

Replacement-Repair Policy Based on a Simulation Model for Multi-State Deteriorating Products Under Warranty

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Abstract. *In this paper, a replacement-repair model is developed to study a warranty servicing policy for a class of multi-state deteriorating and repairable products, based on a computer simulation analysis. In each working state there is a determined probability for transition to each of the subsequent states, given that it has made a transition out of that state. There are two parameters that determine the manufacturer's decision to repair or replace a failed item, assuming that the buyer's claim is valid; the deterioration degree of the item and the length of the residual warranty period. Beside these two parameters, other inputs to the model are: the number of working and failure states, the different rates of transition from each working state, given that the life distribution is exponential, the length of the entire warranty period, the probabilities of expected costs, generated by a mathematical approach, for a few special situations including transition between states and the cost of repair and replacement in each failure state. The output of the model is the mean and standard deviation of the total simulated costs generated for a given set of inputs, such that the best values of the two mentioned parameters can be obtained, based on the statistical test. The model is validated by comparing its output with the optimal expected costs generated by a few special situations where this comparison is possible.*

Keywords: *Warranty; Multi-state; Simulation; Deteriorating product.*

INTRODUCTION

Due to some features of a modern manufacturing environment, such as rapidly changing technologies, global markets, fierce competition and new patterns of consumption, it is necessary to pay special attention to strong tools regarding customer satisfaction. When a customer decides to purchase a product, in addition to some factors such as product price, perceived product quality and reliability and any financing offered by the manufacturer, which, in many instances, may be identical for more than two brands, there are other factors that play a significant role in attracting more customers. Among these, the most important are post-sale services, especially the warranty, which is known to the buyer at the time of purchase as an important factor in product selection. This attention is more evident in the case of new products. Often, customers

are uncertain about new product performance. Here, warranties play an important role in providing product assurance to customers, and different types of warranty are offered, depending on the product and the buyer.

When a supplier warrants a product, he assumes an obligation to the customer and a contract is realized between them. This obligation generates costs to the supplier associated with any product failure, since warranty terms generally require that such items be repaired, replaced or that a cash rebate be given within a predetermined warranty period. The supplier must take these costs into account in pricing the product. There have been numerous attempts to model warranty costs. Summaries of many of these results are given by Hill [1], Hill and Blischke [2] and Blischke [3].

One of the most commonly used warranty policies is the Free Replacement Warranty (FRW). Under FRW, the manufacturer agrees to repair or provide replacements for failed items, free of charge, up to time T , from the time of initial purchase, where T is called the warranty period. There are other types of warranty policy, such as pro-rata and combined warranties, that are referred to in the literature for different products,

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but which are not taken into consideration in this paper.

With a given warranty policy, the manufacturer usually has the option of replacing a failed item with a new one or simply repairing it. Various research results have been reported on how a manufacturer should service a given warranty, utilizing repair or replacement, in order to minimize the total expected cost to the manufacturer during the warranty period. Nguyen and Murthy [4] present models for the evaluation of expected warranty costs and the variance of these costs, considering factors such as; general product lifetime distribution and time dependent repair cost, good-as-new repair or replacement, minimal repair, a mixture of replacement and minimal repair, and imperfect repair. For the FRW with a fixed warranty period, T , Nguyen and Murthy [5] studied a warranty servicing policy to minimize the expected warranty servicing costs. When an item fails and is returned to the manufacturer, the decisions that the manufacturer has to make are:

1. Whether to replace a failed item with a new one or with a repaired item from the stock of repaired items;
2. Whether to repair the failed item received or to discard it;
3. Which type of repair should be carried out.

They propose an optimal servicing strategy for such a warranty policy. Nguyen and Murthy [6] examined the combined FRW with fixed and renewed periods of T and W , respectively. When an item fails within the fixed warranty period, T , for the first time, after performing the optimal amendatory action, it is returned to the customer with a renewed period of warranty (W). They propose an optimal servicing strategy for the warranty policy. Bohoris and Yun [7] provide equations for calculation of the expected value and variance of the manufacturer's total warranty costs under the combined warranty policy, minimal repair, and Weibull lifetime distributions. Chun and Tang [8] determine the warranty price for the FRW policy assuming a constant failure rate and a constant repair cost throughout the warranty period, and manufacturer and customer risk aversion behavior for future repair costs. Murthy et al. [9] propose a warranty policy which incorporates an incentive feature to reduce customer dissatisfaction of defective products. It involves the manufacturer offering compensation, for consumers who experience early product failure, as part of the warranty policy. They study the optimal design of such a policy, considering product quality variations, the warranty policy and the servicing strategy of the manufacturer.

