Transaction E: Industrial Engineering
Vol. 16, No. 1, pp. 1–10

Special Classes of Fuzzy Integer Programming Models with All-Different Constraints

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Abstract. *In this paper, a fuzzy approach is applied to special classes of integer programming problems with all different constraints. In the first model, a fuzzy integer programming model is developed to represent the all-different constraints in mathematical programming. In order to solve the proposed model, a new branching scheme for the Branch and Bound algorithm is also presented. In the second model, a special class of large-scale multi-objective fuzzy integer programming problems with all-different constraints is introduced. A solution method for the proposed model is also developed by using the decomposition technique, weighting method and Branch and Bound algorithm. An illustrative numerical example is also given to clarify the theory and the method discussed in this paper.*

Keywords: *Fuzzy integer programming; All-different constraints; Branch & Bound algorithm.*

INTRODUCTION

In an integer programming model, all constraints are restricted to the forms of equality (=), less-than-or-equality ($\leq$) or greater-than-or-equality ($\geq$). However, in modeling, some real problems, such as the graceful labeling of graphs, the maximum matching problem and the n-queens problem, a situation often arises when variables cannot take the same value. The general form of an integer programming problem with all-different constraints is as follows:

Problem P_1:

$$\max \ Z = c^T X,$$

s.t.

$$AX = b,$$

$$X \geq 0, \ \text{Integer},$$

$$x_p \neq x_q, \ \forall (p,q) \in K,$$

where $X = (x_1, x_2, \cdots, x_n)^T$, $c \in R^n$, $b \in R^m$, $A \in R^{m \times n}$ and $K = \{(p,q) : 1 \leq p < q \leq n\}$.

In the above model, constraints in the form of $x_p \neq x_q$, $\forall (p,q) \in K$ are called all-different constraints. This type of constraint, in mathematical programming models, has been investigated by many researchers during recent years [1-3].

All-different constraints can also be represented as Constraint Satisfaction Problems (CSP). A CSP is a triple $P = (X, D, C)$, where X is a finite set of variables, D assigns to each variable, $x \in X$, a domain, D_x, of all possible values, and each element, $c \in C$, expresses a constraint on some variables, $x_1, x_2, \cdots, x_n$, where $c \subseteq D_{x_1}, D_{x_2}, \cdots, D_{x_n}$.

Definition 1

Let $X = (x_1, x_2, \cdots, x_n)^T \in R^n$; the all-different constraint is defined as follows:

$$\text{all-different } (X) = \{(x_1, \cdots, x_n) \in D_{x_1} \times \cdots \times D_{x_n} : \forall p, q, p \neq q \rightarrow x_p \neq x_q\}.$$

The all-different constraint of the form $x_p \neq x_q$, $\forall (p,q) \in K$ can also be reformulated by the following constraints, as in [2]:

$$x_p - x_q + \delta_{pq} M \leq M - \varepsilon,$$

$$-x_p + x_q + \delta'_{pq} M \leq M - \varepsilon,$$

$$\delta_{pq} + \delta'_{pq} = 1,$$

$$\delta_{pq}, \delta'_{pq} \in \{0,1\}, (p,q) \in K,$$

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Received 1 March 2006; received in revised form 22 April 2007; accepted 6 May 2007

where δ_{pq} and δ'_{pq} are binary variables, M is a large positive number and ε is a very small positive number defined by a decision maker. For a discussion on how to select values for M and ε, see [2].

Although there is a deterministic approach, based on the integer programming model for solving all-different constraints, a huge number of variables should be used in this model to convert all-different constraints. Therefore, the number of variables in the deterministic model increases significantly with the size of all-different constraints in the model. One of the advantages of using a fuzzy approach to solve the problem is the simplicity of applying it to a large number of all-different constraints without dramatically increasing the size of the problem.

In the first part of this paper, a fuzzy integer programming model is developed to represent the all-different constraints in the mathematical programming. Then, a new branching scheme is presented to solve the proposed model by the Branch and Bound algorithm. In the second part of this paper, a large-scale multi-objective integer programming problem with all-different constraints is introduced, in which the fuzzy random variables are used as the coefficients of the constraints. Furthermore, a new algorithm for solving the proposed model is presented. In this algorithm, a weighting method, together with a decomposition algorithm and a modified Branch and Bound method, are used.

PRELIMINARIES

In this section, some basic definitions are introduced. For more details see [4-10].

Definition 2

Let $\tilde{a}_1$ be a fuzzy set on $R = (-\infty, +\infty)$. This fuzzy set is called a level 1 fuzzy point, if its membership function is given as follows:

$$\mu_{\tilde{a}_1}(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$$

$F_p(1) = \{\tilde{a}_1 | \forall a \in R\}$ denotes the family of all level 1 fuzzy points.

