0.6\ \bar{m}/s,$$

$$EI/\bar{m} = 275.4408\ \bar{m}^4/\mathrm{s}^2,$$

$$P/\bar{m}l = 0.2. \tag{15}$$

The vertical axis in Figure 2 shows the dimensionless deflections (introduced by Equation 16) of the point under the moving load, and the horizontal axis depicts the position of the load along the beam. Results reported by Lee [5] are computed by an assumed mode method and are compared with the present result of the Green function approach. As can be observed in Figure 2, there is an excellent agreement between the two results. The dimensionless deflection $\bar{y}$ in these figures is introduced by:

$$\bar{y} = y/y_{st}, \tag{16}$$

where y_{st} is the static transverse deflection at the beam mid span when a concentrated load with amplitude P is applied statically at the beam's mid span. For example, for the clamped-clamped B.C. we have:

$$y_{st} = \frac{Pl^3}{192EI}. \tag{17}$$

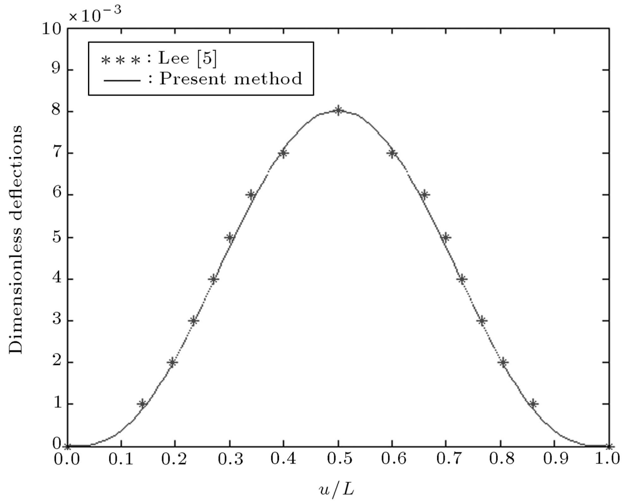

Figure 2. Dimensionless deflections under the moving load for a clamped-clamped beam.

In the second example, results obtained by the present method are compared with those obtained in [8]. In Figure 3, the deflection of a beam with simply supported boundary conditions for the speed parameter $\alpha = 0.25$ (defined by Foda and Abduljabbar [8]) is illustrated. The dimensionless speed parameter in [8] is defined as:

$$\alpha = \nu/\nu_{cr}, \tag{18}$$

where the critical speed is:

$$\nu_{cr} = \frac{2l}{T} = \frac{\pi}{l}\sqrt{\frac{EI}{\bar{m}}}, \tag{19}$$

where EI is the bending rigidity, $\bar{m}$ is the mass per unit length of the beam and T is the period that is related to the lowest mode of beam vibrations. In Figure 3, there is close agreement between the present result and that obtained by Foda [8].

The speed parameter, as mentioned in Equation 18, is the ratio of the speed of the load to the critical speed. In this article, the speed parameter is introduced in the following form:

$$\beta_i = \frac{\nu}{\omega_i l}, \qquad i = 1, 2, 3, \tag{20}$$

where ν is the speed of load in $\bar{m}/\mathrm{s}$ and ω_i is the ith natural frequency of beam in rad/s.

In Figure 4, the deflection for a simply supported beam under constant force, which passes the beam span from left to right in a constant velocity by speed parameters equal to 0.1, 1.0 and 2.5, is presented. It is observed that the response of the beam is harmonic and is symmetric about the mid span of the beam. In this example and in those remaining, $EI/\bar{m}$ is equal to $2.1368\ \bar{m}^4/\mathrm{s}^2$.

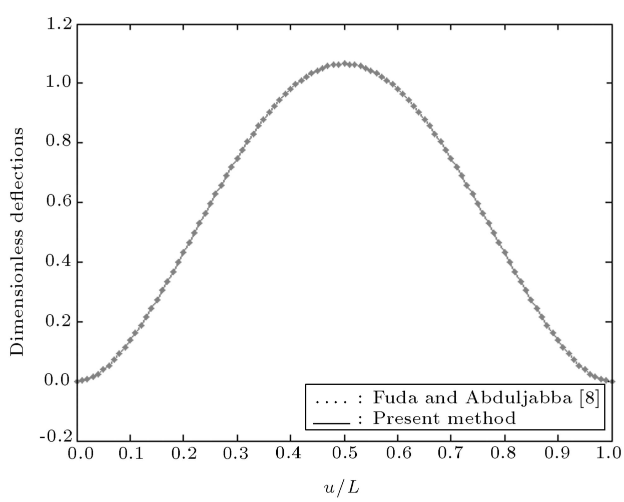

Figure 3. Dimensionless deflection for a simply supported beam.