All research results on warranty servicing strategies reviewed above, assume that an item under war-

ranty may experience only two possible states: working or failure. They do not address the deterioration of the item through time. The change in state of the item under warranty, with the different approaches used here, is modeled by Nguyen and Murthy [4] using imperfect repair and by Nguyen and Murthy [6] using replacement by another repaired item. In these cases, the repaired or replaced items may be considered as being in a different state, because they have a failure rate function different from that of just before the failure. The idea of this problem, perhaps, returns to Derman et al. [10] who studied the optimal replacement problem of a component where there are n types of replacements available, differing only in price and the failure rates of exponential life distributions. Assaf and Levikson [11] and Assaf [12] extend this model to arbitrary and phase-type life distributions. However, they do not model multi-state equipment or items that deteriorate continuously due to usage or age. These models do not consider the natural deterioration of the component and are not suitable for consumer product warranties where, normally, the customer's item is either replaced with a new one or minimally repaired. The customer may dislike having his or her failed item replaced with another used one.

As a more recent contribution, Zuo et al. [13] developed a model for a class of multi-state deteriorating products, wherein each item may gradually deteriorate along a predetermined number of working states. The problem facing the manufacturer is to choose an appropriate action (repair or replace) for each failure state during the warranty period that minimizes the total expected servicing cost during the warranty.

In this paper, a warranty servicing model for a special class of multi-state deteriorating products is developed. There are N different working states, each of which may be experienced by an item during the warranty period. It may fail during each of these states, so there are N possible failure states. The transition between working states has a special rule and the possibility of transitions between non-successive working states, due to some unpredictable events that may take place upon utilization of the item, has been assumed. When an item fails during the warranty period, there are two variables that specify the manufacturer's decision on whether to repair it using minimal repair or to replace it with a new one free of charge to the customer. Those parameters are the degree of deterioration of the failed item and the residual warranty period (from the time of failure to expiration of the warranty). The objective function is minimization of the total expected cost expended by the manufacturer during the warranty period.

The remainder of this paper is organized as follows. First, descriptions and assumptions of the

proposed model are given. Then, a detailed description of the simulation model, along with a mathematical derivation of the warranty servicing cost, is presented. Following that, there is a discussion of a small sampling study performed for model validation. This is accomplished by comparison of the simulation results with some validated results for a few special cases that have been presented in the literature.

DESCRIPTION OF MODEL

There are two types of variable that can be used to model the deterioration of an item with time; continuous or discrete. Although models of deterioration, based on the discrete approach, are an approximation for the real world, where the deterioration is continuous, they have the advantage of being simpler than the models based on a continuous approach and are also easier to analyze.

The discrete approach, involving N discrete working states numbered from 1 through N , is as follows. Working state 1 corresponds to a new item and the degree of deterioration increases with the working state so that working state j corresponds to greater deterioration than working state i if $j > i$. Once the item enters working state j , it can either fail or move to any of the subsequent states. If it enters a failed state, then it can be made operational, either through minimal repair or by replacement. In the former case, it is restored back to working state j and, in the latter case, it is brought back to working state 1. In working state N , when a failure occurs, the item is made operational by replacement, so that the working state becomes 1 after replacement.

The changes in the states are modeled as follows. Once the item enters working state j , it stays in that state for a random length of time, which is given by an exponential distribution with transition parameter μ_j . This implies that the mean time to transition is $1/\mu_j$. The parameter, μ_j , in conformity with the real world, increases as j increases from 1 to N . It means that the expected number of transitions increases as the item proceeds to the latest states. The probability that it moves to working state k ($k > j$) is P_{jk} and the probability that it fails (and moves to failed state j) is

$$\text{given by } \left(1 - \sum_{k=j+1}^N p_{jk} \right).$$

One of the great advantages of such a model is its flexibility, which enables one to cover different models presented in the literature, due to different methods for parameter settings. For example, if one sets $P_{jk} = 0$ for all $k > (j + 1)$, then, the model presented by Zuo et al. [13] is attained and the authors simulation model can be easily verified, which keeps its generality.