Definition 3

Let $\tilde{A}$ be a fuzzy set on R. $\tilde{A}$ is called a fuzzy number if it is satisfied by the following conditions:

(i) $\tilde{A}$ is normal, i.e. $\{x \in R | \tilde{A}(x) = 1\}$ is non-empty;

(ii) $\tilde{A}$ is fuzzy convex, i.e. $\tilde{A}(\alpha x + (1-\alpha)y) \geq \min\{\tilde{A}(x), \tilde{A}(y)\}$ for any $x, y \in R$, $\alpha \in (0,\ 1]$;

(iii) $\tilde{A}$ is upper semi-continuous;

(iv) The support set of $\tilde{A}$ is compact, i.e. $\{x \in R | \tilde{A}(x) > 0\}$ is closed and bounded.

Definition 4

LR fuzzy number $\tilde{A}$ is defined by the following membership function [4,6]:

$$\tilde{A}(x) = \begin{cases} L\left(\frac{A^0 - x}{A^-}\right) & \text{if } x \leq A^0 \\ R\left(\frac{x - A^0}{A^+}\right) & \text{if } x \geq A^0 \end{cases}$$

where A^0 denotes the center (or mode), A^- and A^+ represent the left and right spread, respectively; L, R : $[0,1] \to [0,1]$, with $L(0) = R(0) = 1$, and $L(1) = R(1) = 0$ are strictly decreasing continuous functions. A possible representation of a LR fuzzy number is $\tilde{A} = (A^0, A^-, A^+)_{LR}$.

Let $\tilde{A} = (A^0, A^-, A^+)_{LR}$ be a LR fuzzy number. It is called a triangular fuzzy number and denoted by: $\tilde{A} = (A^0, A^-, A^+)$, if $L(x) = R(x) = 1 - x$. Let $F_N = \{(a^0, a^-, a^+) | \forall a^0 - a^- < a^0 < a^0 + a^+; a^0 \in R, a^-, a^+ \in R^+\}$ be the family of all triangular fuzzy numbers. The family of all left triangular fuzzy numbers can be denoted by $F_L = \{(a^0, a^-, 0) | a^0 - a^- < a^0;\ a^0 \in R, a^- \in R^+\}$.

Similarly, $F_R = \{(a^0, 0, a^+) | a^0 < a^0 + a^+; a^0 \in R, a^+ \in R^+\}$ denotes the family of all right triangular fuzzy numbers.

Note that $\tilde{A} = (A^0, A^-, A^+) = A^0$, if $A^- = A^+ = 0$. It is clear that $F_P(1)$, F_L and F_R are all special cases of F_N.

Therefore, we have $F = F_N \cup F_L \cup F_R \cup F_P(1) = \{(a^0, a^-, a^+) | a^0 - a^- \leq a^0 \leq a^0 + a^+; a^0 \in R, a^-, a^+ \in R^+\}$.

Definition 5 [11,12]

Let (Ω, A, P) be a complete probability space. B denotes the collection of Borel subsets of R and $B \in B$. A Fuzzy Random Variable (FRV) is a Borel measurable function, $X : (\Omega, A) \to (F, d)$.

If X is a fuzzy random variable, then, an α-cut $X_\alpha(\omega) = \{x \in R | X(\omega)(x) \geq \alpha\} = [X_\alpha^-(\omega), X_\alpha^+(\omega)]$ is a random interval for every $\alpha \in (0,1]$ and (R, B) is a Borel measurable iff $X_\alpha^{-1}(B) = \{\omega \in \Omega; X_\alpha(\omega) \cap B \neq \phi\} \in A$.

where $A \subset R$, for any $\omega \in \Omega$, $X(\omega) \in F$, and for any $x \in R$, $X(\omega)(x)$ is a membership function of $X(\omega)$.

Denote the set of all fuzzy random variables on (Ω, A, P) by FRV(Ω).

Lemma 1

Let $X \in$ FRV(Ω), then $X(\omega) = \bigcup_{\alpha \in (0,1]} \alpha . X_\alpha(\omega)$, $\forall \omega \in \Omega$.

Proof

If A is a fuzzy number, then $A = \bigcup_{\alpha \in (0,1]} \alpha . A_\alpha$. Since $(\bigcup_\alpha \alpha . A_\alpha)(x) = \text{Sup}\{\alpha . (A_\alpha)(x) | \alpha \in (0,1]\} = \text{Sup}\{\alpha | x \in A_\alpha\} = A(x)$ for any $x \in R$, then $A = \bigcup_{\alpha \in (0,1]} \alpha . A_\alpha$. Since $X(\omega) \in F$, then the proof is completed.