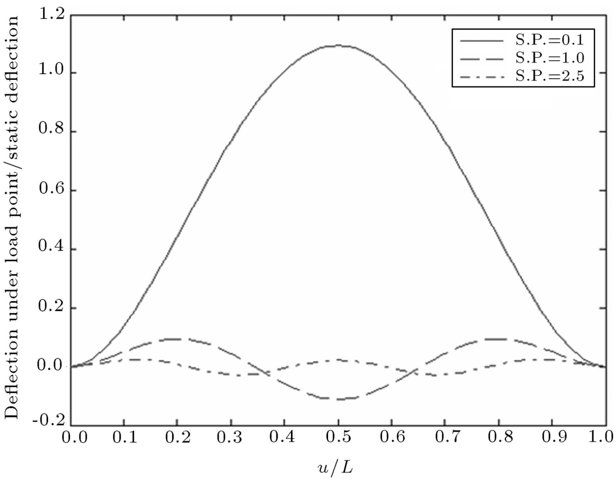

Figure 4. The deflection under the moving load for a simply supported beam with speed parameter, $\beta_1 = 0.1$, 1.0 and 2.5.

In the remaining figures, the effect of varying the dimensionless speed parameters (β_i) on the maximum deflection of beams under different boundary conditions is demonstrated.

The accuracy of Figures 5 to 11 and the values related to the vertical and horizontal axes depends directly on the velocity step sizes in the MATLAB computer code. Of course, as the velocity step size decreases, the peak values (maximum deflections) approach infinity and the critical speed parameters converge to more accurate values. Here, we have set 200 steps for the S.P. values in Figures 5 to 8 and, in order to obtain graphically more clear results, 400 steps were set for the S.P. values in Figures 9 to 11.

From Figures 5 to 8, it can be observed that for C-C, SS-SS, C-F and C-SS boundary conditions, there exist 4, 3, 2 and 3 critical speed parameters, respectively.

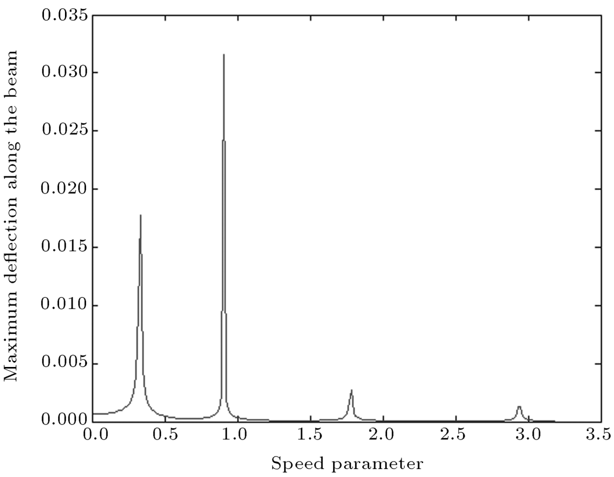

Figure 5. Maximum deflection along the clamped-clamped beam versus speed parameter, β_1.

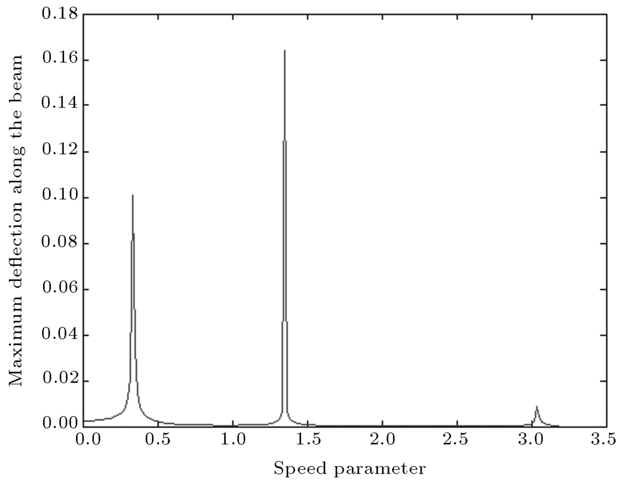

Figure 6. Maximum deflection along the simply supported beam versus speed parameter, β_1.