The item is sold with a FRW policy with warranty

period T . This requires the manufacturer to either repair or provide replacements for failed items, free of charge, up to time T , from the time of the initial purchase. The warranty expires at time T after purchase. It is assumed that the item is repairable and that the manufacturer has the option of either repairing a failed item or replacing it with a new one when it is returned under warranty and the buyer's claim is valid. The repair or replacement time is assumed to be relatively short compared with the mean time between failures and, hence, can be treated as negligible. The optimal choice, based on minimizing the expected cost of warranty service, has the following general form:

The failed item in failure state j is $1 \leq j \leq N$. Whereas the residual warranty time up to the expiration of the warranty period is t , $0 \leq t \leq T$ is replaced with a new one, if, and only if, $k \leq j \leq N$ and $t \geq \alpha$, otherwise, it is minimally repaired, where $2 \leq k \leq N$ and $0 < \alpha \leq T$;

k and α are the decision variables in this optimization problem and the policy is characterized by these two parameters. The manufacturer has to select the parameters, k and α , to minimize the expected cost of servicing the warranty. Under this policy, most items which failed during the warranty period (assuming higher costs for replacement in comparison with repair costs) would normally be minimally repaired. However, if an item fails early in the warranty period and if, for some reason, the degree of deterioration is large, then, it is more economical to replace it with a new one. As a result, the policy effectively avoids (a) unnecessary replacements when the failed item has only experienced minor deterioration and (b) excessive repairs when the failed item has already experienced heavy deterioration and the remaining warranty service time is still long.

Let $C(\alpha, k; T)$ denote the expected warranty servicing cost per item to the manufacturer under this warranty servicing policy. Then, the problem is to determine which optimal k and α values yield the minimum value for $C(\alpha, k; T)$. Initially, the notation used in this paper is introduced. Second, an expression is derived for $C(\alpha, k; T)$ and, then, in the next section, the simulation framework used for finding the best values of the decision variables is explained. For more convenience, the following notation, presented by Zuo et al. [13] is used:

N	number of working states,
T	length of original warranty period,
K	decision variable ($2 \leq k \leq N$),
α	decision variable ($0 < \alpha \leq T$),
$C_m^{(i)}$	cost of minimal repair, given that the item is failed in state i ($i = 1, 2, \dots, N$),

- $C_r^{(i)}$ cost of replacing the failed item, given that it failed in state i ($i = 1, 2, \dots, N$),
- μ_i rate of transition from working state i ($i = 1, 2, \dots, N$),
- P_{ij} probability of the item entering working state j , given that it has made a transition out of working state i ($i = 1, 2, \dots, N - 1$),
- $\left(1 - \sum_{k=i+1}^N P_{ik}\right)$ probability of the item entering failure state i , given that it has made a transition out of working state i ($i = 1, 2, \dots, N - 1$),
- L length of confidence interval,
- A level of significance,
- S standard deviation of sample,
- NOR number of replications.

For more clarification, Figure 1 shows all the possible states of the item and the possible transitions among the states, where circles and squares denote working states and failure states, respectively.

In addition, the following assumptions are made:

- (a) After any failure, the manufacturer faces the customer claim and all failures during the warranty period are valid.
- (b) The time taken for repair or replacement, in comparison with the mean time between failures, is treated as negligible.

Furthermore, it is assumed that, as the item passes to the latest working state, due to usage, where the degree of deterioration gradually

increases, the corresponding rate of transition to the failure state increases too. In other words, it is more likely to make the transition to a failure state instead of a transition to another working state. Assumption (c) shows this condition mathematically:

$$(c) \quad \mu_1 \left(1 - \sum_{j=2}^N P_{1j}\right) < \mu_2 \left(1 - \sum_{j=3}^N P_{2j}\right) < \dots < \mu_N \quad (j = 1, 2, \dots, N - 2).$$

Due to the different patterns of usage of identical items, different patterns of deterioration may take place. As a special pattern, it is assumed that the mean rate of transition between working states decreases, as the distance between states increases and transition between more adjacent states is more likely. In this manner, not only is it possible to consider the possibility of transition between non-successive states, but it is also possible to contemplate real world conditions in this model. It is important to mention that the chances of an item transiting from state 1 to N are very low. Assumption (d) implies this situation mathematically.

$$(d) \quad \mu_j P_{j(j+1)} > \mu_j P_{j(j+2)} > \dots > \mu_j P_{jN},$$

$$(j = 1, 2, \dots, N - 2).$$

Furthermore, it is assumed, as the deterioration degree increases, that the needed amendatory cost will increase. Items that have passed through