Definition 6 **[11,12]**

The expeced value of a fuzzy random variable, X, denoted by $E(X)$, is defined as follows:

$$E(X) = \int_{\Omega} X(\omega)p(d\omega) = \bigcup_{\alpha\in(0,1]} \alpha \int_{\Omega} X_\alpha(\omega)p(d\omega)$$

$$= \bigcup_{\alpha\in(0,1]} \alpha \left[\int_{\Omega} X_\alpha^-(\omega)p(d\omega), \int_{\Omega} X_\alpha^+(\omega)p(d\omega) \right].$$

Therefore, the expectation of a fuzzy random variable is defined as a unique $U \in F$, whose α-cut is $U_\alpha = E(X_\alpha) = [E(X_\alpha^-), E(X_\alpha^+)]$ and $(E(X))_\alpha = E(X_\alpha)$.

Let $X \in \text{FRV}(\Omega)$, then we define the scalar expected value of X, denoted by $Er(X)$, and call it the Er-expected value of X as follows:

$$Er(X) = \frac{1}{2}\int_0^1 [E(X_\alpha^-) + E(X_\alpha^+)]d\alpha,$$

where $E(X_\alpha^-)$ and $E(X_\alpha^+)$ are expected values of $X_\alpha^-(\omega)$ and $X_\alpha^+(\omega)$, respectively.

Corollary 1

Let $X, Y \in \text{FRV}(\Omega)$ and $\lambda \in R$, then:

i) $E(\lambda) = \lambda$;

ii) $E(X + \lambda Y) = E(X) + \lambda E(Y)$;

iii) $Er(X + \lambda Y) = Er(X) + \lambda Er(Y)$.

Definition 7

Let $X, Y \in \text{FRV}(\Omega)$. Then, the relations "$\tilde{\cong}$", "$\tilde{\leq}$" and "$\tilde{\neq}$" are defined, respectively, as follows:

i) $X \tilde{\cong} Y$ iff $Er(X) = Er(Y)$;

ii) $X \tilde{\leq} Y$ iff $Er(X) \leq Er(Y)$;

iii) $X \tilde{\neq} Y$ iff $Er(X) \neq Er(Y)$.

A FUZZY INTEGER PROGRAMMING MODEL WITH ALL-DIFFERENT CONSTRAINTS

Model Definition

In this section, a fuzzy approach to the integer programming problem is presented with all-different constraints. To present a fuzzy form, Verdegay [13] and Chanas [14] approaches have been applied to Problem P_1. They have used the concept of the λ-cut of a fuzzy set in their methods. First, consider the following definition.

Definition 8

Let ε be a tolerance coefficient given by a decision maker for each constraint in the form of $x_p \neq x_q$, $\forall(p,q) \in K$. Now, a membership function for $x_p \neq x_q$ is defined as follows:

$$\mu_{pq} : R^n \to [0,1],$$

$$\mu_{pq}(X) = \begin{cases} 1 & \text{if } |x_p - x_q| \geq \varepsilon \\ |x_p - x_q|/\varepsilon & \text{if } |x_p - x_q| < \varepsilon \\ 0 & \text{if } x_p = x_q \end{cases}$$

where $X \in R^n$ and $(p,q) \in K$.

The membership function of the constraint, $x_p \neq x_q$, as well as its λ-cut, are shown in Figure 1.

Therefore, Problem P_1 can be formulated, as follows, by considering the above membership function:

Problem P_2:

$$\max\ Z = c^T X,$$

s.t.

$$AX = b,$$

$$\forall(p,q) \in K,\ \mu_{pq}(X) \geq \lambda,$$

$$X \geq 0,\ \text{Integer},$$

where we have:

$$\mu_{pq}(X) = \left\{ \begin{array}{cl} 1 & \text{if } |x_p - x_q| \geq \varepsilon \\ |x_p - x_q|/\varepsilon & \text{if } |x_p - x_q| < \varepsilon \\ 0 & \text{if } x_p = x_q \end{array} \right\} \geq \lambda$$

$$\Leftrightarrow |x_p - x_q| \geq \lambda\varepsilon.$$

Therefore, a fuzzy integer programming problem with all-different constraints is generated as follows:

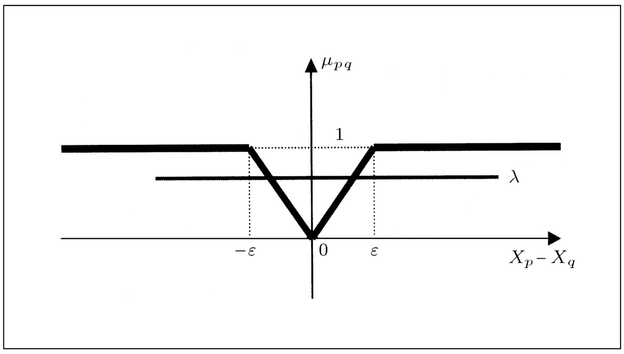

Figure 1. The membership function of $x_p \neq x_q$.