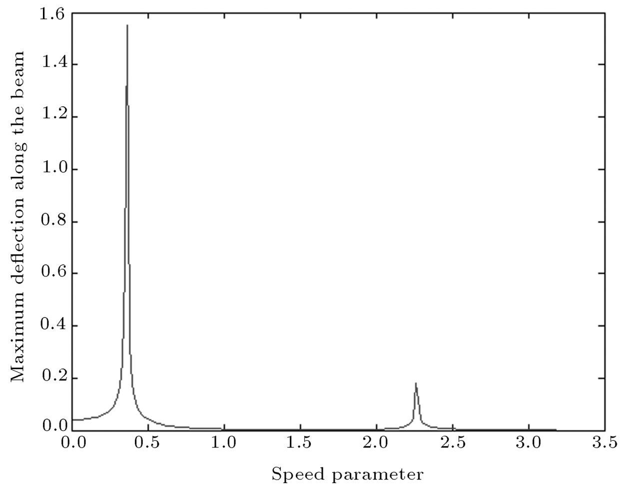

Figure 7. Maximum deflection along the clamped-free beam versus speed parameter, β_1.

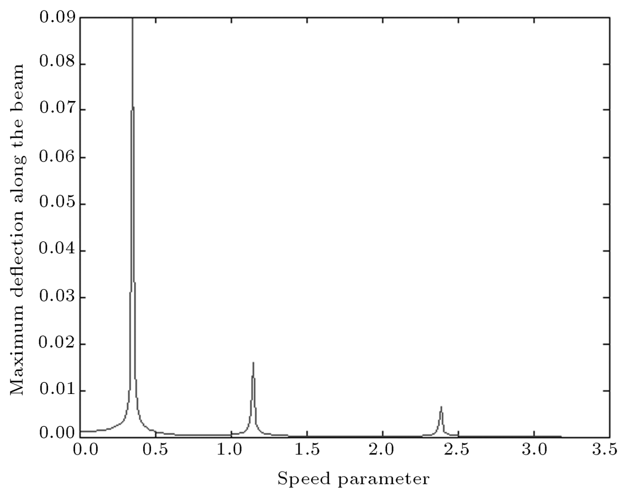

Figure 8. Maximum deflection along the clamped-simply supported beam versus speed parameter, β_1.

The critical values of speed parameter β_1, in which the peak values of maximum deflection occur, are obtained from Figures 5 to 8 and presented in Table 1. From these figures, one can see that the first critical deflection under all boundary conditions falls within the speed parameter range (0.33-0.36). Furthermore, in some cases the first peak and in others the second ones are critical values.

In Figures 9 to 11, the maximum deflection versus speed parameters, β_1, β_2 and β_3 for beams with different boundary conditions are deflected, respectively. The value of these natural frequencies for each boundary condition is presented in Table 2.

In Figure 9, variation of the maximum deflection versus speed parameter, β_1 (related to first natural frequency ω_1), of the beam is shown. As mentioned before, it can be seen more clearly from Figure 9 that critical speed parameters for all boundary conditions fall within the range 0.33 to 0.36. In addition, out of this range, the deflection of a clamped-free (C-F) beam is maximum and that of the C-C is minimum among the four corresponding B.C.'s.

The critical speed parameters (β_i) where the maximum deflection peak values occur in Figures 10 and 11, are presented in Table 3.

From Figures 9 to 11, one can see that the maximum deflections related to β_1, β_2 and β_3, depicted one, two and three peak values, respectively, and the last peak value in each of these figures for all the boundary conditions occurs in the range 0.33 to 0.36. This new observation implies a suitable point for the designers of beam-like structures under moving constant loads, i.e. they should avoid the speed parameters considered to be β_i near the critical values (0.33 to 0.36) in order to prevent resonance.

Table 1. Critical speed parameter, β_1, values for different boundary conditions.

Boundary Conditions	Critical Values of Speed Parameter, β_1			
	1st	2nd	3rd	4th
C-C	0.3285	0.9065	1.7762	2.9361
SS-SS	0.3387	1.3535	3.0456	-
C-F	0.3616	2.2664	-	-
C-SS	0.3527	1.1421	2.3822	-

Table 2. Natural frequencies of bending vibrations of beams with different boundary conditions.