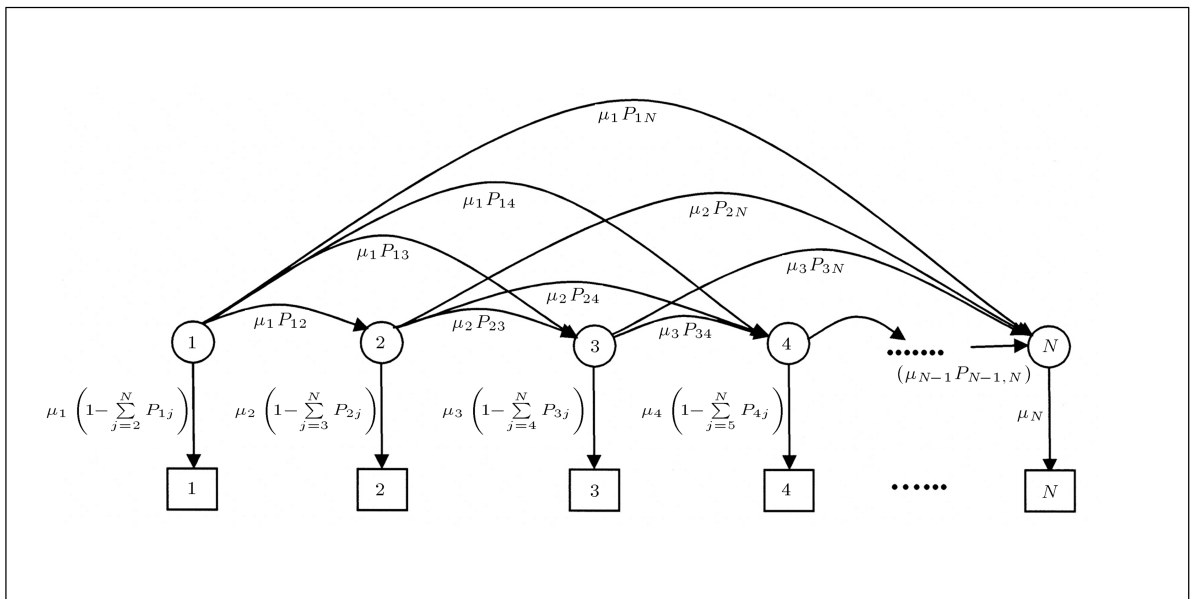


Figure 1. State transition diagram of a warranted item.

too many states become more costly to repair or replace. Also, in case of replacement, this change may have a non-decreasing trend, because many deteriorated items that need to be replaced can no longer be used. The mathematical expressions of these assumptions are presented in (e) and (f) as follows:

$$(e) \quad C_m^{(1)} < C_m^{(2)} < \dots < C_m^{(N-1)} < C_m^{(N)}.$$

$$(f) \quad C_r^{(1)} \leq C_r^{(2)} \leq \dots \leq C_r^{(N-1)} \leq C_r^{(N)}.$$

WARRANTY COST AND SIMULATION MODEL

For special case $N = 2$, i.e., the item has only two possible working states and two possible failure states, the model changes to the model presented by Zuo et al. [13] and the optimal policy can easily be obtained mathematically, as presented in their paper. So, in this case, finding the deterministic solution is possible and simulation is not required to find an approximate solution. However, the simulation solution can be compared with the optimal one as a tool for validation of the simulation model. Therefore, it is assumed that the number of working states is at least 3 and, so the number of failure states.

The general case with $N > 2$ is now considered. Let X_i and I_i represent the time to the first failure and the corresponding failure state, respectively, under the assumption that the item is in working state i ($i = 1, 2, \dots, N$) at time $t = 0$ (general case). Clearly, X_i and I_i are two dependent random variables, where $X_i \in (0, \infty)$ and $I_i = i, i + 1, \dots, N$. In order to derive an expression of warranty servicing cost, one should find the joint probability distribution of these two random variables in a way similar to the method proposed by Zuo et al. [13].

The cumulative and density probability distribution function of the time to the first failure is defined as follows:

$$F_{ij}(x) = \Pr\{X_{i \leq x} \text{ and } I_i = j\},$$

$$j = i, i + 1, \dots, N,$$

$$1 \leq i \leq N, \quad (1)$$

$$f_{ij}(x) = \frac{dF_{ij}(x)}{dx}, \quad 1 \leq i \leq j \leq N. \quad (2)$$

The deterioration process of the item, given that all the time distributions of staying at each working state are exponential, can be regarded as a continuous-time Markov process with $2N$ possible states (see Figure 1). In order to derive the joint probability distribution of