Problem P_3:

$$\max\ Z = c^T X,$$

s.t.

$$AX = b,$$

$$|x_p - x_q| \geq \lambda\varepsilon;\ \forall (p,q) \in K,$$

$$X \geq 0,\ \text{Integer and } 0 < \lambda \leq 1,$$

where λ is a real scalar and ε is a tolerance coefficient.

Theorem 1

The following feasible spaces are equivalent:

$$S_1 = \{X : AX = b, X \geq 0;\ \text{Integer},$$
$$x_p \neq x_q;\ \forall (p,q) \in K\},$$

$$S_2 = \{X : AX = b, X \geq 0;\ \text{Integer},$$
$$|x_p - x_q| \geq \lambda\varepsilon;\ \forall (p,q) \in K, 0 < \lambda \leq 1\}.$$

Proof

Suppose $X \in S_2$. Then, there must exist λ in $(0,1]$ such that $|x_p - x_q| \geq \lambda\varepsilon;\ \forall (p,q) \in K$ where $\varepsilon > 0$. Therefore, we have $x_p - x_q \geq \lambda\varepsilon \vee -x_p + x_q \geq \lambda\varepsilon$. In other words, $x_p \neq x_q;\ \forall (p,q) \in K$ and $X \in S_1$; $S_2 \subseteq S_1$.

Now, suppose that $X \in S_1$. According to the definition of S_1, we have $x_p \neq x_q;\ \forall (p,q) \in K$. Therefore, $x_p - x_q > 0 \vee x_p - x_q < 0;\ \forall (p,q) \in K$ and there must exist δ from $(0,1]$ such that $x_p - x_q \geq \delta \vee x_p - x_q \leq -\delta;\ \forall (p,q) \in K$. In other words, $|x_p - x_q| \geq \delta;\ \forall (p,q) \in K$. Now, if it is assumed that $\delta = \lambda\varepsilon$, where $\varepsilon > 0$, $\lambda \in (0,1]$, we have $X \in S_2$ and $S_1 \subseteq S_2$.

Corollary 2

The optimal solution of Problem 1 and Problem 3 are equal.

The major concern in Problem 3 is the constraints in the form of $|x_p - x_q| \geq \lambda\varepsilon;\ \forall (p,q) \in K$. It is known that these types of constraint can be reformulated as linear constraints by using $n(n-1)$ additional binary variables. Now, a new class of variables will be introduced to develop a strategy to deal with all-different constraints without using binary variables.

Definition 9

Let an integer variable, y, with lower bound $(-L_y)$ and upper bound (U_y) be denoted by $IN(d)$, if it cannot take any value in the interval $(-d,+d)$, where $-L_y \leq -d$ and $d \leq U_y$ and d, L_y, $U_y \geq 0$.

In the relaxation form of an $IN(d)$ variable, it is also assumed that the variable will relax to a real variable restricted to only its lower and upper bound. If $d = 1$, then $IN(1)$ variables become non-zero variables introduced by Hajian [1].

Now Problem P_3 can be converted to the following model:

Problem P_4:

$$\max\ Z = c^T X,$$

s.t.

$$AX = b,$$

$$x_p - x_q = w_{pq}, \forall (p,q) \in K,$$

$$X \geq 0,\ \text{Integer},\ 0 < \lambda \leq 1,$$

$$w_{pq} : IN(\varepsilon\lambda);\ \forall (p,q) \in K.$$

A Branching Strategy for $IN(d)$ Variables

The Branch and Bound method (B&B) is one of the most well-known algorithms for solving mixed integer and discreet programming problems. The Branch and Bound term refers to an enumerative technique that was first used for mixed integer programming problems by Land and Doig [15]. In B&B algorithms, the famous divide and conquer strategy is used. For a detailed discussion of B&B algorithms, the reader is referred to [15,16]. In order to solve an integer programming model with $IN(d)$ variables, non-zero variables, dis-equality constraints or other similar discrete constraints, the branching strategy of the B&B algorithm needed to be modified [1]. A modified B&B algorithm has been developed to solve the presented fuzzy integer programming model with all-different constraints. In this algorithm, a branching strategy is defined, as follows, for special types of the presented variables.

In the first step, the relaxation form of problem P_4, after removing all integral constraints and changing $IN(d)$ variables to relaxed forms, is solved and, if the solution is satisfied by all original constraints and $IN(d)$ variables, then an optimal solution of the original problem is also found. Otherwise, a variable with infeasible value is selected and a branching strategy is performed on the selected variable. If the selected variable, w, is one of the $IN(d)$ type variables, then depending on its current value, w^*, two new sub-problems are created, as illustrated in Figure 2.