Natural Frequency	C-C	SS-SS	C-F	C-SS
ω_1 (Hz)	14	6	2	9
ω_2 (Hz)	39	25	14	32
ω_3 (Hz)	78	57	39	67

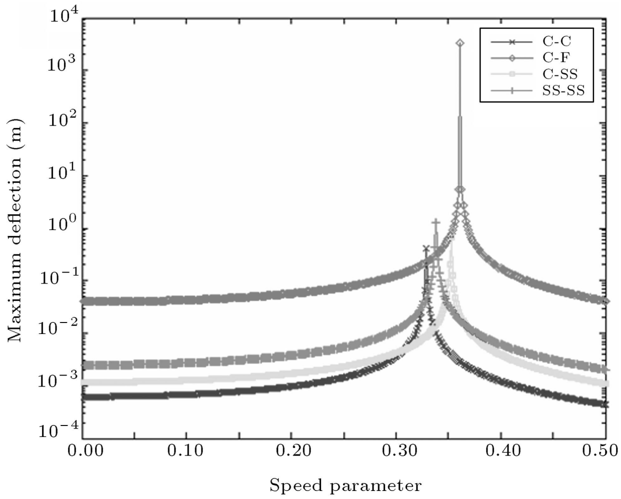

Figure 9. Maximum deflection along the beam versus speed parameter, β_1 .

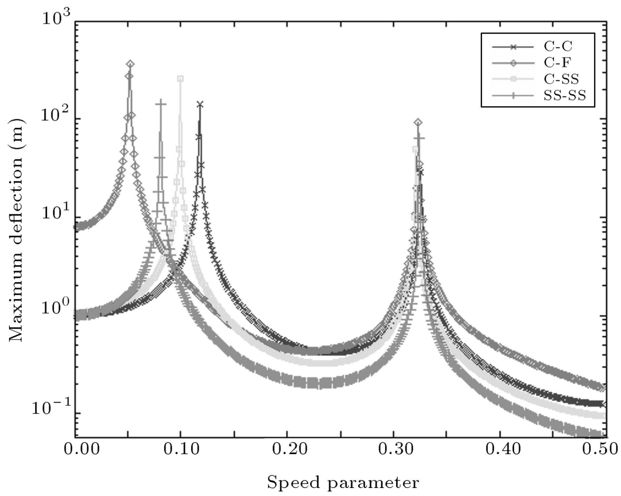

Figure 10. Maximum deflection along the beam versus speed parameter, β_2.

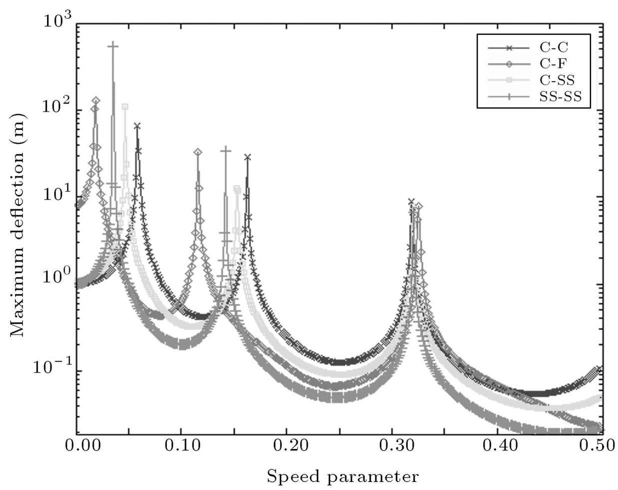

Figure 11. Maximum deflection along the beam versus speed parameter, β_3.

Table 3. Critical speed parameter, β_2 and β_3, values for different boundary conditions.

B.C.	**Speed Parameters**	**Critical Valuse of Speed Parameters**		
	(β_i)	**1st**	**2nd**	**3rd**
C-C	β_2	0.1184	0.3247	-
	β_3	0.0586	0.1630	0.3183
SS-SS	β_2	0.0815	0.3247	-
	β_3	0.0357	0.1426	0.3209
C-F	β_2	0.0522	0.3234	-
	β_3	0.0191	0.1159	0.3259
C-SS	β_2	0.0993	0.3209	-
	β_3	0.0471	0.1528	0.3196

CONCLUSION

An exact and direct modeling technique is presented in this paper for modeling beam structures with various boundary conditions, subjected to a load moving at a constant speed. In order to validate the efficiency of the method presented, quantitative examples are given and results are compared with those available in the literature. In addition, the influence of a variation in the speed parameters of the system on the dynamic response of the beam was studied and the results were given in graphical and tabular form. Maximum deflections versus speed parameter β_i of beams with various boundary conditions are determined. It was observed that the maximum deflections related to β_1, β_2 and β_3, depicted one, two and three peak values, respectively, and the last peak value in each of these figures, for all boundary conditions, occurs in the range 0.33 to 0.36. This new observation implies a suitable point for the designers of beam-like structures under moving constant loads, i.e. they should avoid the speed parameters considered to be β_i near the critical values (0.33 to 0.36) in order to prevent resonance.

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