X and I , one treats the N possible failure states of the item as the N absorbing states of this Markov process. The first failure time of the item is the time at which the process enters an absorbing state. Define:

$$P_i(t) = \Pr \{ \text{The item is in working state } i \text{ at time } t \},$$

$$i = 1, 2, \dots, N,$$

$$Q_i(t) = \Pr \{ \text{The item is in failure state } i \text{ at time } t \},$$

$$i = 1, 2, \dots, N.$$

It is evident that, by this definition, one has:

$$F_{ij}(t) = \Pr \{ \text{The failure time of the item is less$$

than or equal to t and its failure state

$$\text{is } j \text{ at time } t \} = Q_j(t),$$

$$j = i, i + 1, i + 2, \dots, N. \quad (3)$$

One now just needs to find the values of $Q_j(t)$ for any j , which is not a difficult process. By writing out the Kolmogorov equations for this Markov process, one should be able to find the $Q_j(t)$. One has:

$$\frac{d}{dt} \begin{pmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_N(t) \end{pmatrix} = \begin{pmatrix} -\mu_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ P_{12}\mu_1 & -\mu_2 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ P_{1N}\mu_1 & P_{2N}\mu_2 & \dots & P_{N-1,N}\mu_{N-1} & -\mu_N \end{pmatrix} \begin{pmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_N(t) \end{pmatrix}, \quad (4)$$

$$\frac{dQ_i(t)}{dt} = \mu_i \left(1 - \sum_{j=i+1}^N P_{ij} \right) P_i(t),$$

$$i = 1, 2, \dots, N - 1, \quad (5)$$

$$\frac{dQ_N(t)}{dt} = \mu_N P_N(t). \quad (6)$$

In order to reach the mentioned assumptions for the initial situations of the sold item, the initial conditions of these differential equations are adjusted as:

$$P_k(0) = 1, \quad P_i(0) = 0,$$

$$\text{for } i = 1, \dots, k-1, k+1, \dots, N,$$

$$Q_i(0) = 0, \quad \text{for } i = 1, 2, \dots, N.$$

Then, the $Q_i(t)$, obtained by solving Equations 4 to 6, should equal F_{k_j} for all $k \leq j \leq N$. In other words, solving Equations 4 to 6 with these initial conditions, which is compatible with the authors assumptions, leads one to obtaining the joint probability distribution, $f_{k_j}(t)$, for all $1 \leq k \leq j \leq N$. Now, one is able to find the expected warranty servicing cost formulation.

Let $A_i(t)$ represent the expected cost to the manufacturer during the remaining warranty period, given that the item is in working state i ($i = 1, 2, \dots, N$) and the length of the remaining warranty period is t ($0 < t \leq T$). Then, the expected total warranty servicing cost per item to the manufacturer is given by $A_1(T)$.

In the following, the integral equations for $A_i(t)$ are written. Two different cases need to be considered.

Case 1: $t < \alpha$

Taking note that the item has an exponential sojourn time in each working state and that its repair and replacement times are negligible, one has:

$$A_i(t) = \sum_{j=i}^N \int_0^t [C_m^{(j)} + A_j(x)] f_{ij}(t-x) dx, \quad i = 1, 2, \dots, N. \quad (7)$$

Case 2: $t \geq \alpha$

$$A_i(t) = \begin{cases} \omega_i(t) + \sum_{j=i}^k \int_{\alpha}^t [C_m^{(j)} + A_j(x)] f_{ij}(t-x) dx \\ \quad + \sum_{j=k+1}^N \int_{\alpha}^t [C_r^{(j)} + A_1(x)] f_{ij}(t-x) dx \\ \text{for } i = 1, 2, \dots, k \\ \omega_i(t) + \sum_{j=i}^N \int_{\alpha}^t [C_r^{(j)} + A_1(x)] f_{ij}(t-x) dx \\ \text{for } i = k+1, \dots, N \end{cases} \quad (8)$$

where:

$$\omega_i(t) = \sum_{j=i}^N \int_0^{\alpha} [C_m^{(j)} + A_j(x)] f_{ij}(t-x) dx, \quad \alpha \leq t \leq T, \quad i = 1, 2, \dots, N. \quad (9)$$

It is very difficult to obtain closed form solutions of Equations 8 and 9. However, they can be easily evaluated by simulation. The goal is to calculate the warranty servicing cost, $A_1(T)$. A simplified flowchart of the simulation model developed in this study is given in Figure 2. This chart shows the basic dynamics of the modeled replacement-repair process.

Referring to Figure 2, the warranty process begins when the item is placed in service. The next event in the process occurs when the first failure is perceived. According to predefined conditions, related events occur. Then, the warranty process starts over with modified parameters until one of the paths leads to the event "process ends".