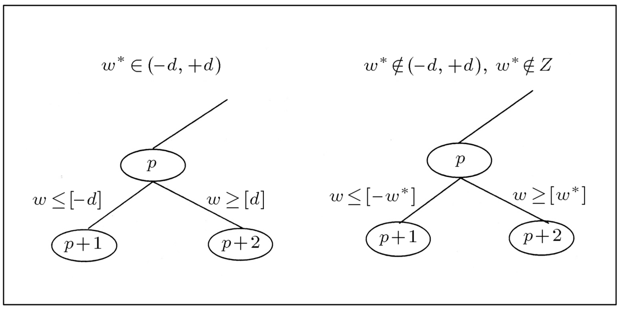

Figure 2. Branching strategy for $IN(d)$ variables.

An Illustrative Example

Consider the following example of an integer programming problem with all-different constraints:

$$\max\ z = 2x_1 + 3x_2 + 4x_4,$$

s.t.

$$x_1 + x_2 + x_3 = 4,$$

$$x_1 + 4x_4 = 8,$$

$$x_1, x_2, x_3, x_4 \ge 0,\ \text{integer},$$

$$\text{all-different } (x_1, x_2, x_3, x_4).$$

The optimal solution of the above problem, without considering all-different constraints, is as follows:

$$X^*_{IP}(x_1, x_2, x_3, x_4) = (0, 4, 0, 2), \qquad Z^*_{IP} = 20.$$

Now, to obtain the optimal solution of the original problem, the above model can be converted to the following fuzzy model, by considering $\varepsilon = 1$:

$$\max\ z = 2x_1 + 3x_2 + 4x_4,$$

s.t.

$$x_1 + x_2 + x_3 = 4,$$

$$x_1 + 4x_4 = 8,$$

$$x_p - x_q = w_{pq},\ \forall (p, q) \in K,$$

$$w_{pq} : IN(\lambda),\ \forall (p, q) \in K,$$

$$x_1, x_2, x_3, x_4 \ge 0,\ \text{integer},$$

$$0 < \lambda \le 1,$$

where $K = \{(p, q) | 1 \le p < q \le 4\}$.

The optimal solution of the above problem is obtained as follows:

$$X^*_{P_4} = (x_1, x_2, x_3, x_4, w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34})$$

$$= (0, 3, 1, 2, -3, -1, -2, 2, 1, -1),$$

$$Z^* = 17.$$

Now, by considering Corollary 2, the optimal solution of the original problem is:

$$X^* = (x_1, x_2, x_3, x_4) = (0, 3, 1, 2),\ \ Z^* = 17.$$

A SPECIAL CLASS OF LARGE-SCALE MULTI-OBJECTIVE FUZZY IP PROBLEMS

Introduction

Large-scale integer programs have been the center of attention of many researchers during the past two decades [7,11]. The fuzzy linear programming problem has also been discussed in many papers [5,10,17]. A fuzzy random variable is a random variable whose actual value is a member of a fuzzy set. The concept of fuzzy random variables was first introduced by Kwakernaak [7]. In this section, a large-scale multi-objective integer programming problem is considered, with all-different constraints and fuzzy random variables as the coefficients for the original constraints, all-different constraints and right-hand side values. Then, an algorithm is presented to solve the problem under consideration. In the proposed algorithm, a weighting method [18], together with a decomposition algorithm [19] and a modified B&B method, is used.

Problem Formulation

The following model is a large-scale, multi-objective integer programming problem with all-different constraints involving fuzzy random variables as the coefficients for the original constraints, all-different constraints and right-hand side values:

Problem P_5:

$$\max\ Z(X) = [z^1(X), \cdots, z^t(X)],$$

s.t.

$$\sum_{j=1}^{n} A_j X_j = b_0,$$

$$\tilde{D}_j X_j \cong b_j + \tilde{\lambda}_j b'_j, \qquad j = 1, \cdots, n,\ n > 1,$$

$$x_j \ge 0,\ \text{integer } \forall j,$$

$$\tilde{\varepsilon}_p x_p \, \widetilde{\neq} \, \tilde{\varepsilon}_q x_q,\ \forall (p, q),$$

where X is the set of decision variables. Let the first set of constraints, i.e. $\sum_{j=1}^{n} A_j X_j = b$, be denoted as a common constraint and the second set as an independent constraint. The set of independent constraints consist of n independent sub-problems. The rest of the parameters and variables are defined as follows, for $i = 1, 2, \cdots, t$ and $j = 1, 2, \cdots, n$:

X_j:	the subset of decision variables in the jth sub-problem,
$z^i(X)$:	the ith term of the objective function, defined by $z^i(X) = \sum_{j=1}^{n} C_j^i X_j$,
C_j^i:	the coefficients of X_j in the ith term of the objective function,
A_j:	a matrix whose elements are the coefficients of X_j in the set of common constraints,
$\tilde{D}_j$:	a matrix of fuzzy random variables, whose elements are the coefficients of variables in the jth sub-problem,
b_0:	a column vector of the right-hand side of the common constraint,
$\tilde{\varepsilon}_p$:	fuzzy random variables with $r \sim N(\mu, \sigma^2)$, $\tilde{\varepsilon}_p(\omega) = (r(\omega) - \varepsilon_p, r(\omega), r(\omega) + \varepsilon_p)$,
$\tilde{\lambda}_j$:	a diagonal matrix of fuzzy random variables in the right-hand side of the jth sub-problem,
b_j, b'_j:	column vectors of the right-hand side coefficients of the jth sub-problem,
m_j:	the number of variables of the sub-problem, j,
r_0:	the number of constraints in the set of the common constraint,
r_j:	the number of constraints of the sub-problem, j.

Problem P_5 has $\sum_{j=0}^{n} r_j$ constraints and $\sum_{j=1}^{n} m_j$ variables in total. Now, the Fuzzy Random Expected Value Model [8,9] is used for model conversion. By using the concept of the Er-expected value of fuzzy random variables and Corollary 1, Problem P_5 can be converted to the following problem:

Problem P_6:

$$\max\ Z(X) = [z^1(X), \cdots, z^t(X)],$$

s.t.

$$\sum_{j=1}^{n} A_j X_j = b_0,$$

$$Er(\tilde{D}_j) X_j = b_j + Er(\tilde{\lambda}_j) b'_j,$$

$$j = 1, \cdots, n, \quad n > 1,$$

$$x_j \geq 0, \text{ integer } \forall j,$$

$$Er(\tilde{\varepsilon}_p) x_p \neq Er(\tilde{\varepsilon}_q) x_q, \ \forall (p, q).$$

Or, similarly;

Problem P_7:

$$\max Z(X) = [z^1(X), \cdots, z^t(X)],$$

s.t.

$$\sum_{j=1}^{n} A_j X_j = b_0,$$

$$D_j X_j = b_j + \lambda_j b'_j,$$

$$j = 1, \cdots, n, \quad n > 1,$$

$$x_j \geq 0, \text{ Integer } \forall j,$$

$$\varepsilon_p x_p \neq \varepsilon_q x_q, \ \forall (p, q).$$

In Problem P_7, all the parameters, λ_j, $j = 1, 2, \cdots, n$, are real numbers and D_j is also a matrix, whose elements are real numbers. Now, the concept of $IN(d)$ variables is applied to the model to generate an Er-large-scale multi-objective integer linear programming problem, as follows:

Problem P_8:

$$\max\ Z(X) = [z^1(X), \cdots, z^t(X)],$$

s.t.

$$\sum_{j=1}^{n} A_j X_j = b_0,$$

$$D_j X_j = b_j + \lambda_j b_j,$$

$$j = 1, \cdots, n, \quad n > 1,$$

$$\varepsilon_p x_p - \varepsilon_q x_q = w_{pq}, \ \forall (p, q),$$

$$x_j \geq 0, \text{ Integer } \forall j,$$

$$w_{pq} : IN(\lambda), \ \forall (p, q).$$

If the weighting method [18] is applied to Problem P_8, then, the following integer linear problem with a single-objective function is generated:

Problem P_9:

$$P(w): \max \sum_{i=1}^{t} w_i z^i(X),$$

s.t.

$$\sum_{j=1}^{n} A_j X_j = b_0,$$

$$D_j X_j = b_j + \lambda_j b'_j,$$

$$j = 1, \cdots, n, \quad n > 1,$$

$$\varepsilon_p x_p - \varepsilon_q x_q = w_{pq}, \ \forall (p, q),$$

$$x_j \geq 0, \text{ Integer } \forall j,$$

$$w_{pq} : IN(\lambda), \ \forall (p, q),$$

where $w_i \geq 0$, $i = 1, \cdots, t$, and $\sum_{i=1}^{t} w_i = 1$. In this method, the weight, w_i, is assigned for the ith objective function.