In order to produce an estimated cost for the warranty of particular specifications, the simulation model must have, as input, the characteristics of the item's life distribution in each working state and the specifications of the warranty provisions. The necessary inputs for a simulation run are:

1. The parameters of life distribution in each working state, μ_i ;
2. The length of the warranty period, T ;

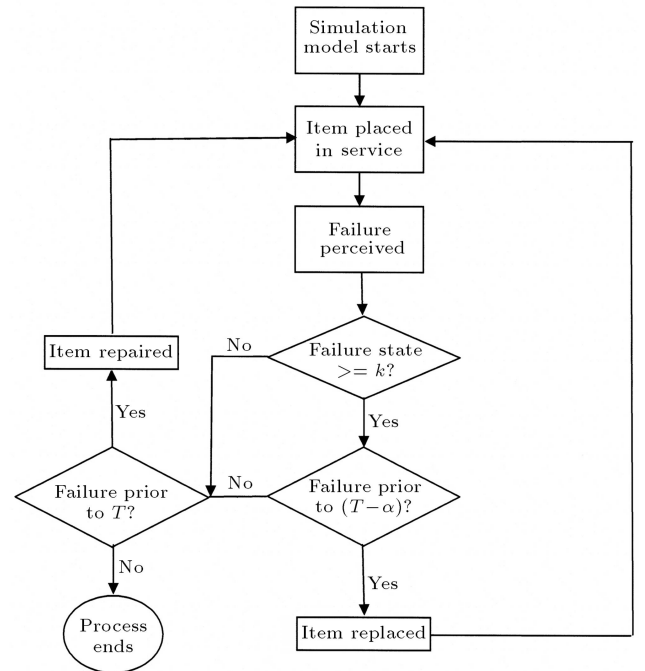


Figure 2. Flow chart of the warranty process.

3. The number of working and failure states, N ;
4. The probabilities of transition between states, P_{ij} ;
5. The cost of repair and replacement in each failure state, $C_m^{(i)}$ and $C_r^{(i)}$;
6. The policy specification parameters, α and K .

The simulation program was executed with VISUAL SLAM (Awesim 3.0) software. A listing of the V.S. source of the modeling statement is available from the authors upon request. The key to successful implementation of any Mont Carlo simulation model is the random number generation of different probability distributions. The V.S. software made this very easy by preparing the predefined and available tools for generating random numbers of different probability distributions.

In order to find the appropriate value of the needed number of replications, it is necessary to determine the level of significance and the length of the confidence interval around the population mean. This length is equivalent to the accepted deviation from the population mean. With respect to the central limit theorem, the sample mean follows a normal distribution for a large number of replications. So, the length of two-sided confidence interval (L) to estimate the population mean, can be shown as:

$$L = 2 \times z_{\alpha/2} \times S/\sqrt{\text{NOR}}. \quad (10)$$

In Equation 10, $z_{\alpha/2}$ indicates the percentile of a standard normal distribution at $(1-\alpha)$ level of significance, and S stands for the standard deviation of the sample. When, for an examined simulation practice with a known number of replications, the calculated value of L becomes less than the predefined accepted range of deviation from the population mean (or equally its estimation), then, the appropriate number of replications has been specified. It is necessary to point out that one needs to find the minimum value of NOR that satisfies the mentioned condition, considering the time required for completing the simulation.

NUMERICAL EXAMPLE

The simulation method described in the previous subsection was run to calculate the warranty servicing cost for the following numerical example, with the inputs, as follows:

$$\begin{aligned} N &= 4, & T &= 3 \text{ years}, & \mu_1 &= 0.5/\text{year}, \\ \mu_2 &= 2/\text{year}, & \mu_3 &= 3/\text{year}, & \mu_4 &= 3.5/\text{year}, \\ P_{12} &= 0.6, & P_{13} &= 0.2, & P_{14} &= 0.1, \\ P_{23} &= 0.5, & P_{24} &= 0.1, & P_{34} &= 0.6, \\ C_r^{(1)} &= \$40, & C_r^{(2)} &= \$50, & C_r^{(3)} &= \$300, \\ C_r^{(4)} &= \$400, & C_m^{(1)} &= \$300, & C_m^{(2)} &= \$500, \\ C_m^{(3)} &= \$600, & C_m^{(4)} &= \$800. \end{aligned}$$

It is necessary to mention that it is assumed that, at most, 5% deviation from the estimated population mean will be accepted and that the level of significance is 95%. Numerical results in this example show that, with nearly 5,000 replications, the mentioned condition will be satisfied. These results are presented in Table 1 and the effect of different values of k and α on the mean warranty cost is shown in Figure 3.

If the mean warranty cost has to be the only criterion for selecting the optimal value of the decision variables, then $K = 3$ and $\alpha = 0.4$ are the optimal values that have minimum warranty costs. As can be seen, for these parameters, the total expected and standard deviations of warranty servicing costs are \$484.58 and \$12.34, respectively. In other words, to minimize the expected warranty servicing cost when the warranty period is 3 years, the manufacturer should minimally repair all failures, except when the item is in failure state 3 or 4 and the remaining warranty period is longer than 0.4 years.

For better understanding, Figure 3 presents a graphical comparison between different values of warranty costs, according to different values of variables K and α .

Small differences between the optimal expected warranty cost and its adjacent values, encouraged the performance of a statistical test, in order to clarify whether or not there is any other parameter setting with a smaller mean warranty cost. It is required to test the null hypothesis, $H_0 : \theta^* \leq \theta$ versus $H_1 : \theta^* > \theta$, where θ^* and θ are the population mean of the warranty cost for an optimal parameter setting and other parameters to be tested, respectively.

For this purpose, it is necessary to define a new level of significance and an appropriate statistic. According to the nature of the performed simulation and the lack of any rational reason for the variance equality of the sample mean for different parameters, it seems that the most appropriate statistic for testing

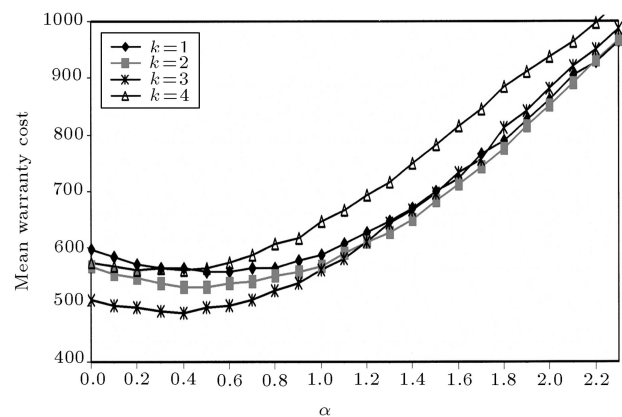


Figure 3. Mean warranty servicing cost as a function of K and α .

Table 1. Numerical results of simulation model including mean warranty cost and mean standard deviation (S.D.).

<i>K</i>								α
4		3		2		1		
Mean S.D.	Mean Warranty Cost	Mean S.D.	Mean Warranty Cost	Mean S.D.	Mean Warranty Cost	Mean S.D.	Mean Warranty Cost	
14.06	573.65	12.49	508.09	13.37	567.66	12.72	598.27	0
13.97	567.73	11.96	497.29	13.13	554.77	12.63	585.96	0.1
14.01	562.89	11.81	494.23	13.17	546.94	12.89	571.18	0.2
14.01	563.59	12.02	494.39	13.17	539.53	12.90	566.34	0.3
14.27	562.41	12.34	484.58*	13.38	532.77	12.86	565.34	0.4
14.60	563.97	12.08	495.31	13.30	532.52	12.96	559.52	0.5
14.87	574.64	12.37	497.87	13.81	537.50	13.20	558.42	0.6
15.24	587.22	12.83	510.15	13.28	542.51	13.27	563.54	0.7
15.88	607.53	13.47	525.79	13.66	551.02	13.82	563.94	0.8
16.81	618.79	14.06	538.13	14.13	558.99	14.35	577.31	0.9
17.55	646.33	15.09	561.52	14.81	567.72	14.95	588.75	1
18.65	665.48	15.78	581.48	15.74	591.85	16.08	606.51	1.2
19.29	692.52	16.80	611.68	16.97	609.59	16.85	628.71	1.3
22.71	716.84	18.34	642.72	18.09	627.94	18.20	648.33	1.4
21.37	748.73	19.31	665.61	19.50	650.48	18.97	671.52	1.5
22.05	781.40	21.14	696.80	20.50	682.53	19.71	700.37	1.6
22.97	814.68	22.08	732.57	21.43	711.61	21.21	723.54	1.7
23.32	843.67	23.50	756.32	23.06	743.79	21.85	764.75	1.8
24.23	885.05	24.17	811.33	24.02	775.92	22.92	789.38	1.9
25.72	912.32	25.10	842.38	25.17	814.33	24.86	824.09	2
26.72	938.58	26.38	881.96	26.40	851.32	25.66	861.84	2.1
27.72	964.47	27.62	922.33	27.70	891.79	26.87	904.06	2.2
29.18	997.38	28.11	952.02	28.82	931.21	28.20	929.04	2.3
30.45	1032.03	28.90	986.25	29.12	967.79	28.77	964.02	2.4
31.32	1045.59	30.06	1022.56	29.91	1001.04	29.92	1001.27	2.5
31.67	1077.49	30.50	1049.92	30.58	1030.88	31.01	1027.54	2.6
32.11	1090.76	31.46	1066.51	31.22	1061.23	32.25	1055.66	2.7
32.70	1100.60	31.85	1090.08	32.43	1090.40	33.24	1095.19	2.8
32.95	1120.21	32.76	1116.33	32.77	1110.19	33.73	1116.94	2.9
33.12	1121.43	33.00	1116.39	33.13	1113.90	34.12	1120.63	3

the mentioned hypothesis is as follows:

$$t = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_x^2}{NOR_x} + \frac{S_y^2}{NOR_y}}}. \tag{11}$$