Problem P_9 is a large-scale integer programming problem. Now, a modified B&B algorithm is developed for solving Problem P_9. In each vertex of the B&B tree, an LP relaxation of problem P_9, after removing all integral constraints and changing $IN(d)$ variables to the relaxed forms, is solved. Since an LP relaxation is still a large-scale problem, a Dantzig-Wolf method is applied to solve it at each step. If we suppose that each convex set, $D_j X_j = b_j + \lambda_j b'_j$, in the relaxed form of Problem P_9 is bounded, then, we have the following relation:

$$X_j = \sum_{k=1}^{k_j} \beta_j^k \hat{X}_j^k, \qquad j = 1, \cdots, n, \qquad \beta_j^k \geq 0,$$

$$\text{and } \sum_{k=1}^{k_j} \beta_j^k = 1 \text{ for all } j,$$

where k_j is the number of extreme points of set j and $\hat{X}_j^k$, $k = 1, \cdots, k_j$ are the extreme points of the jth convex set. Therefore, the master problem in the Dantzig-Wolf procedure for solving the relaxed form of Problem P_9, at each step, has the following form:

Problem P_{10}:

$$\hat{P}(w) = \max \sum_{i=1}^{t} wi \left[\sum_{j=1}^{n} C_j^i \Big(\sum_{k=1}^{k_j} \beta_j^k \hat{X}_j^k\Big) \right],$$

s.t.

$$\sum_{j=1}^{n} \sum_{k=1}^{k_j} A_j \beta_j^k \hat{X}_j^k = b_0,$$

$$\sum_{k=1}^{k_j} \beta_j^k = 1 \text{ for all } j,$$

$$\beta_j^k \geq 0 \text{ for all } j \text{ and } k.$$

Note that β_j^k are decision variables of the master problem. Suppose that π, π^{0j} be the dual multipliers corresponding to the constraints $\sum_{j=1}^{n} \sum_{k=1}^{k_j} A_j \beta_j^k \hat{X}_j^k = b_0$ and $\sum_{k=1}^{k_j} \beta_j^k = 1$, respectively.

Furthermore, the subproblem, j, $j = 1, 2, \cdots, n$, in the Dantzig-Wolf procedure, has also the following form:

Problem P_{11}:

$$\min \ (\pi A_j - \sum_{i=1}^{n} w_i C_j^i) X_j + \pi^{0j},$$

s.t.

$$D_j X_j = b_j + \lambda_j b'_j,$$

$$X_j \geq 0.$$

After solving the relaxed form of Problem P_9 at each vertex of the B&B tree by the above Dantzig-Wolf procedure, if the solution is satisfied by all original integrality constraints and $IN(d)$ variables constraints, then, the corresponding vertex is fathomed. Otherwise, a variable is selected and a branching strategy is performed, as mentioned priviously, on the selected variable.

In the following steps, an algorithm is summarized for solving the model discussed in this section:

1. *Data Entry.* Define a membership function for each fuzzy random variable in Problem P_5 and determine the Er-expected values of the fuzzy random variables in Problem P_6.
2. *Model Structure.*
   - Convert Problem P_5 to Problem P_6 (calculate Er-expected values of fuzzy random variables);
   - Convert Problem P_6 to Problem P_8 (apply the concept of $IN(d)$ variables);
   - Convert Problem P_8 to Problem P_9 (use the weighting method). Problem P_8 is a large scale integer programming model.

3. *Solution Procedure.* Solve Problem P_9 by using a modified B&B method and a Dantzig-Wolf decomposition procedure. Obtain an optimal solution of the original problem.

Theoretically, it has been shown in this section that a large-scale, multi-objective integer programming problem with all-different constraints involving fuzzy random variables can be converted to the integer programming model with a single-objective function. Since both the B&B method and the Dantzig-Wolf decomposition procedure are very powerful methods in solving large-scale models, the proposed algorithm can easily solve the final problem, P_9. However, there still remain several interesting open problems, with respect to the performance of solving real large scale versions of our model. It would be worthwhile to implement some empirical tests on real large scale problems and investigate using alternative weighting methods in converting problem P_8 to problem P_9. Furthermore, it would be desirable to use other decomposition techniques in the solution procedure, which might yield better results. Finally, while the focus of the paper has been on integer variables, it would be useful to extend the results to multi-objective 0-1 models with all-different constraints involving fuzzy random variables

An Illustrative Example

Here, a numerical example is given to clarify the model discussed in this section.

Consider the following two-objective integer programming problem with all-different constraints, where the coefficients of the original variables and their right-hand sides are defined by fuzzy random variables:

$$\max\ Z(X) = [3x_1 + x_2, x_1 + x_2],$$

s.t.

$$-x_1 + 2x_2 \leq 1,$$

$$2\tilde{d}_1 x_1 \leq 4 + 2\tilde{\lambda}_1,$$

$$4\tilde{d}_2 x_2 \leq 12 + 3\tilde{\lambda}_2,$$

$$x_1, x_2 \geq 0 \text{ and integers},$$

$$\tilde{\varepsilon}_1 x_1 \neq \tilde{\varepsilon}_2 x_2.$$

In this example, it is assumed that fuzzy random variables are given in Table 1 as follows.

Furthermore, it is assumed that the membership function corresponding to the fuzzy random variable λ, $\tilde{d}$ and $\tilde{\varepsilon}$ is given in the following form:

Table 1. Data for fuzzy random variables.