In Equation 11, \bar{X} and \bar{Y} indicate the sample means concerning the optimal and compared parameter settings. It is known that, for a large number of observations, the asymptotic distribution of this statistic follows a standard normal distribution. If the computed

statistic, based on the numerical results, is smaller than, or equal to, z_α (the percentile of a standard normal distribution at (1- α) level of significance), then, the null hypothesis will not be rejected and current parameters remain optimal. Otherwise, there is not enough evidence to select the current solution as the best possible one.

Among the presented results in Table 1, Equation 11 was computed for comparing two sets of samples at a 95% level of significance, using MINITAB software;

the first with $\alpha = 0.4$ and $k = 3$ (current optimal solution) and the second with $\alpha = 0.3$ and $k = 3$ (the nearest adjacent). The following software output proved the authors claim, regarding the optimality of $\alpha = 0.4$ and $k = 3$, in comparison with $\alpha = 0.3$ and $k = 3$:

Test of $\theta^* \leq \theta$ VS $\theta^* > \theta$

Variable	N	Mean	SE Mean
$\alpha = 0.4, k = 3$	5000	484.58	12.34
$\alpha = 0.3, k = 3$	5000	494.39	12.02

Variable	T	P -Value
$\alpha = 0.3, k = 3$	0.56	0.2877

Similar comparisons were performed for other comparable alternatives and the same results were gained. So, the mentioned total expected warranty cost, that is \$484.58, will remain optimal.

As stated previously, by implementing some changes in the model, one can reach some useful results and this model may be transformed to some similar models presented in the literature. So, the model verification seems very easy. For this, another example was chosen that is, to some extent, similar to a previous example presented by Zuo et al. [13]. They used a numerical approach to find the optimal policy for the warranty model. The inputs are as follows:

$$\begin{aligned}
 N = 4, & \quad T = 3 \text{ years}, & \mu_1 = 0.5/\text{year}, \\
 \mu_2 = 2/\text{year}, & \mu_3 = 3/\text{year}, & \mu_4 = 3.5/\text{year}, \\
 P_{12} = 0.9, & P_{13} = P_{14} = 0, & P_{23} = 0.6, \\
 P_{24} = 0, & P_{34} = 0.6, & C_r^{(1)} = \$40, \\
 C_r^{(2)} = \$50, & C_r^{(3)} = \$300, & C_r^{(4)} = \$400, \\
 C_m^{(1)} = \$300, & C_m^{(2)} = \$500, & C_m^{(3)} = \$600, \\
 C_m^{(4)} = \$800.
 \end{aligned}$$

After running the model in V.S., fortunately, the result was completely similar to that presented in their paper. Their optimal parameters were $K = 3$ and $\alpha = 0.5$, with a total warranty servicing cost of \$408. So, to minimize the expected warranty servicing cost when the warranty period is 3 years, the manufacturer should minimally repair all failures, except when the item is in failure state 3 or 4 and the remaining warranty period is longer than 0.5 years. Similarly, the numerical results of this simulation, based on 5,000 replications, obtains $K = 3$ and $\alpha = 0.5$ with a nearly equal mean cost of warranty and 7.68 for the standard deviation.

CONCLUSIONS

In this paper, a warranty servicing policy has been developed for a product with N working states and N failure states. The policy is characterized by two parameters. For the special case where $N = 2$, the

optimal parameter values can be calculated analytically. When $N > 2$, a simulation procedure has been proposed. The model can be used to minimize the warranty servicing costs of multi-state deteriorating products, using minimal repairs and replacements. The model reported in this paper can be extended in several ways. Some of the many issues that need further study are listed below:

1. It has been assumed that the sojourn times are exponential. One can relax this assumption and treat the sojourn time to follow more general distributions, with parameters varying with respect to the state. The power of V.S. software in generating different kinds of random number, makes it easy to analyze very different items, which have their own special specifications.
2. The FRW policy is one of the many different warranty policies presented in the literature. One can carry out similar analyses to obtain the optimal repair-replacement strategies for other warranty policies.
3. In the authors' model, the deterioration is characterized discretely and gradually through N states. A natural extension is to characterize the deterioration as a continuous variable. This will imply an infinite state space.

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