	$\boldsymbol{\mu}$	$\boldsymbol{\beta}$	$\boldsymbol{\lambda}$
d_1	1	3.8	1.8
λ_1	1	4.3	2.3
d_2	3	2.5	0.5
λ_2	3	3.7	1.7

$$\mu(x) = \begin{cases} \frac{x-r(\omega)+\beta}{\beta}, & r(\omega) - \beta \leq x \leq r(\omega) \\ \frac{r(\omega)+\gamma-x}{\gamma}, & r(\omega) \leq x \leq r(\omega) + \gamma \end{cases}$$

Suppose that $r \sim N(\mu, \sigma^2)$ is a normal random variable with the expectation μ and variance σ^2 on Ω and:

$$\tilde{d} = (r(\omega) - \beta, r(\omega), r(\omega) + \gamma),$$

$$\tilde{\lambda} = (r(w) - \beta, r(\omega), r(\omega) + \gamma).$$

Therefore, we have:

$$\tilde{d}^-_\alpha(\omega) = \tilde{\lambda}^-_\alpha(\omega) = r(\omega) + \beta(\alpha - 1),$$

$$\tilde{d}^+_\alpha(\omega) = \tilde{\lambda}^+_\alpha(\omega) = r(\omega) + \gamma(1 - \alpha),$$

$$\tilde{d}_\alpha(\omega) = \tilde{\lambda}_\alpha(\omega) = [r(\omega) + \beta(\alpha - 1), r(\omega) + \gamma(1 - \alpha)],$$

$$E(\tilde{d}_\alpha(\omega)) = E(\tilde{\lambda}_\alpha(\omega)) = [\mu + \beta(\alpha - 1), \mu + \gamma(1 - \alpha)],$$

$$\begin{aligned} Er(\tilde{d}(\omega)) &= Er(\tilde{\lambda}(\omega)) \\ &= \frac{1}{2}\int_0^1 [2\mu + \alpha(\beta - \gamma) - (\beta - \gamma)]d\alpha \\ &= \mu - \frac{1}{4}(\beta - \gamma), \end{aligned}$$

$$\tilde{\varepsilon}_p(\omega) = (r(\omega) - \varepsilon_p, r(\omega), r(\omega) + \varepsilon_p),$$

then:

$$Er(\tilde{\varepsilon}(\omega)) = \frac{1}{2}\int_0^1 [2\mu + 0]d\alpha = \mu.$$

Now, by using $IN(d)$ variables, the model can be written as follows:

$$\max\ Z(X) = [3x_1 + x_2, x_1 + x_2],$$

s.t.

$-x_1 + 2x_2 + s_1 = 1,$

$x_1 + s_2 = 2,$

$x_2 + s_3 = 1.95,$

$\mu x_1 - \mu x_2 = w_{12},$

$x_1, x_2 \geq 0,$ integers,

$w_{pq} : IN(\lambda).$

If we assume $w_1^* = w_2^* = 0.5$ in a weighting method, then, the problem has the following form:

$\max\ Z(X) = 2x_1 + x_2,$

s.t.

$-x_1 + 2x_2 + s_1 = 1,$

$x_1 + s_2 = 2,$

$x_2 + s_3 = 1.95,$

$\mu x_1 - \mu x_2 = w_{12},$

$x_1, x_2 \geq 0,$ integers,

$w_{pq} : IN(\lambda).$

The above problem is solved by a modified B&B method. At each node of the B&B tree, there is a relaxed LP problem, which can be solved by a Dantzig-Wolf decomposition method. The optimal solution of the problem is $X^* = (x_1, x_2) = (2, 1)$, which is the optimal solution to the original problem, according to Corollary 2.

CONCLUDING REMARKS

In this paper, a fuzzy approach was applied to special classes of integer programming problems with all-different constraints. First, a fuzzy integer programming model was introduced to represent the all-different constraints in mathematical programming. Then, a special class of large-scale multi-objective fuzzy integer programming problems with all-different constraints was presented. A solution method for the proposed model was also developed by using a decomposition technique, the weighting method and the B&B algorithm.

One of the advantages of using these algorithm is the simplicity of applying them to a large number of all-different constraints without dramatically increasing the size of the problem. Because this paper is an early step towards the study of developing a fuzzy approach for the all-different constraint problem, it is also necessarily restricted to simple assumptions. For example, it may be that, in some circumstances, the selection of a tolerance coefficient and membership function might be better described by new rules. To see whether the authors' algorithm is also a powerful method in more practical and large scale cases, further research is also needed. It could be worthwhile to study further the possibility of changing the weighting method used in this article. Another interesting area of further study would be the implementation of the proposed method for solving the different types of real examples involving all-different constraints, such as the graceful labeling problem.

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